Inequalities in Information Theory
A Brief Introduction

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Part I

Basic Concepts and Inequalities
Outline

1. Basic Concepts
2. Basic inequalities
3. Bounds on Entropy
The Entropy

**Definition**

1. The Shannon information content of an outcome $x$ is defined to be

$$h(x) = \log_2 \frac{1}{P(x)}$$

2. The entropy of an ensemble $X$ is defined to be the average Shannon information content of an outcome:

$$H(X) = \sum_{x \in \mathcal{X}} P(X) \log_2 \frac{1}{P(X)}$$

3. Conditional Entropy: the entropy of a r.v., given another r.v.

$$H(X|Y) = -\sum_i \sum_j p(x_i, y_j) \log_2 p(x_i|y_j)$$
The Entropy

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The Entropy

The Joint Entropy

The joint entropy of $X; Y$ is:

$$H(X, Y) = \sum_{x \in X, y \in Y} p(x, y) \log_2 \frac{1}{p(x, y)}$$

(3)

Remarks

- The entropy $H$ answers the question of what is the ultimate data compression.
- The entropy is a measure of the average uncertainty in the random variable. It is the number of bits on the average required to describe the random variable.

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Reference for \[2\] Thomas and \[4\] David

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The joint entropy of $X; Y$ is:

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The Mutual Information

Definition

The mutual information is the reduction in uncertainty when given another r.v., for two r.v. $X$ and $Y$ this reduction is

$$I(X; Y) = H(X) - H(X|Y) = \sum_{x,y} p(x, y) \log \frac{p(x, y)}{p(x)p(y)}$$  \hspace{1cm} (4)

- The capacity of channel is

$$C = \max_{p(x)} I(X; Y)$$
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The relationships

Figure: The relationships between Entropy and Mutual Information

The relative entropy

**Definition**

The relative entropy or Kullback Leibler distance between two probability mass functions \( p(x) \) and \( q(x) \) is defined as

\[
D(p \parallel q) = \sum_{x \in X} p(x) \log \frac{p(x)}{q(x)} = E_p \log \frac{p(X)}{q(X)}.
\]  

1. The relative entropy and mutual information

\[
I(X; Y) = D(p(x, y) \parallel p(x)p(y))
\]

2. Pythagorean decomposition: let \( X = AU \), then

\[
D(p_x \parallel p_u) = D(p_x \parallel \tilde{p}_x) + D(\tilde{p}_x \parallel p_u).
\]
Basic Concepts

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$$I(X; Y) = D(p(x, y) \parallel p(x)p(y)) \quad (6)$$

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Conditional definitions

**Conditional mutual information**

\[ I(X; Y|Z) = H(X|Z) - H(X|Y,Z) \]  
\[ = E_{p(x,y,z)} \log \frac{p(X,Y|Z)}{p(X|Z)p(Y|Z)}. \]

**Conditional relative entropy**

\[ D(p(y|x) \parallel q(y|x)) = \sum_x p(x) \sum_y p(y|x) \log \frac{p(y|x)}{q(y|x)} \]
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(10)

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\]

(11)
Differential entropy

**Definition 1**

The **differential entropy** \( h(X_1, X_2, ..., X_n) \), sometimes written \( h(f) \), is defined by

\[
h(X_1, X_2, ..., X_n) = - \int f(x) \log f(x) \, dx
\]  

(12)

**Definition 2**

The **relative entropy** between probability densities \( f \) and \( g \) is

\[
D(f \parallel g) = - \int f(x) \log(f(x)/g(x)) \, dx
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Differential entropy

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Chain Rules

1. Chain rule for entropy

\[ H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \ldots, X_1). \] (14)

2. Chain rule for information

\[ I(X_1, X_2, \ldots, X_n; Y) = \sum_{i=1}^{n} I(X_i; Y|X_{i-1}, \ldots, X_1). \] (15)

3. Chain rule for entropy

\[ D(p(x, y) \parallel q(x, y)) = D(p(x) \parallel q(x)) + D(p(y|x) \parallel q(y|x)). \] (16)
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# Chain Rules

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(16)
Outline

1. Basic Concepts
2. Basic inequalities
3. Bounds on Entropy
Jensen’s inequality

**Definition**
A function $f$ is said to be convex if

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \quad (17)$$

for all $0 \leq \lambda \leq 1$ and all $x_1$ and $x_2$ in the convex domain of $f$.

**Theorem**
If $f$ is convex, then

$$f(EX) \leq Ef(x) \quad (18)$$

**Proof**
We consider discrete distributions only. The proof is given by induction. For a two mass point distribution, by definition. for $k$ mass points, let $p'_i = p_i/(1 - p_k)$ for $i \leq k - 1$, the result can be derived easily.
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**Log sum inequality**

**Theorem**

For positive numbers, \(a_1, a_2, \ldots, a_n\) and \(b_1, b_2, \ldots, b_n\),

\[
\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left( \sum_{i=1}^{n} a_i \right) \log \left( \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \right) \tag{19}
\]

with equality iff \(\frac{a_i}{b_i} = \text{constant}\).

**Proof**

We substitute discrete distribution parameters in Jensen’s Inequality by \(\alpha_i = b_i / \sum_{j=1}^{n} b_j\) and the variables by \(t_i = a_i / b_i\), we obtain the inequality.
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**Proof**

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By Jensen’s inequality and Log Sum inequality, we can easily prove following basic conclusions:

\[ 0 \leq H(X) \leq \log |\mathcal{X}| \quad (20) \]

\[ D(p \parallel q) \geq 0 \quad (21) \]

Further more,

\[ I(X; Y) \geq 0 \quad (22) \]

Note: the conditions when the equalities holds.
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Further more,

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Note: the conditions when the equalities holds.
Inequalities in Entropy Theory (cont.)

- Conditioning reduces entropy:

\[ H(X|Y) \leq H(X) \]

- The chain rule and independence bound on entropy:

\[
H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \ldots, X_1) \leq \sum_{i=1}^{n} H(X_i) \tag{23}
\]

- Note: the conclusions continue to hold for differential entropy.

- If \( X \) and \( Y \) are independent, then

\[ h(X + Y) \geq h(Y) \]
Inequalities in Entropy Theory (cont.)

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Theorem

$D(p \parallel q)$ is convex in the pair $(p, q)$, i.e., if $(p_1, q_1)$ and $(p_2, q_2)$ are two pairs of probability mass functions, then

$$D(\lambda p_1 + (1 - \lambda) p_2 \parallel \lambda q_1 + (1 - \lambda) q_2) \leq \lambda D(p_1 \parallel q_1) + (1 - \lambda) D(p_2 \parallel q_2)$$

for all $0 \leq \lambda \leq 1$.

Apply the log sum inequality to the term on the left hand side of (24).
Convexity & concavity entropy theory

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for all $0 \leq \lambda \leq 1$.

- Apply the log sum inequality to the term on the left hand side of (24).
Theorem

$H(p)$ is a concave function of $p$.

Let $u$ be the uniform distribution on $|\mathcal{X}|$ outcomes, then the concavity of $H$ then follows directly from then convexity of $D$, since the following equality holds.

$$H(p) = \log |\mathcal{X}| - D(p \parallel u)$$  \hspace{1cm} (25)
Convexity & concavity in entropy theory (cont.)

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- Let $u$ be the uniform distribution on $|\mathcal{X}|$ outcomes, then the concavity of $H$ then follows directly from the convexity of $D$, since the following equality holds.

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Convexity & concavity in entropy theory (cont.)

**Theorem**

Let $(X, Y) \sim p(x, y) = p(x)p(y|x)$. The mutual information $I(X; Y)$ is a concave function of $p(x)$ for fixed $p(y|x)$ and a convex function of $p(y|x)$ for fixed $p(X)$.

- The detailed proof can be found in [[2] Thomas, section 2.7]. An alternative proof is given in [1], P51-52.
Convexity & concavity in entropy theory (cont.)

Theorem

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Outline

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2. Basic inequalities
3. Bounds on Entropy
$\ell_1$ bound on entropy

**Theorem**

Let $p$ and $q$ be two probability mass functions on $\mathcal{X}$ such that

$$\| p - q \|_1 = \sum_{x \in \mathcal{X}} | p(x) - q(x) | \leq \frac{1}{2}.$$

Then

$$| H(p) - H(q) | \leq - \| p - q \|_1 \log \frac{\| p - q \|_1}{| \mathcal{X} |}. \quad (26)$$
Proof of $L_1$ bound on entropy

Consider the function $f(t) = -t \log t$, it is concave and positive on $[0, 1]$, since $f(0) = f(1) = 0$.

1. Let $0 \leq \nu \leq \frac{1}{2}$, for any $0 \leq t \leq 1 - \nu$, we have

$$| f(t) - f(t + \nu) | \leq \max\{ f(\nu), f(1 - \nu) \} = -\nu \log \nu. \quad (27)$$

2. Let $r(x) = | p(x) - q(x) |$. Then

$$| H(p) - H(q) | = \sum_{x \in X} ( -p(x) \log p(x) + q(x) \log q(x) ) \leq \sum_{x \in X} \left| ( -p(x) \log p(x) + q(x) \log q(x) ) \right| \quad (28)$$

$$\leq \sum_{x \in X} \left| ( -p(x) \log p(x) + q(x) \log q(x) ) \right| \quad (29)$$
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$$\leq \sum_{x \in \mathcal{X}} (-p(x) \log p(x) + q(x) \log q(x))$$  \hspace{1cm} (29)
Proof of $L_1$ bound on entropy

By using (27), we have

$$Left \leq \sum_{x \in \mathcal{X}} -r(x) \log r(x)$$

$$= \| p - q \|_1 \sum_{x \in \mathcal{X}} - \frac{r(x)}{\| p - q \|_1} \log \frac{r(x)}{\| p - q \|_1} \| p - q \|_1$$

$$= -\| p - q \|_1 \log \| p - q \|_1 + \| p - q \|_1 H \left( \frac{r(x)}{\| p - q \|_1} \right)$$

$$\leq -\| p - q \|_1 \log \| p - q \|_1 + \| p - q \|_1 \log |\mathcal{X}|.$$
The lower bound of relative entropy

**Theorem**

\[
D(P_1 \parallel P_2) \geq \frac{1}{2 \ln 2} \| P_1 - P_2 \|_1^2. 
\]  

(34)

**Proof**

(1) Binary case. Consider two binary distribution with parameter \( p \) and \( q \) with \( p \leq q \). We will show that

\[
p \log \frac{p}{q} + (1 - p) \log \frac{1 - p}{1 - q} \geq \frac{4}{2 \ln 2} (p - q)^2.
\]

Let

\[
g(p, q) = p \log \frac{p}{q} + (1 - p) \log \frac{1 - p}{1 - q} - \frac{4}{2 \ln 2} (p - q)^2.
\]
The lower bound of relative entropy

Proof (cont.)

Then

\[ \frac{\partial g(p, q)}{\partial q} \leq 0 \]

since \( q(1 - q) \leq \frac{1}{4} \) and \( q \leq p \). For \( q = p \), \( g(p, q) = 0 \), and hence \( g(p, q) \geq 0 \) for \( q \leq p \), which proves the binary case.
The lower bound of relative entropy

Proof (cont.)

(2) For the general case, for any two distribution $P_1$ and $P_2$, let $A = \{ x : P_1(x) > P_2(x) \}$. Define $Y = \phi(X)$, the indicator of the set $A$, and let $\hat{P}_1$ and $\hat{P}_2$ be the distribution of $Y$. By the data processing inequality ([2] Thomas, section 2.8) applied to relative entropy, we have

$$D(P_1 \parallel P_2) \geq D(\hat{P}_1 \parallel \hat{P}_2) \geq \frac{4}{2\ln 2} (P_1(A) - P_2(A))^2 = \frac{1}{2\ln 2} \| P_1 - P_2 \|_1^2.$$
Part II

Entropy in Statistics
Outline

4 Entropy in Markov chain

5 Bounds on entropy on distributions
Data processing inequality and its corollaries

Data processing inequality

If \( X \rightarrow Y \rightarrow Z \), then

\[
I(X; Y) \geq I(X; Z). \tag{35}
\]

Corollary

In particular, if \( Z = g(Y) \), we have

\[
I(X; Y) \geq I(X; g(Y)). \tag{36}
\]

Corollary

If \( X \rightarrow Y \rightarrow Z \), then

\[
I(X; Y|Z) \geq I(X; Y). \tag{37}
\]
Data processing inequality and its corollaries

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If $X \rightarrow Y \rightarrow Z$, then

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Theorem

For a Markov Chain:

1. Relative entropy $D(\mu_n \parallel \mu'_n)$ decreases with time.
2. Relative entropy $D(\mu_n \parallel \mu)$ between a distribution and the stationary distribution decreases with time.
3. Entropy $H(X_n)$ increases if the stationary distribution is uniform.
4. The conditional entropy $H(X_n|X_1)$ increases with time for a stationary Markov chain.
5. Shuffles increase entropy.
Entropy in Markov chain

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Theorem

For a Markov Chain:

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Theorem

For a *Markov Chain*:

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4. The conditional entropy \( H(X_n|X_1) \) increases with time for a stationary Markov chain.
5. Shuffles increase entropy.
Proof for item 1

Let $\mu_n$ and $\mu'_n$ be two probability distributions on the state space of a Markov chain at time $n$, corresponding to $p$ and $q$ as joint mass functions. By the chain rule:

$$D(p(x_n, x_{n+1}) \parallel q(x_n, x_{n+1}))$$

$$= D(p(x_n) \parallel q(x_n)) + D(p(x_{n+1}|x_n) \parallel q(x_{n+1}|x_n))$$

$$= D(p(x_{n+1}) \parallel q(x_{n+1})) + D(p(x_n|x_{n+1}) \parallel q(x_n|x_{n+1}))$$
Proof for item 1 (cont.)

Since the probability transition function \( p(x_{n+1}|x_n) = q(x_{n+1}|x_n) \) from the Markov chain, hence \( D(p(x_{n+1}|x_n) \parallel q(x_{n+1}|x_n)) = 0 \), and also \( D(p(x_n|x_{n+1}) \parallel q(x_n|x_{n+1})) \geq 0 \), we have

\[
D(p(x_n) \parallel q(x_n)) \geq D(p(x_{n+1}) \parallel q(x_{n+1}))
\]
or

\[
D(\mu_n \parallel \mu'_n) \geq D(\mu_{n+1} \parallel \mu'_{n+1}).
\]
Proof for item 2

Let $\mu'_n = \mu$, and $\mu'_{n+1} = \mu$, $\mu$ can be any stationary distribution. By item 1, the inequality holds.

Remarks

The monotonically non-increasing non-negative sequence $D(\mu_n \parallel \mu)$ has 0 as its limit if the stationary distribution is unique.

Remark on item 3

Let the stationary distribution $\mu$ be uniform, then by

$$D(\mu_n \parallel \mu) = \log |\mathcal{X}| - H(\mu_n) = \log |\mathcal{X}| - H(X_n)$$

we know the conclusion holds.
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Proof for item 4

\[ H(X_n|X_1) \geq H(X_n|X_1, X_2) = H(X_n|X_2) = H(X_{n-1}|X_1) \]

Remarks on item 5

If \( T \) is a shuffle permutation of cards and \( X \) is the initial random position, and if \( T \) is independent of \( X \), then

\[ H(TX) \geq H(X) \]

where \( TX \) is the permutation by the shuffle \( T \) on \( X \).

- Proof

\[ H(TX) \geq H(TX|T) = H(T^{-1}TX|T) = H(X|T) = H(X) \]

- Reference for [[2]Thomas, section 4.4.]
Proof for item 4

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Proof

\[ H(TX) \geq H(TX|T) = H(T^{-1}TX|T) = H(X|T) = H(X) \]

Reference for [[2]Thomas, section 4.4.]
Entropic in Markov chain

Theorem (Fano’s inequality)
For any estimator $\hat{X}$ such that $X \rightarrow Y \rightarrow \hat{X}$, with $P_e = Pr(X \neq \hat{X})$, we have

$$H(P_e) + P_e \log(|\mathcal{X}|) \geq H(X|\hat{X}) \geq H(X|Y)$$

(38)

This inequality can be weakened to

$$1 + P_e \log |\mathcal{X}| \geq H(X|Y)$$

(39)

or

$$P_e \geq \frac{H(X|Y) - 1}{\log |\mathcal{X}|}.$$ 

(40)
Proof of Fano’s inequality

Proof

Define an error random variable,

\[ E = \begin{cases} 
1, & \text{if } \hat{X} \neq X \\
0, & \text{if } \hat{X} = X 
\end{cases} \]

Then,

\[ H(E, X|\hat{X}) = H(X|\hat{X}) + H(E|X, \hat{X}) = H(E|\hat{X}) + H(X|E, \hat{X}). \]

since

\[ H(X|E, \hat{X}) = Pr(E = 0)H(X|\hat{X}, E = 0) + Pr(E = 1)H(X|\hat{X}, E = 1) \leq (1 - P_e)0 + P_e \log |X| . \]
Proof of Fano’s inequality

Proof (cont.)

By the data-processing inequality, we have $I(X; \hat{X}) \geq I(X; Y)$ since $X \rightarrow Y \rightarrow \hat{X}$ is a Markov chain, and therefore $H(X|\hat{X}) \geq H(X|Y)$. Thus we have (38) holds.

- For any two random variables $X$ and $Y$, if the estimator $g(Y)$ takes values in the set $X$, we can strengthen the inequality slightly by replacing $\log |\mathcal{X}|$ with $\log (|\mathcal{X}| - 1)$.
Proof of Fano’s inequality

Proof (cont.)

By the data-processing inequality, we have $I(X; \hat{X}) \geq I(X; Y)$ since $X \to Y \to \hat{X}$ is a Markov chain, and therefore $H(X|\hat{X}) \geq H(X|Y)$. Thus we have (38) holds.

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Empirical probability mass function

**Theorem**

Let $X_1, X_2, \ldots, X_n$ be i.i.d. $\sim p(x)$. Let $\tilde{p}_n$ be the empirical probability mass function of $X_1, X_2, \ldots, X_n$. Then

$$ED(\hat{p}_n \parallel p) \leq ED(\hat{p}_{n-1} \parallel p)$$  \hspace{1cm} (41)

**Proof**

Use $D(\hat{p}_n \parallel p) = E_{\hat{p}_n} \log \frac{\hat{p}_n}{p(x)} = E_{\hat{p}_n} \log \hat{p}_n - \log p(x)$, we have

$$E_p D(\hat{p}_n \parallel p) = H(p) - H(\hat{p}_n),$$

then by item 3 in Markov Chain.
Outline

4 Entropy in Markov chain

5 Bounds on entropy on distributions
Entropy of a multivariate normal distribution

Lemma

Let $X_1, X_2, \ldots, X_n$ have a multivariate normal distribution with mean $\mu$ and covariance matrix $K$. Then

$$h(X_1, X_2, \ldots, X_n) = h(\mathcal{N}(\mu, K)) = \frac{1}{2} \log(2\pi e)^n |K| \text{ bits,}$$

where $|K|$ denotes the determinant of $K$. (42)
Bounds on differential entropies

**Theorem**

Let the random vector \( \mathbf{X} \in \mathbb{R}^n \) have zero mean and covariance \( \mathbf{K} = \mathbb{E}\mathbf{XX}^t \), i.e., \( K_{ij} = \mathbb{E}X_iX_j, 1 \leq j, j \leq n \). Then

\[
h(\mathbf{X}) \leq \frac{1}{2} \log \left(2\pi e\right)^n |\mathbf{K}|,
\]

with equality iff \( \mathbf{X} \sim \mathcal{N}(0, \mathbf{K}) \).
Bounds on differential entropies

Proof

Let \( g(x) \) be any density satisfying \( \int g(x)x_i x_j \, dx = K_{ij} \) for all \( i, j \). Let \( \phi_K \sim \mathcal{N}(0, K) \). Note that \( \log \phi_K(x) \) is a quadratic form and \( \int x_i x_j \phi_K(x) \, dx = K_{ij} \). Then

\[
0 \leq D(g \parallel \phi_K) = \int g \log(g / \phi_K) = -h(g) - \int g \log \phi_K = -h(g) - \int \phi_K \log \phi_K = -h(g) + h(\phi_K)
\]

since \( h(\phi_K) = \frac{1}{2} \log (2\pi e)^n |K| \), the conclusion holds.
Bounds on discrete entropies

Theorem

\[
H(p_1, p_2, \ldots) \leq \frac{1}{2} \log(2\pi e) \left( \sum_{i=1}^{\infty} p_i i^2 - \left( \sum_{i=1}^{\infty} i p_i \right)^2 + \frac{1}{12} \right)
\]  

(44)

Proof

Define new r.v. \( X \), with the distribution \( Pr(X = i) = p_i \), \( U \sim \mathcal{U}(0, 1) \), define \( \tilde{X} \) by \( \tilde{X} = X + U \). Then

\[
H(X) = - \sum_{i=1}^{\infty} p_i \log p_i
\]

\[
= - \sum_{i=1}^{\infty} \left( \int_{i}^{i+1} f_{\tilde{X}}(x) \, dx \right) \log \left( \int_{i}^{i+1} f_{\tilde{X}}(x) \, dx \right)
\]
Bounds on discrete entropies

Proof (cont.)

\[
H(X) = - \sum_{i=1}^{\infty} \int_{i}^{i+1} f_{\tilde{X}}(x) \log f_{\tilde{X}}(x) \, dx
= - \int_{1}^{\infty} f_{\tilde{X}}(x) \log f_{\tilde{X}}(x) \, dx
= h(\tilde{X})
\]

since \( f_{\tilde{X}}(x) = p_i \) for \( i \leq x < i + 1 \). Hence

\[
h(\tilde{X}) \leq \frac{1}{2} \log(2\pi e) \text{Var}(\tilde{X}) = \frac{1}{2} \log(2\pi e)(\text{Var}(X) + \text{Var}(U))
= \frac{1}{2} \log(2\pi e) \left( \sum_{i=1}^{\infty} p_i i^2 - \left( \sum_{i=1}^{\infty} ip_i \right)^2 + \frac{1}{12} \right).
\]
Entropy and fisher information

- The Fisher information matrix is a measure of the minimum error in estimating a parameter vector of a distribution.

- The Fisher information matrix of the distribution of $X$ with a parameter vector $\theta$ is defined as

$$ J(\theta) = E\{ \left[ \frac{\partial}{\partial \theta} \log f_\theta(X) \right] \left[ \frac{\partial}{\partial \theta} \log f_\theta(X) \right]^T \} $$ (45)

  for any $\theta \in \Theta$.

- If $f_\theta$ is twice differentiable in $\theta$, and alternative expression is

$$ J(\theta) = -E \left[ \frac{\partial^2}{\partial \theta \partial \theta^T} \log f_\theta(X) \right]. $$ (46)

- Reference in [5].
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  (45)

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  $$ J(\theta) = -E \left[ \frac{\partial^2}{\partial \theta \partial \theta^T} \log f_{\theta}(X) \right]. $$

  (46)

- Reference in [5].
Fisher information of a distribution

- Let $X$ be any r.v. with density $f(x)$, for a location parameter $\theta$, the fisher information w.r.t. $\theta$ is given by

$$J(\theta) = \int_{-\infty}^{\infty} f(x - \theta) \left[ \frac{\partial}{\partial \theta} \ln f(x - \theta) \right]^2 dx.$$  

- As the differentiation w.r.t. $x$ is equivalent to $\theta$, so we can rewrite the Fisher information as

$$J(X) = J(\theta) = \int_{-\infty}^{\infty} f(x) \left[ \frac{\partial}{\partial x} \ln f(x) \right]^2 dx$$

$$= \int_{-\infty}^{\infty} f(x) \left[ \frac{\partial}{\partial x} \frac{f(x)}{f(x)} \right]^2 dx.$$
Cramér-Rao inequality

**Theorem**

The mean-squared error of any unbiased estimator $T(X)$ of the parameter $\theta$ is lower bounded by the reciprocal of the Fisher information:

$$\text{Var}[T(X)] \geq [J(\theta)]^{-1}.$$  \hspace{1cm} (47)

**Proof**

By Cauchy-Schwarz inequality,

$$\text{Var}[T(X)]\text{Var} \left( \frac{\partial \log f}{\partial \theta} \right) \geq \text{Cov}^2 \left( T(X), \frac{\partial \log f}{\partial \theta} \right)$$

Then

$$\text{Cov}^2 \left( T(X), \frac{\partial \log f}{\partial \theta} \right) = E \left( T(X) \frac{\partial \log f}{\partial \theta} \right) = \frac{\partial}{\partial \theta} E_\theta(T(X)) = 1.$$
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Then

$$\text{Cov}^2\left( T(X), \frac{\partial \log f}{\partial \theta} \right) = \mathbb{E}\left( T(X) \frac{\partial \log f}{\partial \theta} \right) = \frac{\partial}{\partial \theta} \mathbb{E}_\theta(T(X)) = 1.$$
Theorem

Let $X$ be any random variable with a finite variance with a density $f(x)$. Let $Z$ be an independent normally distributed random variable with zero mean and unit variance. Then

$$\frac{\partial}{\partial t} h_e(X + \sqrt{t}Z) = \frac{1}{2} J(X + \sqrt{t}Z),$$  \hspace{1cm} (48)

where $h_e$ is the differential entropy to base $e$. In particular, if the limit exists as $t \to 0$,

$$\frac{\partial}{\partial t} h_e(X + \sqrt{t}Z) \bigg|_{t=0} = \frac{1}{2} J(X).$$  \hspace{1cm} (49)
Proof

Let $Y_t = X + \sqrt{t}Z$. Then the density of $Y_t$ is

$$g_t(y) = \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} \, dx.$$ 

It’s easy to verify that

$$\frac{\partial}{\partial t} g_t(y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} g_t(y).$$ 

(50)
Proof

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- It’s easy to verify that

$$\frac{\partial}{\partial t} g_t(y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} g_t(y).$$ (50)
Proof

Since \( h_e(Y_t) = -\int_{-\infty}^{\infty} g_t(y) \ln g_t(y) dy \) Differentiating, by \( \int g_t(y) dy = 1 \) and (50), then integrate by parts, we obtain

\[
\frac{\partial}{\partial t} h_e(Y_t) = -\frac{1}{2} \left[ \frac{\partial g_t(y)}{\partial y} \ln g_t(y) \right]_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} \left[ \frac{\partial}{\partial y} g_t(y) \right]^2 \frac{1}{g_t(y)} dy.
\]

The first term above goes to 0 at both limit, and by definition, the first term is \( \frac{1}{2} J(Y_t) \). Thus the theorem is prove.
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Since \( h_e(Y_t) = -\int_{-\infty}^{\infty} g_t(y) \ln g_t(y) \, dy \) Differentiating, by \( \int g_t(y) \, dy = 1 \) and (50), then integrate by parts, we obtain

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\]

The first term above goes to 0 at both limit, and by definition, the first term is \( \frac{1}{2} J(Y_t) \). Thus the theorem is prove.
Part III

Some important theories deduced from entropy
Outline

6 Entropy rates of subsets

7 The Entropy power inequality
Entropy on subsets

**Definition: Average Entropy Rate**

Let \((X_1, X_2, \ldots, X_n)\) have a density, and for every \(S \subseteq \{1, 2, \ldots, n\}\), denote by \(X(S)\) the subset \(\{X_i : i \in S\}\). Let

\[
h_k^{(n)} = \frac{1}{\binom{n}{k}} \sum_{S:|S|=k} \frac{h(X(S))}{k}.
\]  

(51)

Here \(h_k^{(n)}\) is the average entropy in bits per symbol of a randomly drawn \(k\)-element subset of \((X_1, X_2, \ldots, X_n)\).

- The average conditional entropy rate and average mutual information rate can be defined similarly on \(h(X(S)|X(S^c))\) and \(I(X(S); X(S^c))\).
Entropy rates of subsets

Entropy on subsets

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- The average conditional entropy rate and average mutual information rate can be defined similarly on \(h(X(S)|X(S^c))\) and \(I(X(S); X(S^c))\).
Entropy rates of subsets

Entropy on subsets

Theorem

1. For average entropy rate,

\[ h_1^{(n)} \geq h_2^{(n)} \geq \ldots \geq h_n^{(n)}. \]  \hspace{1cm} (52)

2. For average conditional entropy rate,

\[ g_1^{(n)} \leq g_2^{(n)} \leq \ldots \leq g_n^{(n)}. \]  \hspace{1cm} (53)

3. For average mutual information,

\[ f_1^{(n)} \geq f_2^{(n)} \geq \ldots \geq f_n^{(n)}. \]  \hspace{1cm} (54)
Proof for Theorem, item 1

- We first prove $h_n^{(n)} \leq h_{n-1}^{(n)}$. Since for $i = 1, 2, \ldots, n$,

\[
h(X_1, X_2, \ldots, X_n) = h(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)
+ h(X_i | X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)
\leq h(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)
+ h(X_i | X_1, X_2, \ldots, X_{i-1})
\]

- Adding these $n$ inequalities and using the chain rule, we obtain

\[
\frac{1}{n} h(X_1, X_2, \ldots, X_n) \leq \frac{1}{n} \sum_{i=1}^{n} \frac{h(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)}{n-1}
\]

Thus $h_n^{(n)} \leq h_{n-1}^{(n)}$ holds.
Proof for Theorem, item 1

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$$h(X_1, X_2, \ldots, X_n) = h(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$$
$$+ h(X_i|X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$$
$$\leq h(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$$
$$+ h(X_i|X_1, X_2, \ldots, X_{i-1})$$

- Adding these $n$ inequalities and using the chain rule, we obtain

$$\frac{1}{n} h(X_1, X_2, \ldots, X_n) \leq \frac{1}{n} \sum_{i=1}^{n} \frac{h(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)}{n-1}$$

Thus $h_n^{(n)} \leq h_{n-1}^{(n)}$ holds.
For each $k$-element subset, $h_k^{(k)} \leq h_{k-1}^{(k)}$,

and hence the inequality remains true after taking the expectation over all $k$-element subsets chosen uniformly from the $n$ elements.
For each $k$-element subset, $h_k^{(k)} \leq h_{k-1}^{(k)}$,

and hence the inequality remains true after taking the expectation over all $k$-element subsets chosen uniformly from the $n$ elements.
Entropy on subsets

Proof for Theorem, item 2 and 3

(1) We prove $g_{n}^{(n)} \leq g_{n-1}^{(n)}$ first. By

$$h(X_1, X_2, \ldots, X_n) \leq \sum_{i=1}^{n} h(X_i)$$

$$(n-1)h(X_1, X_2, \ldots, X_n) \geq \sum_{i=1}^{n} (h(X_1, X_2, \ldots, X_n) - h(X_i))$$

$$= \sum_{i=1}^{n} h(X_1, X_2, \ldots, X_{i-1}, X_i, \ldots, X_n|X_i).$$

Similar as the proof of item 1, we have $g_k^{(k)} \leq g_{k-1}^{(k)}$.

(2) Since $I(X(S); X(S^c)) = h(X(S)) - h(X(S)|X(S^c))$, item 3 holds.
Outline

6 Entropy rates of subsets

7 The Entropy power inequality
The Entropy power inequality

If $\mathbf{X}$ and $\mathbf{Y}$ are independent random $n$-vectors with densities, then

$$2^n h(\mathbf{X} + \mathbf{Y}) \geq 2^n h(\mathbf{X}) + 2^n h(\mathbf{Y}).$$

(55)

Remarks

For normal distributions, since $2^{2n h(\mathbf{X})} = (2\pi e)^{\frac{1}{2}}\sigma_{\mathbf{X}}^2$, we have a new statement of the entropy power inequality.
The Entropy power inequality

**Theorem**
If \( X \) and \( Y \) are independent random \( n \)-vectors with densities, then

\[
2^{\frac{2}{n}} h(X+Y) \geq 2^{\frac{2}{n}} h(X) + 2^{\frac{2}{n}} h(Y).
\]

(55)

**Remarks**
For normal distributions, since \( 2^{2h(X)} = (2\pi e)\sigma_X^2 \), we have a new statement of the entropy power inequality.
The entropy power inequality

Theorem: the entropy power inequality

For two independent random variables $X$ and $Y$,

$$h(X + Y) \geq h(X' + Y')$$

where $X'$ and $Y'$ are independent normal random variables with $h(X') = h(X)$ and $h(Y') = h(Y)$. 
The Entropy power inequality

Definitions

- **The set sum** $A + B$ of two sets $A, B \subset \mathcal{R}^n$ is defined as the set 
  \[ \{x + y : x \in A, y \in B\} \].

- Example: The set sum of two spheres of radius 1 at the origins is a sphere of radius 2 at the origin.

- Let the $L_r$ norm of the density be defined by 
  \[ \| f \|_r = \left( \int f^r(x) dx \right)^{\frac{1}{r}} \].

- The Rényi entropy $h_r(X)$ of order $r$ is defined as 
  \[ h_r(X) = \frac{1}{1 - r} \log \left[ \int f^r(x) dx \right] \] \hspace{1cm} (56)
  for $0 < r < \infty, r \neq 1$. 
Remarks on definition

Remarks

- If we take the limit as $r \to 1$, we obtain the Shannon entropy function

$$h(X) = h_1(x) = - \int f(x) \log f(x) \, dx.$$  

- If we take the limit as $r \to 0$, we obtain the logarithm of the support set,

$$h_0 = \log(\mu\{x : f(x) > 0\}).$$

- Thus the zeroth order Rényi entropy gives the measure of the support set of the density of $f$. 

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Inequalities in Information Theory

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The Brunn-Minkowski inequality

Theorem: Brunn-Minkowski inequality

The volume of the set sum of two sets $A$ and $B$ is greater than the volume of the set sum of two spheres $A'$ and $B'$ with the same volume as $A$ and $B$, respectively, i.e.,

$$V(A + B) \geq V(A' + B')$$

where $A'$ and $B'$ are spheres with $V(A') = V(A)$ and $V(B') = V(B)$. 
The Rényi Entropy Power

**Definition**

The Rényi entropy power $V_r(X)$ of order $r$ is defined as

$$V_r(X) = \begin{cases} 
\left[ \int f^r(x)dx \right]^{\frac{2}{n}r'}, & 0 < r \leq \infty, r \neq 1, \frac{1}{r} + \frac{1}{r'} = 1 \\
\exp[\frac{2}{n}h(X)], & r = 1 \\
\mu(\{x : f(x) > 0\})^{\frac{2}{n}}, & r = 0
\end{cases}$$

**Theorem**

For two independent random variables $X$ and $Y$ and any $0 \leq r < \infty$ and any $0 \leq \lambda \leq 1$, let $p = \frac{r}{r+\lambda(1-r)}$, $q = \frac{r}{r+(1-\lambda)(1-r)}$, we have

$$\log V_r(X + Y) \geq \lambda \log V_p(X) + (1 - \lambda) \log V_q(Y) + H(\lambda)$$

$$+ \left(\frac{1+r}{1-r}\right) \left[ H \left(\frac{r + \lambda(1-r)}{1+r} \right) - H \left(\frac{r}{1+r} \right) \right].$$
Remarks on the Rényi Entropy Power

- The Entropy power inequality. Taking the limit of (58) as \( r \to 1 \) and setting \( \lambda = \frac{V_1(X)}{V_1(X)+V_1(Y)} \), we obtain

\[
V_1(X + Y) \geq V_1(X) + V_1(Y).
\]

- The Brunn-Minkowski inequality. Similarly letting \( r \to 0 \) and choosing \( \lambda = \frac{\sqrt{V_0(X)}}{\sqrt{V_0(X)}+\sqrt{V_0(Y)}} \), we obtain

\[
\sqrt{V_0(X + Y)} \geq \sqrt{V_0(X)} + \sqrt{V_0(Y)}
\]

Now let \( A \) and \( B \) be the support set of \( X \) and \( Y \). Then \( A + B \) is the support set of \( X + Y \), and the equation above reduces to

\[
[\mu(A + B)]^{1/n} \geq [\mu(A)]^{1/n} + [\mu(B)]^{1/n},
\]

which is the Brunn-Minkowski inequality.
Part IV

Important applications
The Method of Types

Combinatorial Bounds on Entropy
**Basic concepts**

**Definition**

1. The **type** $P_x$ of a sequence $x_1, x_2, \ldots, x_n$ is the relative proportion of occurrences in $\mathcal{X}$, i.e., $P_x(a) = N(a|\mathbf{x})/n$ for all $a \in \mathcal{X}$.

2. Let $\mathcal{P}_n$ denote the set of types with a sequence of $n$ symbols.

3. If $P \in \mathcal{P}_n$, then the type class of $P$, denoted $T(P)$ is defined as:

$$T(P) = \{ \mathbf{x} \in \mathcal{X}^n : P_x = P \}$$
Theorem: the probability of $x$

If $X_1, X_2, \ldots, X_n$ are drawn i.i.d. $\sim Q(x)$, then the probability of $x$ depends only on its type and is given by

$$Q^{(n)}(x) = 2^{-n(H(P_x) + D(P_x \| Q))} \tag{59}$$

Proof

$$Q^{(n)}(x) = \prod_{i=1}^{n} Q(X_i) = \prod_{a \in \mathcal{X}} Q(a)^{N(a|x)}$$

$$= \prod_{a \in \mathcal{X}} Q(a)^{nP_x(a)} = \prod_{a \in \mathcal{X}} 2^{nP_x \log Q(a)}$$

$$= 2^n \sum_{a \in \mathcal{X}} (-P_x(a) \log \frac{P_x(a)}{Q(a)} + P_x(a) \log P_x(a)).$$
The Method of Types

Bound on number of types

Theorem: the probability of $x$

If $X_1, X_2, \ldots, X_n$ are drawn i.i.d. $\sim Q(x)$, then the probability of $x$ depends only on its type and is given by

$$Q^{(n)}(x) = 2^{-n(H(P_x) + D(P_x \parallel Q))} \quad (59)$$

Proof

$$Q^{(n)}(x) = \prod_{i=1}^{n} Q(X_i) = \prod_{a \in \mathcal{X}} Q(a)^{N(a|x)}$$

$$= \prod_{a \in \mathcal{X}} Q(a)^{nP_x(a)} = \prod_{a \in \mathcal{X}} 2^{nP_x} \log Q(a)$$

$$= 2^n \sum_{a \in \mathcal{X}} (-P_x(a) \log \frac{P_x(a)}{Q(a)} + P_x(a) \log P_x(a)).$$
The Method of Types

Size of type class $T(P)$

Theorem

$|\mathcal{P}_n| \leq (n+1)|x|$.  

(60)

Theorem

For any type of $P \in \mathcal{P}_n$,

$$\frac{1}{(n+1)|x|} 2^{nH(P)} \leq |T(P)| \leq 2^{nH(P)}.$$  

(61)
Size of type class \( T(P) \)

**Theorem**

\[ | \mathcal{P}_n | \leq (n + 1)|\mathcal{X}|. \]  \hspace{1cm} (60)

**Theorem**

For any type of \( P \in \mathcal{P}_n \),

\[ \frac{1}{(n + 1)|\mathcal{X}|} 2^{nH(P)} \leq | T(P) | \leq 2^{nH(P)}. \]  \hspace{1cm} (61)
Size of type class $T(P)$

**Proof**

By (59), if $x \in T(P)$, then $P^{(n)}(x) = 2^{-nH(P)}$, we have

$$1 \geq P^{(n)}(T(P)) = \sum_{x \in T(P)} P^{(n)}(x) = \sum_{x \in T(P)} 2^{-nH(P)} = |T(P)| \cdot 2^{-nH(P)}.$$

For the lower bound, we use the fact $P^{(n)}(T(P)) \geq P^{(n)}(T(\hat{P}))$, for all $\hat{P} \in \mathcal{P}_n$ without proof.

$$1 = \sum_{Q \in \mathcal{P}_n} P^{(n)}(T(Q)) \leq \sum_{Q \in \mathcal{P}_n} P^{(n)}(T(P)) \leq (n + 1)^{|X|} P^{(n)}(T(P)) = (n + 1)^{|X|} |T(P)| \cdot 2^{-nH(P)}.$$
Probability of type class

**Theorem**

For any \( P \in P_n \) and any distribution \( Q \), the probability of the type class \( T(P) \) under \( Q^{(n)} \) is

\[
\frac{1}{(n+1)|X|} 2^{-nD(P\|Q)} \leq |Q^{(n)}(T(P))| \leq 2^{-nD(P\|Q)}.
\]  

(62)

**Proof**

\[
Q^{(n)}(T(P)) = \sum_{x \in T(P)} Q^{(n)}(x) = \sum_{x \in T(P)} 2^{-n(D(P\|Q) + H(P))} = |T(P)| 2^{-n(D(P\|Q) + H(P))}
\]

Then use the bounds on \(|T(P)|\) derived in last theorem.
The Method of Types

Probability of type class

**Theorem**

For any $P \in P_n$ and any distribution $Q$, the probability of the type class $T(P)$ under $Q^{(n)}$ is

$$\frac{1}{(n+1)|\mathcal{X}|} 2^{-nD(P\|Q)} \leq |Q^{(n)}(T(P))| \leq 2^{-nD(P\|Q)}. \quad (62)$$

**Proof**

$$Q^{(n)}(T(P)) = \sum_{x \in T(P)} Q^{(n)}(x) = \sum_{x \in T(P)} 2^{-n(D(P\|Q)+H(P))}$$

$$= |T(P)| 2^{-n(D(P\|Q)+H(P))}$$

Then use the bounds on $|T(P)|$ derived in last theorem.
Summarize

- We can summarize the basic theorems concerning types in four equations:

\[ |\mathcal{P}_n| \leq (n + 1)|\mathcal{X}|, \quad (63) \]
\[ Q^{(n)}(x) = 2^{-n(H(P_x)+D(P_x\|Q))}, \quad (64) \]
\[ |T(P)| \equiv 2^{nH(P)}, \quad (65) \]
\[ Q^{(n)}(T(P)) \equiv 2^{-nD(P\|Q)}. \quad (66) \]

- There are only a polynomial number of types and an exponential number of sequences of each type.

- We can calculate the behavior of long sequences based on the properties of the type of the sequence.
Outline

8 The Method of Types

9 Combinatorial Bounds on Entropy
Tight bounds on the size of \( \binom{n}{k} \)

**Lemma**

For \( 0 < p < 1 \), \( q = 1 - p \), such that \( np \) is an integer,

\[
\frac{1}{\sqrt{8npq}} \leq \binom{n}{np} 2^{-nH(p)} \leq \frac{1}{\sqrt{\pi npq}}.
\]  
(67)
Tight bounds on the size of \( \binom{n}{k} \)

Proof of Lemma

Applying a strong form of Stirling’s approximation, which states that

\[
\sqrt{2\pi} n \left( \frac{n}{e} \right)^n \leq n! \leq \sqrt{2\pi} n \left( \frac{n}{e} \right)^n e^{\frac{1}{12n}}.
\]

we obtain

\[
\binom{n}{np} \leq \frac{\sqrt{2\pi} n \left( \frac{n}{e} \right)^n e^{\frac{1}{12n}}}{\sqrt{2\pi} np \left( \frac{np}{e} \right)^{np} \sqrt{2\pi} nq \left( \frac{nq}{e} \right)^{nq}}
\]

\[
= \frac{1}{\sqrt{2\pi} npq} \frac{1}{p^{np} q^{nq}} e^{\frac{1}{12n}}
\]

\[
< \frac{1}{\sqrt{\pi} npq} 2^{nH(p)}
\]

Since \( e^{\frac{1}{12n}} < e^{\frac{1}{12}} < \sqrt{2} \). The lower bound is obtained similarly.
Combinatorial Bounds on Entropy

Tight bounds on the size of $\binom{n}{k}$

Proof of Lemma (cont.)

$$\binom{n}{np} \geq \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{-\left(\frac{1}{12np} + \frac{1}{12nq}\right)}}{\sqrt{2\pi np} \left(\frac{np}{e}\right)^{np} \sqrt{2\pi nq} \left(\frac{nq}{e}\right)^{nq}}$$

$$= \frac{1}{\sqrt{2\pi npq}} \frac{1}{p^{np} q^{nq}} e^{-\left(\frac{1}{12np} + \frac{1}{12nq}\right)}$$

$$< \frac{1}{\sqrt{2\pi npq}} 2^{nH(p)} e^{-\left(\frac{1}{12np} + \frac{1}{12nq}\right)}$$

If $np \geq 1$, and $nq \geq 3$, then $e^{-\left(\frac{1}{12np} + \frac{1}{12nq}\right)} > e^{-\frac{1}{9}} = 0.8948 > \frac{\sqrt{\pi}}{2} = 0.8862$. For $np = 1$, $nq = 1$ or $2$, and $np = 2$, $nq = 2$ can easily be verified that the inequality still holds. Thus we proved the Lemma.
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Thank You!!!
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