Exercise Sheet 4: Interactive Proofs in the Propositional Calculus

1 Introduction

1.1 Purpose

The purpose of this exercise is to make you familiar with the practice of interactive proofs with the Rodin Platform on the Proposition Calculus.

1.2 Your Task

We distribute to you a Rodin development named "04_prop": it contains 12 contexts, each of which with one theorem. You will be asked to prove these theorems by using 5 successive, so-called, *tactic profile*: "prp_profile_1", ...,"prp_profile_5". Each of them is a more powerful extension of the previous one.

A tactic profile is made of a sequence of *elementary tactics*. Each elementary tactic is made of one (or more) *inference rule* or *re-writing rule*: these are described in corresponding sections where we present the various profiles.

1.3 Grading

The grading of this exercise will be made on the basis of the proofs of the 12 theorems *using the first tactic profile* "prp_profile_1" only. Hand out your proofs in a corresponding renamed export of the development: "04_prop_XXX".

2 Some Red Operator Buttons

When a tactic profile cannot discharge a theorem, you are asked to use the *red operator buttons* that are visible on the prover user interface. A red operator button is usually made of several elementary tactics (defined in a menu) that you can invoke interactively by depressing the corresponding button in the menu. These elementary tactics are described in the following subsections.

2.1 Goal Red Operators (as labeled in the Rodin Platform)

Deduction

$$\frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{G}}{\mathbf{H} \vdash \mathbf{P} \Rightarrow \mathbf{G}}$$

Conjunction Introduction. This is not a red button, it is a button situated on the left of the goal.

$$\frac{\mathbf{H} \vdash \mathbf{P} \qquad \mathbf{H} \vdash \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \land \mathbf{Q}}$$

Disjunction to Implication

$$\frac{\mathbf{H} \vdash \neg\,\mathbf{P} \Rightarrow \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \lor \mathbf{Q}}$$

Equivalence Rewrite

$$\frac{\mathbf{H} \vdash \mathbf{P} \Rightarrow \mathbf{Q} \qquad \mathbf{H} \vdash \mathbf{Q} \Rightarrow \mathbf{P}}{\mathbf{H} \vdash \mathbf{P} \Leftrightarrow \mathbf{Q}}$$

2.2 Hypotheses Red Operators (as labeled in the Rodin Platform)

Proof by Cases

$$\frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{G} \qquad \mathbf{H}, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \mathbf{P} \vee \mathbf{Q} \vdash \mathbf{G}}$$

Contradict (special case). It is not a red button. It is situated on the left of each hypothesis.

$$\frac{\mathbf{H}, \mathbf{P}, \neg\,\mathbf{Q} \vdash \mathbf{P}}{\mathbf{H}, \mathbf{P}, \neg\,\mathbf{P} \vdash \mathbf{Q}}$$

Do Case Distinction on this Implication

$$\frac{\mathbf{H}, \neg\,\mathbf{P} \vdash \mathbf{G} \qquad \mathbf{H}, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \mathbf{P} \Rightarrow \mathbf{Q} \vdash \mathbf{G}}$$

Remove Negation in Hypotheses

$$\frac{\mathbf{H}, \neg\,\mathbf{P} \vee \neg\,\mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \neg\,(\mathbf{P} \wedge \mathbf{Q}) \vdash \mathbf{G}} \qquad \frac{\mathbf{H}, \neg\,\mathbf{P}, \neg\,\mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \neg\,(\mathbf{P} \vee \mathbf{Q}) \vdash \mathbf{G}} \qquad \frac{\mathbf{H}, \mathbf{P}, \neg\,\mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \neg\,(\mathbf{P} \Rightarrow \mathbf{Q}) \vdash \mathbf{G}} \qquad \frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{G}}{\mathbf{H}, \neg\,\neg\,\mathbf{P} \vdash \mathbf{G}}$$

Equivalence Rewrite

$$\frac{\mathbf{H}, \mathbf{P} \Rightarrow \mathbf{Q}, \mathbf{Q} \Rightarrow \mathbf{P} \vdash \mathbf{G}}{\mathbf{H}, \mathbf{P} \Leftrightarrow \mathbf{Q} \vdash \mathbf{G}}$$

3 Tactic Profile 1

It contains the following unique elementary tactic:

Goal in Hypotheses

$$\frac{}{\mathbf{H}, \mathbf{P} \vdash \mathbf{P}}$$

With this elementary tactic profile, you will have to depress many red operator buttons of the prover user interface. Always use FIRST the goal buttons (section 2.1). When you cannot do anything anymore on the goal buttons start to use the hypotheses buttons (section 2.2). If some red buttons reappears in the goal while treating the hypothesis buttons, then stop using these hypothesis buttons until you cannot do anything again on the goal buttons, and so on.

4 Tactic Profile 2

This profile extends Profile 1 by adding some elementary tactics for the goal. The added elementary tactics are the following:

Implicative Goal (deduction)

$$\frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{G}}{\mathbf{H} \vdash \mathbf{P} \Rightarrow \mathbf{G}}$$

Conjunctive Goal (conjunction introduction)

$$\frac{\mathbf{H} \vdash \mathbf{P} \qquad \mathbf{H} \vdash \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \wedge \mathbf{Q}}$$

Remove Disjunction in a Disjunctive Goal (Disjunction to Implication)

$$\frac{\mathbf{H} \vdash \neg\,\mathbf{P} \Rightarrow \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \vee \mathbf{Q}}$$

Remove all Equivalences in Goal (Equivalence Rewrite)

$$\mathbf{P} \Leftrightarrow \mathbf{Q} == (\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{P})$$

As a consequence, it is now not necessary to depress any red operator in the goal (as mentioned in section 2.1): this is done automatically by the tactic profile 2. You will have to depress hypothesis buttons only (as mentioned in section 2.2).

5 Tactic Profile 3

This profile extends Profile 2 by adding some elementary tactics for the hypotheses. The added elementary tactics are the following:

Find Contradictory Hypotheses (Contradict)

$$\overline{\mathbf{H}, \mathbf{P}, \neg\,\mathbf{P} \vdash \mathbf{G}}$$

Put in Negation Normal Form (Remove Negation in Hypotheses)

$$\neg\,(\mathbf{P} \wedge \mathbf{Q}) == \neg\,\mathbf{P} \vee \neg\,\mathbf{Q}$$

$$\neg\,(\mathbf{P} \vee \mathbf{Q}) == \neg\,\mathbf{P} \wedge \neg\,\mathbf{Q}$$

$$\neg\,(\mathbf{P} \Rightarrow \mathbf{Q}) == \mathbf{P} \wedge \neg\,\mathbf{Q}$$

$$\neg\,\neg\,\mathbf{P} == \mathbf{P}$$

Remove all Equivalences in Hypotheses (Equivalence Rewrite)

$$\mathbf{P} \Leftrightarrow \mathbf{Q} == (\mathbf{P} \Rightarrow \mathbf{Q}) \wedge (\mathbf{Q} \Rightarrow \mathbf{P})$$

As a consequence, the only red operators one has to depress in the hypotheses are the following:

Proof by Cases

$$\frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{G} \qquad \mathbf{H}, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \mathbf{P} \vee \mathbf{Q} \vdash \mathbf{G}}$$

Do Case Distinction on this Implication

$$\frac{\mathbf{H}, \neg\,\mathbf{P} \vdash \mathbf{G} \qquad \mathbf{H}, \mathbf{Q} \vdash \mathbf{G}}{\mathbf{H}, \mathbf{P} \Rightarrow \mathbf{Q} \vdash \mathbf{G}}$$

6 Tactic Profile 4

This profile extends Profile 3 by adding more elementary tactics for the hypotheses. The added elementary tactics are the following:

Generalized Modus Ponens

$$\frac{\mathbf{H}(\top) \vdash \mathbf{G}(\top)}{\mathbf{H}(\mathbf{P}), \mathbf{P} \vdash \mathbf{G}(\mathbf{P})} \qquad \frac{\mathbf{H}(\neg\top) \vdash \mathbf{G}(\neg\top)}{\mathbf{H}(\mathbf{P}), \neg\mathbf{P} \vdash \mathbf{G}(\mathbf{P})}$$

Simplification Rewriter

$$\mathbf{P} \wedge \top == \mathbf{P} \qquad \top \wedge \mathbf{P} == \mathbf{P}$$

$$\mathbf{P} \wedge \neg\top == \neg\top \qquad \neg\top \wedge \mathbf{P} == \neg\top$$

$$\mathbf{P} \vee \top == \top \qquad \top \vee \mathbf{P} == \top$$

$$\mathbf{P} \vee \neg\top == \mathbf{P} \qquad \neg\top \vee \mathbf{P} == \mathbf{P}$$

$$\mathbf{P} \Rightarrow \top == \top \qquad \top \Rightarrow \mathbf{P} == \mathbf{P}$$

$$\mathbf{P} \Rightarrow \neg\top == \neg\mathbf{P} \qquad \neg\top \Rightarrow \mathbf{P} == \top$$

You will notice that all proofs are now automatically discharged except the last two ones (discharged with one click)

7 Tactic Profile 5

This profile extends Profile 4 by adding a call to the automatic prover P0. All proofs are now discharged automatically