Asymptotic Behavior of Periodic Cohen-Grossberg Neural Networks with Delays

Wei Lin
linwei@usc.edu
Department of Mathematics, University of Southern California, Los Angeles, CA 90089, U.S.A.

Without assuming the positivity of the amplification functions, we prove some M-matrix criteria for the $\mathbb{R}_+^n$-global asymptotic stability of periodic Cohen-Grossberg neural networks with delays. By an extension of the Lyapunov method, we are able to include neural systems with multiple nonnegative periodic solutions and nonexponential convergence rate in our model and also include the Lotka-Volterra system, an important prototype of competitive neural networks, as a special case. The stability criteria for autonomous systems then follow as a corollary. Two numerical examples are provided to show that the limiting equilibrium or periodic solution need not be positive.

1 Introduction

Since the seminal work of Cohen and Grossberg (1983), many efforts have been devoted to investigating the asymptotic behavior of the Cohen-Grossberg neural networks. The results in Cohen and Grossberg have been improved and extended in many directions. Ye, Michel, and Wang (1995) proved that the Cohen-Grossberg network with transmission delays is still globally stable as long as the delays are sufficiently small. Without assuming the symmetry of interconnections, Wang and Zou (2002a) and Chen and Rong (2003) provided some delay-independent conditions for global stability. Exponential stability was discussed in Wang and Zou (2002b), Chen and Rong (2004), and Cao and Liang (2004), where the amplification functions were assumed to have positive lower and upper bounds. Lu and Chen (2003) presented some new results by dropping the upper boundedness of the amplification functions, and dropping the positive lower bounds as well if exponential stability is not desired. The dynamics of Cohen-Grossberg networks with discontinuous activation functions was studied in Lu and Chen (2005).

As a model for competitive neural networks, the Cohen-Grossberg neural network includes the Lotka-Volterra system, which is widely used for describing the dynamics of interacting populations (see, e.g., Murray, 2002), as a special case. However, very few of the existing results for

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Cohen-Grossberg networks can be applied directly to the Lotka-Volterra system, and some important characteristics of competitive neural networks are excluded due to assumptions that are too restrictive. To illustrate this, consider a simple Lotka-Volterra system,

$$\dot{x}_i = x_i\left(r_i - \sum_{j=1}^{n} w_{ij}x_j\right), \qquad i = 1, \ldots, n, \tag{1.1}$$

where x_i are population densities, r_i are natural growth rates, and w_{ij} are interaction weights.

First, most stability analyses require the system to have a unique equilibrium; however, this is not true for system 1.1. It is easily seen that if the system of equations

$$r_i - \sum_{j=1}^{n} w_{ij}x_j = 0, \qquad i = 1, \ldots, n$$

has a unique solution with $x_i \neq 0$ for all i, then system 1.1 has exactly 2^n equilibria. In the existing literature, a common way to guarantee the uniqueness of the equilibrium is to assume that the amplification functions (x_i in system 1.1) are always positive, even at an equilibrium, and this restriction rules out the Lotka-Volterra system.

Second, many stability results involve exponential stability and thus do not apply to the Lotka-Volterra system, since the convergence of that system may not be exponential. A simple illustrative example is the one-dimensional case of system 1.1 with $r = 0$:

$$\dot{x} = -wx^2.$$

Solving this equation with the initial condition $x(0) = x_0 > 0$ gives the exact solution

$$x = \frac{1}{wt + x_0^{-1}},$$

which converges to the equilibrium $x = 0$ at a power rate. To obtain an exponential stability result, one has to assume that the amplification functions have positive lower bounds, which is not feasible for analyzing system 1.1 unless additional assumptions are made on the positivity of the limiting equilibrium.

In a paper most related to this letter, Lu and Chen (2007) studied the $\mathbb{R}^n_+$-global convergence of a delayed Cohen-Grossberg network to a nonnegative, but not necessarily positive, equilibrium, and their results apply as well to the Lotka-Volterra system. However, their method relies on the theory of nonlinear complementary problems and is difficult to be generalized to nonautonomous (e.g., periodic) systems.

The purpose of this letter is to investigate the $\mathbb{R}_+^n$-global asymptotic stability of periodic Cohen-Grossberg neural networks with delays by an extension of the Lyapunov method. The main advantages of our method are twofold: first, multiple equilibria or periodic solutions are allowed, and it is not needed to prove the existence and uniqueness of the equilibrium or periodic solution; second, our method applies equally well to the autonomous case and the periodic case. Using this method, we are able to include the Lotka-Volterra system as a special case in our results. Moreover, the M-matrix form of our stability criteria makes them easy to verify and leads to an intuitive interpretation.

The rest of the letter is organized as follows. In section 2, we specify our model and make some preparations for further analysis. In section 3, we establish our main stability results. Two numerical examples are given in section 4, and section 5 concludes this letter.

2 Preliminaries

In this section, we specify our model and give some preliminary results needed in further analysis. We consider a periodic Cohen-Grossberg neural network with delays described by the system of differential equations,

$$\dot{x}_i(t) = a_i(x_i(t)) \Bigg[-b_i(x_i(t), t) + \sum_{j=1}^{n} w_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} v_{ij}(t) g_j(x_j(t-\tau_{ij})) + I_i(t) \Bigg], \quad i = 1, \ldots, n, \tag{2.1}$$

where $a_i(x)$ are amplification functions, $b_i(x,t)$ are self-inhibition terms, $f_i(x)$ and $g_i(x)$ are activation functions, $w_{ij}(t)$ and $v_{ij}(t)$ are connection weights, $\tau_{ij} > 0$ are transmission delays, and $I_i(t)$ are external inputs. Throughout this letter, we assume $a_i(x)$, $b_i(x,t)$, $f_i(x)$, $g_i(x)$, $w_{ij}(t)$, $v_{ij}(t)$, and $I_i(t)$ are continuous, and $b_i(x,t)$, $w_{ij}(t)$, $v_{ij}(t)$, and $I_i(t)$ are periodic in t with period $\omega > 0$. In addition, we make the following assumptions:

A1: $a_i(x) > 0$ for all $x > 0$, and for any $\beta > \alpha > 0$,

$$\int_0^{\alpha} \frac{1}{a_i(x)}\, dx = \infty \quad \text{and} \quad \int_{\alpha}^{\beta} \frac{1}{a_i(x)}\, dx < \infty.$$

A2: There are continuous ω-periodic functions $B_i(t) > 0$ such that

$$\frac{b_i(x,t) - b_i(y,t)}{x - y} \geq B_i(t)$$

for all x, y, and t;

A3: There are constants $F_i > 0$ and $G_i > 0$ such that

$$|f_i(x) - f_i(y)| \le F_i|x - y| \quad \text{and} \quad |g_i(x) - g_i(y)| \le G_i|x - y|$$

for all x and y.

Denote $\tau = \max_{i,j} \tau_{ij}$. The initial condition with system 2.1 is (IC) $x_i(t) = \theta_i(t) > 0$ for $-\tau \le t \le 0$, where $\theta : [-\tau, 0] \to \mathbb{R}^n$ is continuous.

When $a_i(x) = x$, $b_i(x, t) = B_i(t)x$, $f_i(x) = x$, and $g_i(x) = x$, system 2.1 reduces to a periodic Lotka-Volterra system with delays

$$\dot{x}_i(t) = x_i(t)\left[-B_i(t)x_i(t) + \sum_{j=1}^{n} w_{ij}(t)x_j(t) + \sum_{j=1}^{n} v_{ij}(t)x_j(t - \tau_{ij}) + I_i(t)\right], \quad i = 1, \dots, n, \tag{2.2}$$

for which assumptions A1 to A3 are all satisfied.

When b_i, w_{ij}, v_{ij}, and I_i do not depend on t, system 2.1 reduces to the autonomous system

$$\dot{x}_i(t) = a_i(x_i(t))\left[-b_i(x_i(t)) + \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} v_{ij} g_j(x_j(t - \tau_{ij})) + I_i\right], \quad i = 1, \dots, n, \tag{2.3}$$

for which assumption A2 can be replaced:

A2′: There are constants $B_i > 0$ such that

$$\frac{b_i(x) - b_i(y)}{x - y} \ge B_i$$

for all x and y.

Assumption A1 and the initial condition (IC) ensure that the orbits of system 2.1 always stay in the open positive orthant, although some components of the solution may approach zero as $t \to \infty$. More precisely, we have the following lemma:

Lemma 1 (Positivity). *Assume assumption A1 holds. The solution of system 2.1 with the IC satisfies $x_i(t) > 0$ for all $t > 0$ and all i.*

Proof. Suppose there exist $T > 0$ and index i such that $x_i(T) = 0$ and $x_i(t) > 0$ for $0 < t < T$. Then we have

$$\int_0^T \left[-b_i(x_i(t), t) + \sum_{j=1}^n w_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^n v_{ij}(t) g_j(x_j(t - \tau_{ij})) + I_i(t) \right] dt$$
$$= \int_0^T \frac{\dot{x}_i(t)}{a_i(x_i(t))}\, dt = \int_{x_i(0)}^{x_i(T)} \frac{1}{a_i(u)}\, du = -\int_0^{\theta_i(0)} \frac{1}{a_i(u)}\, du = -\infty.$$

But the left side is bounded since all the functions in the integrand are continuous on $[0, T]$, yielding a contradiction. Therefore, $x_i(t) > 0$ for all $t > 0$ and all i.

For the periodic system 2.1, we are interested in the convergence of the system toward a periodic orbit, while for the autonomous system 2.3, convergence to an equilibrium is desired. Adopting the terminology in Lu and Chen (2007), we have the following definitions:

Definition 1. *An ω-periodic solution $x^*(t)$ of system 2.1 with $x_i^*(t) \geq 0$ for all t and all i is said to be $\mathbb{R}_+^n$-globally asymptotically stable if for any IC the solution $x(t)$ of system 2.1 satisfies $\|x(t) - x^*(t)\| \to 0$ as $t \to \infty$.*

Definition 2. *An equilibrium x^* of system 2.3 with $x_i^* \geq 0$ for all i is said to be $\mathbb{R}_+^n$-globally asymptotically stable if for any IC the solution of system 2.3 satisfies $\|x(t) - x^*\| \to 0$ as $t \to \infty$.*

We also need some basic facts about an M-matrix. This will allow the stability criteria to be formulated in a simple form and make a variety of tools available for checking them. For the proof of the following proposition and many other properties of M-matrices, the reader is referred to Chapter 6 in Berman and Plemmons (1994).

Proposition 1. *Let $C = (c_{ij}) \in \mathbb{R}^{n\times n}$ with $c_{ij} \leq 0$ for all $i \neq j$. That C is a nonsingular M-matrix is equivalent to either of the following:*

a. There are positive constants $\xi_1, \xi_2, \ldots, \xi_n$ such that $\sum_{j=1}^n \xi_j c_{ij} > 0$ for all i.
b. There are positive constants $\eta_1, \eta_2, \ldots, \eta_n$ such that $\sum_{j=1}^n \eta_j c_{ji} > 0$ for all i.

3 Stability Results

The aim of this section is to establish our stability results for the periodic system 2.1; stability results for the Lotka-Volterra system 2.2 and the autonomous system 2.3 then follow as corollaries. The Lyapunov method is a common approach to proving the stability of an equilibrium, in which the equilibrium is usually translated to the origin. However, system 2.3 may have multiple equilibria, and it is not trivial at all to see which of them is the limiting equilibrium. The LaSalle invariance principle does not require determining the limiting equilibrium, but its conclusion is weaker, guaranteeing only the convergence of the system to a set of points for which the orbit derivative equals zero. Now we show that by imposing a stronger inequality condition on the orbit derivative, the Lyapunov method can be extended to a form particularly suited for our purpose. Specifically, we have the following:

Proposition 2. *Suppose $V : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $x : \mathbb{R}_+ \to \mathbb{R}^n$ is bounded and uniformly continuous. If there is a constant $c > 0$ such that*

$$\dot{V}(t) \leq -c\|x(t+\omega) - x(t)\| \tag{3.1}$$

for all $t \geq 0$, then there exists an ω-periodic function $x^*(t)$ such that $\|x(t) - x^*(t)\| \to 0$ as $t \to \infty$.

Proof. Integrating both sides from 0 to T gives

$$V(T) - V(0) \leq -c \int_0^T \|x(t+\omega) - x(t)\|\, dt,$$

or

$$\int_0^T \|x(t+\omega) - x(t)\|\, dt \leq \frac{1}{c}[V(0) - V(T)].$$

Letting $T \to \infty$, we have

$$\int_0^\infty \|x(t+\omega) - x(t)\|\, dt \leq \frac{1}{c} V(0) < \infty.$$

Note that this can be rewritten as

$$\sum_{n=1}^{\infty} \int_0^\omega \|x(t+n\omega) - x(t+(n-1)\omega)\|\, dt < \infty.$$

By the Cauchy criterion, the sequence $\{x(t+n\omega)\}_{n=1}^{\infty}$ converges in $L^1[0,\omega]$. Since $x(t)$ is bounded and uniformly continuous, $\{x(t+n\omega)\}$ is uniformly bounded and equicontinuous. Then by the Arzelà-Ascoli theorem, there exists a subsequence $\{x(t+n_k\omega)\}$ that converges uniformly on any compact set in $\mathbb{R}$; denote the limit function by $x^*(t)$. It is clear that $x^*(t)$ is also the limit of $\{x(t+n\omega)\}$ in $L^1[0,\omega]$, that is,

$$\lim_{n\to\infty}\int_0^{\omega}\|x(t+n\omega)-x^*(t)\|\,dt=0.$$

It is easily seen that $x(t+n\omega)\to x^*(t)$ uniformly on any compact set in $\mathbb{R}$.

Since

$$x^*(t+\omega)=\lim_{n\to\infty}x(t+(n+1)\omega)=\lim_{n\to\infty}x(t+n\omega)=x^*(t),$$

$x^*(t)$ is ω-periodic. Letting $t=t_0+n\omega$, where $0\le t_0<\omega$, we have

$$\|x(t)-x^*(t)\|=\|x(t_0+n\omega)-x^*(t_0)\|.$$

Then the uniform convergence of $\{x(t+n\omega)\}$ on $[0,\omega]$ implies $\|x(t)-x^*(t)\|\to 0$ as $t\to\infty$.

Remark 1. In the autonomous case, inequality 3.1 can be replaced by $\dot{V}(t)\le -c\|\dot{x}(t)\|$, and the boundedness and uniform continuity need not be assumed. Following the same argument, we obtain

$$\int_0^{\infty}\|\dot{x}(t)\|\,dt\le\frac{1}{c}V(0)<\infty.$$

Then

$$\|x(T)-x(S)\|=\left\|\int_S^T\dot{x}(t)\,dt\right\|\le\int_S^T\|\dot{x}(t)\|\,dt\to 0\quad\text{as}\quad S,T\to\infty,$$

and the convergence of $x(t)$ follows immediately from the Cauchy criterion.

Although proposition 2 ensures the convergence of every orbit to a periodic function, the limiting function may not be the same for different orbits. In order to establish a global stability result, we also need to prove that every two orbits are synchronized asymptotically. This is provided by the following simple but essential proposition:

Proposition 3. *Suppose* $V : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous and* $x : \mathbb{R}_+ \to \mathbb{R}^n$ *and* $y : \mathbb{R}_+ \to \mathbb{R}^n$ *are bounded and uniformly continuous. If there is a constant* $c > 0$ *such that*

$$\dot{V}(t) \leq -c\|x(t) - y(t)\|$$

for all $t \geq 0$*, then* $\|x(t) - y(t)\| \to 0$ *as* $t \to \infty$*.*

Proof. As in the proof of proposition 2, we obtain

$$\int_0^{\infty} \|x(t) - y(t)\|\, dt \leq \frac{1}{c} V(0) < \infty.$$

By the boundedness and uniform continuity of $x(t)$ and $y(t)$, one easily sees that $\|x(t) - y(t)\| \to 0$ as $t \to \infty$.

To apply propositions 2 and 3, we need to guarantee the boundedness and uniform continuity of the solution, which are proved in the following lemma. Denote $\underline{B}_i = \inf_t B_i(t)$, $\underline{b}_i(0) = \inf_t b_i(0, t)$, $\overline{w}_{ij} = \sup_t |w_{ij}(t)|$, $\overline{v}_{ij} = \sup_t |v_{ij}(t)|$, $\overline{I}_i = \sup_t I_i(t)$, and the matrices $\underline{B} = \operatorname{diag}(\underline{B}_1, \ldots, \underline{B}_n)$, $F = \operatorname{diag}(F_1, \ldots, F_n)$, $G = \operatorname{diag}(G_1, \ldots, G_n)$, $\overline{W} = (\overline{w}_{ij})$, and $\overline{V} = (\overline{v}_{ij})$.

Lemma 2 (Boundedness and uniform continuity). *Assume assumptions A1 to A3 hold and the matrix*

$$C = \underline{B} - \overline{W}F - \overline{V}G$$

is a nonsingular M-matrix. Then the solution of system 2.1 with the IC is bounded and uniformly continuous on $\mathbb{R}_+$*.*

Proof. By proposition 1, that C is a nonsingular M-matrix is equivalent to that there are positive constants $\xi_1, \ldots, \xi_n$ such that

$$\xi_i \underline{B}_i - \sum_{j=1}^{n} \xi_j \overline{w}_{ij} F_j - \sum_{j=1}^{n} \xi_j \overline{v}_{ij} G_j > 0$$

for all i. Now let

$$\sigma = \min_i \left\{ \xi_i \underline{B}_i - \sum_{j=1}^{n} \xi_j \overline{w}_{ij} F_j - \sum_{j=1}^{n} \xi_j \overline{v}_{ij} G_j \right\},$$

$$J = \max_i \left\{ |\underline{b}_i(0)| + \sum_{j=1}^{n} \overline{w}_{ij} |f_j(0)| + \sum_{j=1}^{n} \overline{v}_{ij} |g_j(0)| + |\overline{I}_i| \right\},$$

and $M_i = \lambda\xi_i$ for some $\lambda > 0$. If $x_i(t)$ ever exceeded M_i, $x_i(t)$ would attain M_i at some time t due to its continuity. Suppose $x_i(t_0) = M_i$ for some index i and $x_j(t) \le M_j$ for all $t \le t_0$ and all j. Then we have

$$\begin{aligned}\frac{\dot{x}_i(t_0)}{a_i(x_i(t_0))} &= -b_i(x_i(t_0), t_0) + \sum_{j=1}^{n} w_{ij}(t_0) f_j(x_j(t_0)) \\ &\quad + \sum_{j=1}^{n} v_{ij}(t_0) g_j(x_j(t_0 - \tau_{ij})) + I_i(t_0) \\ &\le -(\underline{B}_i M_i + \underline{b}_i(0)) + \sum_{j=1}^{n} \overline{w}_{ij}(F_j M_j + |f_j(0)|) \\ &\quad + \sum_{j=1}^{n} \overline{v}_{ij}(G_j M_j + |g_j(0)|) + \overline{I}_i \\ &= -\lambda\left(\xi_i \underline{B}_i - \sum_{j=1}^{n} \xi_j \overline{w}_{ij} F_j - \sum_{j=1}^{n} \xi_j \overline{v}_{ij} G_j\right) - \underline{b}_i(0) \\ &\quad + \sum_{j=1}^{n} \overline{w}_{ij} |f_j(0)| + \sum_{j=1}^{n} \overline{v}_{ij} |g_j(0)| + \overline{I}_i \\ &\le -\lambda\sigma + J < 0,\end{aligned}$$

provided λ is sufficiently large. It follows that $\dot{x}_i(t_0) < 0$ since $a_i(x_i(t_0)) > 0$. Therefore, $x_i(t)$ cannot exceed M_i, that is, $x(t)$ is bounded. Moreover, the right side of system 2.1 is bounded, showing that $\dot{x}(t)$ is bounded and $x(t)$ is uniformly continuous.

We are now ready to prove the main result of this letter.

Theorem 1 ($\mathbb{R}^n_+$-global asymptotic stability). *Assume assumptions A1 to A3 hold, and the matrix*

$$C = \underline{B} - \overline{W}F - \overline{V}G$$

is a nonsingular M-matrix. Then system 2.1 has an ω-periodic solution that is $\mathbb{R}^n_+$-globally asymptotically stable.

Proof. By proposition 1, that C is a nonsingular M-matrix is equivalent to that there are positive constants $\eta_1, \ldots, \eta_n$ such that

$$\eta_i \underline{B}_i - \sum_{j=1}^{n} \eta_j \overline{w}_{ji} F_i - \sum_{j=1}^{n} \eta_j \overline{v}_{ji} G_i > 0$$

for all i. Let

$$c = \min_i \left\{ \eta_i \underline{B}_i - \sum_{j=1}^n \eta_j \overline{w}_{ji} F_j - \sum_{j=1}^n \eta_j \overline{v}_{ji} G_j \right\}.$$

We now provide a Lyapunov function. Denote $\Delta x_i(t) = x_i(t+\omega) - x_i(t)$, $\Delta f_i(x_i(t)) = f_i(x_i(t+\omega)) - f_i(x_i(t))$, $\Delta g_i(x_i(t)) = g_i(x_i(t+\omega)) - g_i(x_i(t))$, and $\Delta b_i(x_i(t), t) = b_i(x_i(t+\omega), t+\omega) - b_i(x_i(t), t)$. Let

$$V(t) = \sum_{i=1}^n \eta_i \left| \int_{x_i(t)}^{x_i(t+\omega)} \frac{1}{a_i(u)}\, du \right| + \sum_{i,j=1}^n \eta_i \int_{t-\tau_{ij}}^t \overline{v}_{ij} G_j |\Delta x_j(s)|\, ds.$$

Then we have

$$\begin{aligned}
\dot{V}(t) = {} & \sum_{i=1}^n \eta_i \operatorname{sgn}(\Delta x_i(t)) \left[\frac{\dot{x}_i(t+\omega)}{a_i(x_i(t+\omega))} - \frac{\dot{x}_i(t)}{a_i(x_i(t))} \right] \\
& + \sum_{i,j=1}^n \eta_i \left[\overline{v}_{ij} G_j |\Delta x_j(t)| - \overline{v}_{ij} G_j |\Delta x_j(t-\tau_{ij})| \right] \\
= {} & \sum_{i=1}^n \eta_i \operatorname{sgn}(\Delta x_i(t)) \left[-\Delta b_i(x_i(t), t) + \sum_{j=1}^n w_{ij}(t) \Delta f_j(x_j(t)) \right. \\
& \left. + \sum_{j=1}^n v_{ij}(t) \Delta g_j(x_j(t-\tau_{ij})) \right] \\
& + \sum_{i,j=1}^n \eta_i \left[\overline{v}_{ij} G_j |\Delta x_j(t)| - \overline{v}_{ij} G_j |\Delta x_j(t-\tau_{ij})| \right] \\
\le {} & -\sum_{i=1}^n \eta_i B_i |\Delta x_i(t)| + \sum_{i,j=1}^n \eta_j \overline{w}_{ji} F_i |\Delta x_i(t)| \\
& + \sum_{i,j=1}^n \eta_j \overline{v}_{ji} G_i |\Delta x_i(t-\tau_{ji})| \\
& + \sum_{i,j=1}^n \eta_j \left[\overline{v}_{ji} G_i |\Delta x_i(t)| - \overline{v}_{ji} G_i |\Delta x_i(t-\tau_{ji})| \right] \\
= {} & -\sum_{i=1}^n \left(\eta_i B_i - \sum_{j=1}^n \eta_j \overline{w}_{ji} F_i - \sum_{j=1}^n \eta_j \overline{v}_{ji} G_i \right) |\Delta x_i(t)| \\
\le {} & -c \|\Delta x(t)\|.
\end{aligned}$$

Applying proposition 2 yields the existence of an ω-periodic function $x^*(t)$ such that $\|x(t) - x^*(t)\| \to 0$ as $t \to \infty$. It is easily seen that $x^*(t)$ is also a solution of system 2.1.

We now show that the convergence is global. Let $x(t)$ and $y(t)$ be any two solutions of system 2.1. Denote $\delta x_i(t) = x_i(t) - y_i(t)$, $\delta f_i(x_i(t)) = f_i(x_i(t)) - f_i(y_i(t))$, $\delta g_i(x_i(t)) = g_i(x_i(t)) - g_i(y_i(t))$, and $\delta b_i(x_i(t), t) = b_i(x_i(t), t) - b_i(y_i(t), t)$. Let

$$V(t) = \sum_{i=1}^{n} \eta_i \left| \int_{y_i(t)}^{x_i(t)} \frac{1}{a_i(u)}\, du \right| + \sum_{i,j=1}^{n} \eta_i \int_{t-\tau_{ij}}^{t} \overline{v}_{ij} G_j |\delta x_j(s)|\, ds.$$

Proceeding as above, we obtain $\dot{V}(t) \le -c\|\delta x(t)\|$, and it follows from proposition 3 that $\|x(t) - y(t)\| \to 0$ as $t \to \infty$, proving the uniqueness of $x^*(t)$.

Applying theorem 1 to the Lotka-Volterra system 2.2 immediately yields the following:

Corollary 1. *Assume that the matrix*

$$C = \underline{B} - \overline{W} - \overline{V}$$

is a nonsingular M-matrix. Then system 2.2 has an ω-periodic solution that is $\mathbb{R}^n_+$-globally asymptotically stable.

The stability of the autonomous system 2.3 also follows as a consequence of theorem 1. Denote the matrices $B = \mathrm{diag}(B_1, \ldots, B_n)$, $W = (|w_{ij}|)$, and $V = (|v_{ij}|)$.

Corollary 2. *Assume assumptions A1, A2′, and A3 hold, and the matrix*

$$C = B - WF - VG$$

is a nonsingular M-matrix. Then system 2.3 has an equilibrium that is $\mathbb{R}^n_+$-globally asymptotically stable.

Proof. Taking an arbitrary $\omega > 0$ and applying theorem 1, we obtain the existence of an ω-periodic solution $x^*_\omega(t)$ such that $\|x(t) - x^*_\omega(t)\| \to 0$ as $t \to \infty$. For any positive integer k, we can similarly obtain $x^*_{\omega/k}(t)$. Then $\|x^*_\omega(t) - x^*_{\omega/k}(t)\| \le \|x(t) - x^*_\omega(t)\| + \|x(t) - x^*_{\omega/k}(t)\| \to 0$ as $t \to \infty$. Thus,

$$\begin{aligned}\|x^*_\omega(t) - x^*_{\omega/k}(t)\| &= \|x^*_\omega(t + n\omega) - x^*_{\omega/k}(t + kn \cdot \omega/k)\| \\ &= \|x^*_\omega(t + n\omega) - x^*_{\omega/k}(t + n\omega)\| \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

yielding that $x^*_\omega(t) \equiv x^*_{\omega/k}(t)$. Then we have

$$\dot{x}^*_\omega(t) = \lim_{k\to\infty} \frac{x^*_\omega(t+\omega/k) - x^*_\omega(t)}{\omega/k} = \lim_{k\to\infty} \frac{x^*_{\omega/k}(t+\omega/k) - x^*_{\omega/k}(t)}{\omega/k} = 0$$

for all t. Hence, $x^*_\omega(t)$ is a constant function and is $\mathbb{R}^n_+$-globally asymptotically stable.

Remark 2. Corollary 2 can also be proved directly by defining the Lyapunov function,

$$V(t) = \sum_{i=1}^n \eta_i \left| -b_i(x_i(t)) + \sum_{j=1}^n w_{ij} f_j(x_j(t)) + \sum_{j=1}^n v_{ij} g_j(x_j(t-\tau_{ij})) + I_i \right| + \sum_{i,j=1}^n \eta_i \int_{t-\tau_{ij}}^t |v_{ij}| G_j |\dot{x}_j(s)|\, ds.$$

and using the proposition indicated in remark 1.

Remark 3. Lu, Xu, and Yang (2006) studied a class of Cohen-Grossberg neural networks including the Lotka-Volterra system. However, due to a too restrictive assumption $a_i(s) > 0$ for $s \in \mathbb{R}$, their global asymptotic stability results do not apply to the Lotka-Volterra system. Instead, the global attracting region of the Lotka-Volterra system was estimated, as illustrated in their example 3. In that example, simulations showed that the unique positive equilibrium of the system is in fact globally asymptotically stable, and how to justify this was raised as an unsolved question. This can now be answered by our corollary 2: the criterion matrix associated with the system in their example is a nonsingular M-matrix, and the system therefore has an $\mathbb{R}^n_+$-globally asymptotically stable equilibrium.

Remark 4. Assuming $0 < \underline{\alpha}_i \le a_i(u) \le \overline{\alpha}_i$, Chen (2006) proved that under an M-matrix condition similar to that in our corollary 2, a Cohen-Grossberg neural network with delays has a unique equilibrium. He further obtained an M-matrix stability condition involving $\underline{\alpha}_i$ and $\overline{\alpha}_i$, which was shown in the proof of his theorem 2 to be stronger than his existence condition. Now our corollary 2 shows that his existence condition in fact suffices for the $\mathbb{R}^n_+$–global asymptotic stability, even when the assumption on positive lower and upper bounds is dropped.

4 Examples

In this section, we give two numerical examples showing that the limiting equilibrium or periodic solution need not be positive under our stability criteria.

4.1 Example 1. Consider the autonomous system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[-4x_1(t) + \tanh x_1(t - .2) - 2\tanh x_2(t - 1) + 1] \\ \dot{x}_2(t) &= x_2(t)[-2x_2(t) + \tanh x_2(t - .2) - \tanh x_1(t - 1) + 1]. \end{aligned} \tag{4.1}$$

Noting that $(\tanh x)' \le 1$, we have $G_i = 1$ for all i, and the criterion matrix

$$C = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

A naive method for checking if a matrix is a nonsingular M-matrix is to check if all of its leading principal minors are positive, which is the case in this example. By corollary 2, system 4.1 has an equilibrium that is $\mathbb{R}^n_+$-globally asymptotically stable. The dynamical behavior of the system with some randomly selected initial conditions is shown in Figure 1; it is clear that the first component of the system approaches zero in the limit.

4.2 Example 2. Consider the periodic system

$$\begin{aligned} \dot{x}_1(t) = x_1(t)[&-(5 + \cos t)x_1(t) + (1 + \sin t)\tanh x_2(t - 1) \\ &- (2 + \cos t)\tanh x_3(t - 1) + 2 + \sin t], \\ \dot{x}_2(t) = x_2(t)[&-(5 + \sin t)x_2(t) + (1 + \sin t)\tanh x_1(t - 1) \\ &- (1 + \cos t)\tanh x_3(t - 1) + 2 + \cos t], \\ \dot{x}_3(t) = x_3(t)[&-(7 + \sin t)x_3(t) - (1 + \cos t)\tanh x_1(t - 1) \\ &- (1 + \sin t)\tanh x_2(t - 1) + \cos t], \end{aligned} \tag{4.2}$$

with the criterion matrix

$$C = \begin{bmatrix} 4 & -2 & -3 \\ -2 & 4 & -2 \\ -2 & -2 & 6 \end{bmatrix}.$$

Again, we can verify that C is a nonsingular M-matrix by checking its leading principal minors. It follows from theorem 1 that system 4.2 has a 2π-periodic solution that is $\mathbb{R}^n_+$-globally asymptotically stable. The dynamical behavior of the system with some randomly selected initial conditions is

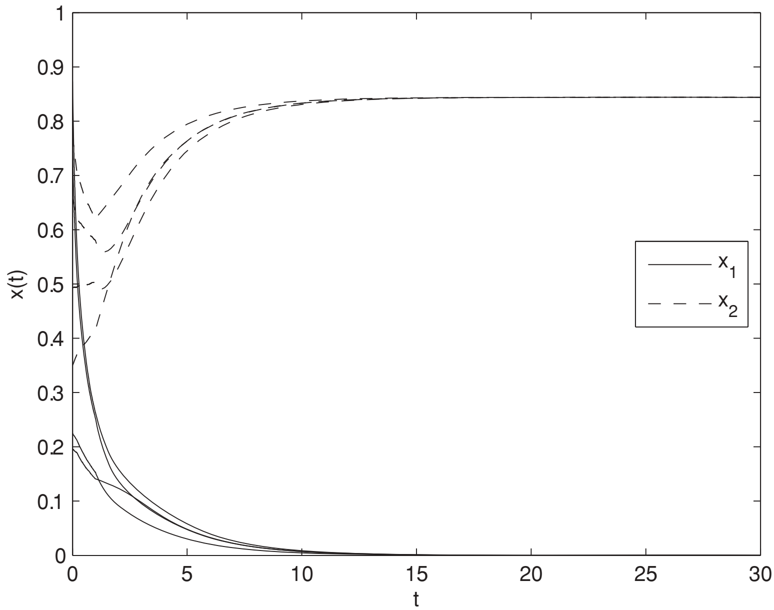

Figure 1: Convergence of system 4.1 to an equilibrium.

shown in Figure 2, where we see that the system converges to a periodic orbit in the plane $x_3 = 0$.

5 Conclusion

Without assuming the positivity of the amplification functions, we have proved some M-matrix criteria for the $\mathbb{R}^n_+$-global asymptotic stability of periodic Cohen-Grossberg neural networks with delays, which include the Lotka-Volterra system as a special case. The stability criteria for autonomous systems then follow as a corollary. Under our criteria, the system converges to an equilibrium or periodic solution that is nonnegative but need not be positive. Moreover, the system is allowed to have multiple equilibria or periodic solutions and a nonexponential convergence rate.

The M-matrix form of our criteria not only facilitates verification but also gives rise to an intuitive interpretation. Roughly speaking, the results state that the system is stable if the self-inhibition terms dominate, up to scaling factors, the interaction effects. It is worth pointing out that our criteria are delay independent and input independent, showing that delays and inputs neither harm nor contribute to the stability in this case. However, inputs do

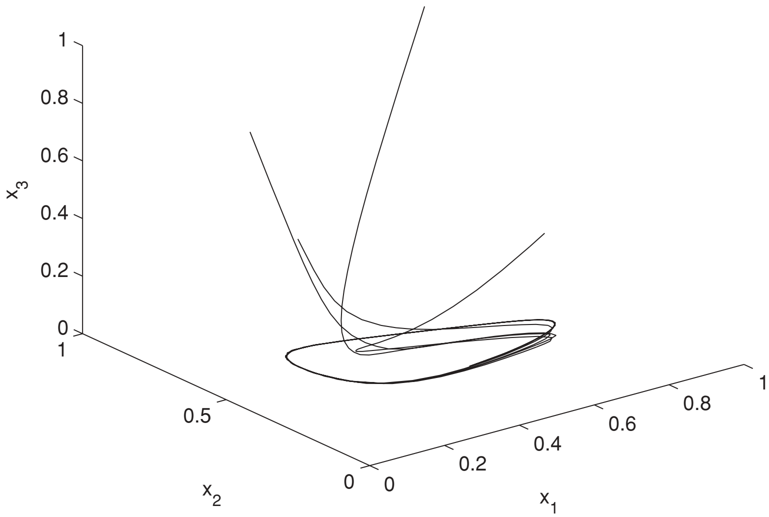

Figure 2: Convergence of system 4.2 to a periodic orbit.

play an important role in determining the limiting equilibrium or periodic solution.

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