Math 867 Spring 2015 Homework 2

Due: June 3, 2015

1. (Concentration of sample covariances) Prove inequality (A.4) in Rothman, Levina and Zhu (2009): for uniformly sub-Gaussian X_{1j} and sufficiently small $t > 0$,

$$P\left(\max_{i,j} |\hat{\sigma}_{ij} - \sigma_{ij}| > t\right) \le C_1 p^2 e^{-nC_2t^2} + C_3 p e^{-nC_4 t}.$$

2. (Sign consistency of covariance thresholding) Let $\Psi = \{(i,j) \colon \sigma_{ij} \neq 0\}$ be the support of $\boldsymbol{\Sigma}$. *Correct* and prove the second part of Theorem 2 in Rothman, Levina and Zhu (2009): if in addition $|\sigma_{ij}| \ge \tau$ for all $(i,j) \in \Psi$ and $\sqrt{n}(\tau - \lambda) \to \infty$, then with probability tending to 1,

$$\operatorname{sgn}(s_\lambda(\hat{\sigma}_{ij})) = \operatorname{sgn}(\sigma_{ij}) \quad \text{for all } (i,j) \in \Psi.$$

3. (Implementation of graphical Lasso) Implement the graphical Lasso algorithm in Friedman, Hastie and Tibshirani (2008) using R, MATLAB, or any other programming language that you prefer. You may use either coordinate descent or ADMM to solve the inner Lasso problem.
   (a) Print out all source code.
   (b) Design a small simulation study to check if your program works as expected.
   (c) Conduct a timing experiment on a 1000-node problem. Describe your computing environment (CPU and memory) and report the CPU time in seconds.

4. (Unitarily invariant norms) Let a matrix $\mathbf{A}$ be partitioned as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

   and let $\|\cdot\|$ be a unitarily invariant norm. Show that $\|\mathbf{A}_{ij}\| \le \|\mathbf{A}\|$ for all $i, j = 1, 2$.

5. (RSC condition) For an approximately low-rank matrix $\boldsymbol{\Theta}^*$ with $\sum_{j=r+1}^m \sigma_j(\boldsymbol{\Theta}^*) > 0$, prove the following claims in Negahban and Wainwright (2011):
   (a) The set of all $\boldsymbol{\Delta}$ satisfying the constraint

$$\|\boldsymbol{\Delta}_{\mathcal{B}^r}\|_* \le 3\|\boldsymbol{\Delta}_{\mathcal{A}^r}\|_* + 4\sum_{j=r+1}^m \sigma_j(\boldsymbol{\Theta}^*)$$

   includes an open ball around the origin.
   (b) If the operator $\mathfrak{X}$ fails to satisfy the RSC condition

$$\frac{1}{2N}\|\mathfrak{X}(\boldsymbol{\Delta})\|_2^2 \ge \kappa(\mathfrak{X})\|\boldsymbol{\Delta}\|_F^2$$

   for all $\boldsymbol{\Delta}$, then it will also fail to satisfy it under the constraint in part (a).