Math 867 Spring 2015 Homework 3

Due: June 24, 2015

1. (Multiple testing) In a multiple testing situation with both N_0 and N_1 positive, show that

$$E\left(\frac{a}{N_0}\right) = \bar{\alpha} \quad \text{and} \quad E\left(\frac{b}{N_1}\right) = \bar{\beta},$$

where $\bar{\alpha}$ and $\bar{\beta}$ are the average size and power of the null and non-null cases, respetively. (Refer to Figure 4.1 of Efron's book for the meanings of a and b.)

2. (Benjamini–Hochberg procedure) For $t \in (0, 1]$, let $a(t)$ be the number of null cases with $p_i \le t$, and let $A(t) = a(t)/t$. Verify (4.19) in Efron's book:

$$E\{A(s) \mid A(t)\} = A(t) \qquad \text{for } s \le t.$$

3. (Two-sample mean test) Consider the test statistic in Chen and Qin (2010):

$$T_n = \frac{1}{n_1(n_1-1)} \sum_{\substack{i,j=1 \\ i \ne j}}^{n_1} X_{1i}^T X_{1j} + \frac{1}{n_2(n_2-1)} \sum_{\substack{i,j=1 \\ i \ne j}}^{n_2} X_{2i}^T X_{2j} - \frac{2}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} X_{1i}^T X_{2j}.$$

Assume

$$(\mu_1 - \mu_2)^T \mathbf{\Sigma}_i (\mu_1 - \mu_2) = o(n^{-1} \operatorname{tr}\{(\mathbf{\Sigma}_1 + \mathbf{\Sigma}_2)^2\}), \qquad i = 1, 2.$$

Show that under H_1,

$$\operatorname{Var}(T_n) = \left\{ \frac{2}{n_1(n_1-1)} \operatorname{tr}(\mathbf{\Sigma}_1^2) + \frac{2}{n_2(n_2-1)} \operatorname{tr}(\mathbf{\Sigma}_2^2) + \frac{4}{n_1 n_2} \operatorname{tr}(\mathbf{\Sigma}_1 \mathbf{\Sigma}_2) \right\} \{1 + o(1)\},$$

where the $o(1)$ term vanishes under H_0.

4. (Scaled Lasso) Show that the loss function

$$L_{\lambda_0}(\boldsymbol{\beta}, \sigma) = \frac{\|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2}{2n\sigma} + \frac{\sigma}{2} + \lambda_0 \|\boldsymbol{\beta}\|_1$$

is jointly convex in $(\boldsymbol{\beta}, \sigma)$.