Exercise Sheet 6: Interactive Proofs in the Set Theory

1 Introduction

1.1 Purpose

The purpose of this exercise is to make you familiar with the practice of interactive proofs with the Rodin Platform on the Set Theory.

1.2 Your Task

We distribute to you a Rodin development named "06_set": it contains 14 contexts (set_0 to set_93), each of which with one theorem except set_93 that contains two theorems. You will be asked to prove these theorems by using 5 successive *tactic profile*: "set_profile_1", "set_profile_2", "set_profile_3", "set_profile_4, and "set_profile_5. Each of them (except "set_profile_4"is a slight extension of the previous one.

1.3 Grading

The grading of this exercise will be made on the basis of the proofs of the 15 theorems using the tactic profiles as follows:

1. Use "set_profile_1" for theorems in set_0 and set_1
2. Use "set_profile_2" for the theorem in set_2
3. Use "set_profile_3" for the remaining theorems except for the two theorems in set_93 where you have to use "set_profile_4"

Hand out your proofs in a corresponding renamed export of the development: "06_set_XXX" (where XXX is your name).

In the last section, I give some hints to help discharging some of the proofs.

2 Some Red Operator Buttons

In this section, I present more red operator buttons besides the ones we introduced in the previous exercise.

2.1 Other Red Operators in the Goal or in the Hypotheses (as labeled in the Rodin Platform)

Remove membership (incomplete)

$$
\begin{array}{lcl}
E \in A \cup B & == & E \in A \ \lor \ E \in B \\
E \in A \cap B & == & E \in A \ \land \ E \in B \\
E \in A \setminus B & == & E \in A \ \land \ E \notin B \\
E \in \varnothing & == & \bot \\
r \in S \leftrightarrow T & == & r \subseteq S \times T \\
E \mapsto F \in S \times T & == & E \in S \ \land F \in T \\
E \mapsto F \in r^{-1} & == & F \mapsto E \in r \\
E \in r[A] & == & \exists x \cdot x \in A \ \land \ x \mapsto E \in r \\
E \mapsto F \in f \,;\, g & == & \exists x \cdot A \mapsto x \in f \ \land \ x \mapsto F \in g \\
E \mapsto F \in \mathrm{id} & == & E = F
\end{array}
$$

Remove inclusion

$$A \subseteq B \ == \ \forall x \cdot x \in A \ \Rightarrow \ x \in B$$

3 Tactic Profiles

3.1 set_profile_1

The tactic profile "set_profile_1" is a slight extension of the tactic profile "pred_profile_2" used in the previous exercise.

3.2 set_profile_2

This profile extends "set_profile_1" by adding an elementary tactic for the goal and an elementary tactic for the hypotheses. It corresponds to the removing of membership and the removing of inclusion. Their names are "Remove all Membership/Inclusion in goal" and "Remove all Membership/Inclusion in hypotheses".

3.3 set_profile_3

This profile extends "set_profile_2" by taking account of equalities

3.4 set_profile_4

This profile is the same as "set_profile_2" where the removing of set membership or inclusion has been removed

3.5 set_profile_5

This profile extends "set_profile_3" by calling the automatic prover P0.

4 Hints

4.1 For the theorem in context "set_5"

1. Use profile "set_profile_3"

2. Do a proof by contradiction by pressing the blue button "ct" on the hypothesis $\neg\, y \mapsto z \in r$.

3. You are left to prove $y \mapsto z \in r$. For this use the big quantified hypothesis. Warning: this hypothesis will disappear after that, this is normal.

4. You are left to prove $\exists\, x \cdot y \mapsto x \in r \;\land\; x \mapsto z \in r$. For this, instantiate x with x.

5. You are left to prove $y \mapsto x \in r$. For this use again the big quantified hypothesis (you can get it by pressing the button "Show cache hypotheses")

6. Do the rest of the proof by yourself

4.2 Hints for theorems in contexts set_8 and followers

1. When you have hypotheses of the form $A = B$ or $\neg\, A = B$ where A and B are sets you might push the red operator "=". One option in the corresponding menu is "Set equality rewrites". It is often interesting to use that option. It transforms the set equality into two inclusions (Extensionality Axiom).

2. You might instantiate universal quantifications by using the button "Instantiate universal followed by **modus tollens**". It is useful when you have an hypothesis of the form $\forall\, x \cdot P(x) \;\Rightarrow\; Q(x)$ and another hypothesis of the form $\neg\, Q(E)$. It will generate the new hypothesis $\neg\, P(E)$.

3. In set_93, you have to use the "set_profile_4". In this context, you have two axioms that correspond to the two theorems you proved in set_91 and set_92. Use these axioms in proving the two theorems thm1 and thm2. Moreover, in thm2, you can use thm1 proved before it.