Group Actions on Surfaces

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Definition of Group Action

Definition

An action of a group *G on a manifold M is a continuous (or differentiable) function $\phi : G \times M \to M$ satisfying*

- $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$
- $\phi(e, x) = x$ *for all x where e is the identity of G.*

A homeomorphism $f : M \to M$ defines an action of $\mathbb{Z}$ on on M by $\phi(n, x) = f^n(x)$.

We will be interested in actions of discrete non-compact groups such as $SL(n, \mathbb{Z})$ is the group of $n \times n$ integer matrices with determinant 1.

Group Actions

Let $\mathrm{Homeo}(M)$ and $\mathrm{Diff}(M)$ denote the groups of orientation preserving homeomorphisms and diffeomorphisms of the compact manifold M.

Definition (Alternate)

An action of a group G on a manifold M is a homomorphism

$$\phi : G \to \mathrm{Homeo}(M)$$

or

$$\phi : G \to \mathrm{Diff}(M).$$

A Motivating Conjecture

Conjecture (R. Zimmer [21])

Any C^∞ volume preserving action of $SL(n, \mathbb{Z})$ on a compact manifold with dimension less than n, factors through an action of a finite group.

We are really interested in results valid for all finite index subgroups of $SL(n, \mathbb{Z})$.

Theorem (D. Witte [20])

Let $\mathcal{G}$ be a finite index subgroup of $SL(n, \mathbb{Z})$ with $n \geq 3$. Any homomorphism

$$\phi : \mathcal{G} \to \mathrm{Homeo}(S^1)$$

has a finite image.

Example

The group $SL(3,\mathbb{Z})$ acts analytically on S^2 by projectivizing the standard action on $\mathbb{R}^3$.

S^2 is the set of unit vectors in $\mathbb{R}^3$. If $x \in S^2$ and $g \in SL(3,\mathbb{Z})$, we can define $\phi(g) : S^2 \to S^2$ by

$$\phi(g)(x) = \frac{gx}{|gx|}.$$

Question

Let $\mathcal{G}$ be a finite index subgroup of $SL(4,\mathbb{Z})$. Does every homomorphism from $\mathcal{G}$ to $\mathrm{Diff}(S^2)$ or $\mathrm{Homeo}(S^2)$ have a finite image? What about other surfaces?

The Heisenberg group

Example

The group of integer matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

is called the Heisenberg group*.*

If

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Their *commutator* $f = [g, h] := g^{-1}h^{-1}gh$ is

$$f = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and it commutes with } g \text{ and } h.$$

This implies

$$[g^n, h^n] = f^{n^2}.$$

Distortion in Groups

Definition (Gromov)

An element g in a finitely generated group G is called a distortion element *if it has infinite order and*

$$\liminf_{n\to\infty} \frac{|g^n|}{n} = 0,$$

where $|g|$ denotes the minmal word length of g in some set of generators. If $\mathcal{G}$ is not finitely generated then g is distorted if it is distorted in some finitely generated subgroup.

Example

In the subgroup G of $SL(2,\mathbb{R})$ *generated by*

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1}BA = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = B^4 \text{ and } A^{-n}BA^n = B^{4^n}$$

so B is distorted.

Example

In the Heisenberg group the identity

$$[g^n, h^n] = f^{n^2}.$$

shows f *is distorted since it implies* $|f^{n^2}| \leq 4n$.

Example (G. Mess)

Consider the subgroup of $\mathrm{Aff}(\mathrm{T}^2)$ *generated by the automorphism given by*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and a translation $T(x) = x + w$ *where* $w \neq 0$ *is parallel to the unstable manifold of* A*. The element* T *is distorted.*

Proof: Let λ be the expanding eigenvalue of A. The element $h_n = A^n T A^{-n}$ satisfies $h_n(x) = x + \lambda^n w$ and $g_n = A^{-n} T A^n$ satisfies $g_n(x) = x + \lambda^{-n} w$. Hence $g_n h_n(x) = x + (\lambda^n + \lambda^{-n}) w$. Since $trA^n = \lambda^n + \lambda^{-n}$ is an integer we conclude $T^{trA^n} = g_n h_n$, so $|T^{trA^n}| \leq 4n + 2$. Thus

$$\lim_{n \to \infty} \frac{|T^{trA^n}|}{trA^n} = 0,$$

so T is distorted. □

Question

Can one characterize the dynamics of distortion elements in $\mathrm{Homeo}(S^1)$ *or* $\mathrm{Diff}(S^2)$ *or in area preserving diffeomorphisms of* S^2? *What about irrational rotations of* S^1 *or* S^2 *in the area preserving or analytic case.*

Theorem (D. Calegari)

There is a C^0 *action of the Heisenberg group on* S^2 *whose center generated by an irrational rotation.*

The example of Calegari for the Heisenberg group acting on S^2 is not conjugate to a C^1 example.

Proof: For $a \in \mathbb{R}$ let

$$S(x, y) = (x + y, y),$$
$$T_a(x, y) = (x + a, y), \text{ and}$$
$$U(x, y) = (x, y + 1)$$

be maps of $\mathbb{R}^2$. Since U and S commute with T_a they induce homeomorphisms $\hat{U}$, $\hat{T}_a$ and $\hat{S}$ of the infinite cylinder $\mathbb{R}^2/T_\theta$ (identifying (x, y) with $(x + \theta, y)$. If θ is irrational then $\hat{T}_1$ is an irrational rotation of C.

It is easy to check that $[U, S] = T_1$ so $[\hat{U}, \hat{S}] = \hat{T}_1$. Hence the group generated by $\hat{U}$ and $\hat{S}$ is isomorphic to the Heisenberg group $\mathcal{H}$. Compactifying the two ends of C by adding points gives an action of $\mathcal{H}$ by *homeomorphisms* on S^2. □

Theorem (D. Calegari and M. Freedman [1])

An irrational rotation of S^2 *is distorted in* $\mathrm{Diff}^\infty(S^2)$.

Theorem (D. Calegari and M. Freedman [1])

An irrational rotation of S^1 *is distorted in* $\mathrm{Diff}^1(S^1)$.

Question

Is an irrational rotation of S^1 *distorted in* $\mathrm{Diff}^r(S^1)$ *for* $r \geq 2$?

Recall the definition:

Definition (Gromov)

An element g in a finitely generated group G is called a distortion element *if it has infinite order and*

$$\liminf_{n\to\infty} \frac{|g^n|}{n} = 0,$$

where $|g|$ denotes the minmal word length of g in some set of generators. If $\mathcal{G}$ is not finitely generated then g is distorted if it is distorted in some finitely generated subgroup.

Many Lattices have Distortion

Theorem (Lubotzky-Mozes-Ragunathan [12])

Suppose Γ is a non-uniform irreducible lattice in a semi-simple Lie group $\mathcal{G}$ with $\mathbb{R}-$rank ≥ 2. Suppose further that $\mathcal{G}$ is connected, with finite center and no nontrivial compact factors. Then Γ has distortion elements, in fact, elements whose word length growth is at most logarithmic.

Interval Exchange Transformations

Definition

An interval exchange transformation *(IET) is an invertible map* $\phi : \mathrm{T}^1 \to \mathrm{T}^1$ *of the circle* $T^1 = \mathbb{R}/\mathbb{Z}$ *which acts as a piecewise translation on a finite collection of subintervals.*

Theorem (Novak [14])

If $d(f)$ *denotes the number of discontinuities of an IET* f *then* $d(f^n)$ *is either bounded or has linear growth in* n*.*

Interval Exchange Transformations

Theorem (Novak [14])

Let $\mathcal{E}$ denote the group of interval exchange transformations on T^1. Then there are no distortion elements in $\mathcal{E}$.

Corollary

Many finitely generated groups are not isomorphic to subgroups of $\mathcal{E}$.

Question

Is $\mathcal{F}_2$, the free group on two generators, isomorphic to a subgroup of $\mathcal{E}$.

Margulis' normal subgroup theorem

Definition

A group is called almost simple *if every normal subgroup is finite or has finite index.*

Theorem (Margulis)

Assume Γ *is an irreducible lattice in a semi-simple Lie group with* $\mathbb{R}-$*rank* ≥ 2*, e.g. any finite index subgroup of* $SL(n, \mathbb{Z})$ *with* $n \geq 3$*. Then any normal subgroup of* Γ *is either finite and in the center of* Γ *or has finite index. In particular* Γ *is almost simple.*

Proposition

If $\mathcal{G}$ *is a finitely generated almost simple group which contains a distortion element and* $\mathcal{H} \subset \mathcal{G}$ *is a normal subgroup, then the only homomorphism from* $\mathcal{H}$ *to* $\mathbb{R}$ *is the trivial one.*

Thurston's stability theorem

Theorem (Thurston [19])

Suppose $\mathcal{G}$ is a finitely generated group,

$$\phi : G \to \mathrm{Diff}^1(M^n)$$

is a homomorphism and there is $x_0 \in M$ such that for all $g \in \mathcal{G}$

$$\phi(g)(x_0) = x_0 \text{ and } D\phi(g)(x_0) = I.$$

Then either ϕ is trivial or there is a non-trivial homomorphism from $\mathcal{G}$ to $\mathbb{R}$.

The proof we give is due to W. Schachermayer [18].

Proof of Thurston's stability theorem

Let $\{g_i\}$ be a set of generators for $\phi(\mathcal{G})$. WLOG assume $M = \mathbb{R}^m$ and $x_0 = 0$ is not in the interior of $\mathrm{Fix}(\phi(\mathcal{G}))$. For $g \in \phi(\mathcal{G})$ let $\widehat{g}(x) = g(x) - x$, so $g(x) = x + \widehat{g}(x)$ and $D\widehat{g}(0) = 0$. We compute

$$\begin{aligned}
\widehat{gh}(x) &= g(h(x)) - x \\
&= h(x) - x + g(h(x)) - h(x) \\
&= \widehat{h}(x) + \widehat{g}(h(x)) \\
&= \widehat{h}(x) + \widehat{g}(x + \widehat{h}(x)) \\
&= \widehat{g}(x) + \widehat{h}(x) + (\widehat{g}(x + \widehat{h}(x)) - \widehat{g}(x)).
\end{aligned}$$

Hence for all $g, h \in \mathcal{G}$ and for all $x \in \mathbb{R}^m$

$$\widehat{gh}(x) = \widehat{g}(x) + \widehat{h}(x) + (\widehat{g}(x + \widehat{h}(x)) - \widehat{g}(x)). \qquad (1)$$

Choose a sequence $\{x_n\}$ in $\mathbb{R}^m$ converging to 0 such that for some i we have $|\widehat{g}_i(x_n)| \neq 0$ for all n. Possible since 0 is not in the interior of $\mathrm{Fix}(\phi(\mathcal{G}))$.

Let $M_n = \max\{|\widehat{g}_1(x_n)|, \dots, |\widehat{g}_k(x_n)|\}$. Passing to a subsequence we may assume that for each i the limit

$$L_i = \lim_{n\to\infty} \frac{\widehat{g}_i(x_n)}{M_n}$$

exists and that $\|L_i\| \leq 1$. For some i we have $\|L_i\| = 1$; say for $i = 1$.

If $g \in \mathcal{G}$ and the limit

$$L = \lim_{n\to\infty} \frac{\widehat{g}(x_n)}{M_n}$$

exists then for each i we will show that

$$\lim_{n\to\infty} \frac{\widehat{g_i g}(x_n)}{M_n} = L_i + L. \tag{2}$$

By Equation (1) it suffices to show

$$\lim_{n\to\infty} \frac{\widehat{g}_i(x_n + \widehat{g}(x_n)) - \widehat{g}_i(x_n)))}{M_n} = 0. \tag{3}$$

By the mean value theorem

$$\lim_{n\to\infty} \left\| \frac{\widehat{g}_i(x_n + \widehat{g}(x_n)) - \widehat{g}_i(x_n)))}{M_n} \right\|$$
$$\leq \lim_{n\to\infty} \sup_{t\in[0,1]} \|D\widehat{g}_i(z_n(t))\| \left\| \frac{\widehat{g}(x_n)}{M_n} \right\|,$$

where $z_n(t) = x_n + t\widehat{g}(x_n)$. But

$$\lim_{n\to\infty} \frac{\widehat{g}(x_n)}{M_n} = L \text{ and } \lim_{n\to\infty} sup_{t\in[0,1]} \|D\widehat{g}_i(z_n(t))\| = 0,$$

so Equation (3) holds. Defining $\Theta : \phi(\mathcal{G}) \to \mathbb{R}^m$ by

$$\Theta(g) = \lim_{n\to\infty} \frac{\widehat{g}(x_n)}{M_n}$$

gives a homomorphism from $\phi(\mathcal{G})$ to $\mathbb{R}^m$.

Rotation Numbers

Definition

If $\mathcal{G}$ is a group, a function $\phi : \mathcal{G} \to \mathbb{R}$ is called a quasi-morphism if there is $D > 0$ such that $|\phi(gh) - \phi(g) - \phi(h)| < D$ for all $g, h \in \mathcal{G}$.

Let $f : S^1 \to S^1$ be a degree one homeomorphism with lift $F : \mathbb{R} \to \mathbb{R}$

Proposition

For $x_0 \in \mathbb{R}$ define the function $\phi : \mathbb{Z} \to \mathbb{R}$ by $\phi(n) = F^n(x_0) - x_0$. Then ϕ is a quasi-morphism, in fact, $|\phi(n+m) - \phi(n) - \phi(m)| < 1$ for all $n, m \in \mathbb{Z}$. Moreover $|\phi(kn) - k\phi(n)| \leq k$ for all $k, n \in \mathbb{Z}$.

Rotation Numbers

Proposition

For any $x_0 \in \mathbb{R}$ the limit

$$\tau(x_0, F) = \lim_{n \to \infty} \frac{F^n(x_0) - x_0}{n}$$

exists and is independent of x_0. (Only because we are on S^1.)

Definition

*The **translation number** of F is $\tau(F) = \tau(x_0, F)$ and the **rotation number** of f is $\rho(f) = (\tau(F) + \mathbb{Z}) \in \mathbb{R}/\mathbb{Z}$.*

$\rho(f)$ is independent of the choice of lift F.

Measure and Rotation numbers

In general the function $\rho : \mathrm{Homeo}(S^1) \to \mathbb{R}/\mathbb{Z}$ is not a homomorphism, but

Proposition

If μ is a Borel measure on S^1 then

$$\rho : \mathrm{Homeo}_\mu(S^1) \to \mathbb{R}/\mathbb{Z}$$

is a homomorphism, where $\mathrm{Homeo}_\mu(S^1)$ denotes the group of orientation preserving homeomorphism which preserve the measure μ.

N.B.: For definitive results on C^1 actions on S^1 see E. Ghys, [9].

Theorem (Toy Theorem)

Suppose $\mathcal{G}$ is a finitely generated almost simple group and has a distortion element and suppose μ is a finite probability measure on S^1. If

$$\phi : \mathcal{G} \to Diff_\mu(S^1)$$

is a homomorphism then $\phi(\mathcal{G})$ is finite.

Proof:

- The rotation number $\rho : Diff_\mu(S^1) \to \mathbb{R}/\mathbb{Z}$ is a homomorphism.
- If f is distorted $\rho(f^n) = 0$ for some $n > 0$ so $\mathrm{Fix}(f^n)$ is non-empty.
- $supp(\mu) \subset \mathrm{Fix}(f^n)$
- $\mathcal{G}_0 := \{g \in \mathcal{G} \mid \phi(g)$ pointwise fixes $supp(\mu)\}$ is infinite and normal, and hence finite index.
- $\phi(\mathcal{G}_0)$ is trivial by Thurston stability.

□

Distortion and Measure

Theorem (F-Handel [5])

Suppose that S is a closed oriented surface, that f is a distortion element in $\mathrm{Diff}(S)_0$ and that μ is an f-invariant Borel probability measure.

1. *If S has genus at least two then $\mathrm{Per}(f) = \mathrm{Fix}(f)$ and $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f)$.*
2. *If $S = T^2$ and $\mathrm{Per}(f) \neq \emptyset$, then all points of $\mathrm{Per}(f)$ have the same period, say n, and $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f^n)$*
3. *If $S = S^2$ and if f^n has at least three fixed points for some smallest $n > 0$, then $\mathrm{Per}(f) = \mathrm{Fix}(f^n)$ and $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f^n)$.*

Theorem (F-Handel [5])

Suppose S is a closed oriented surface of genus at least one and μ is a Borel probabilty measure on S with infinite support. Suppose $\mathcal{G}$ is finitely generated, almost simple and has a distortion element. Then any homomorphism

$$\phi : \mathcal{G} \to \mathrm{Diff}_\mu(S)$$

has finite image.

This result was previously known in the special case of symplectic diffeomorphisms and Lebesgue measure by a result of L. Polterovich [17].

The result above also holds even when *supp*(μ) is finite if $\mathcal{G}$ is a Kazhdan group (aka $\mathcal{G}$ has property T).

Proof:

• If f is distorted $supp(\mu) \subset \mathrm{Fix}(f)$, so $\mathrm{Fix}(f)$ is an infinite closed set.

• Let $\mathcal{G}_0 := \{g \in \mathcal{G} \mid \phi(g)$ pointwise fixes $supp(\mu)\}$. It is infinite and normal, and hence finite index in $\mathcal{G}$.

• Let $x \in \mathrm{Fix}(f)$. There is a common eigenvector with eigenvalue 1 for $Dg_x : TM_x \to TM_x$ for every $g \in \phi(\mathcal{G}_0)$.

• $Dg_x = Id$ for every $g \in \phi(\mathcal{G}_0)$.

• $\phi(\mathcal{G}_0)$ is trivial by Thurston stability.

• $\mathcal{G}/ker(\phi)$ is finite. □

Heisenberg again

Theorem (F-Handel [5])

Suppose S is a closed oriented surface with Borel probabilty measure μ and $\mathcal{G}$ is a finitely generated, almost simple group with a subgroup isomorphic to the Heisenberg group. Then any homomorphism

$$\phi : \mathcal{G} \to \mathrm{Diff}_\mu(S)$$

has finite image.

In general there seem to be strong parallels between results about $\mathrm{Diff}(S^1)_0$ and $\mathrm{Diff}_\mu(S)_0$. In addition to our results above there is Witte's theorem

Theorem (D. Witte [20])

Let $\mathcal{G}$ be a finite index subgroup of $SL(n, \mathbb{Z})$ with $n \geq 3$. Any homomorphism

$$\phi : \mathcal{G} \to \mathrm{Homeo}(S^1)$$

has a finite image.

Also there are the following results

Theorem (Hölder)

Suppose $\mathcal{G}$ is a subgroup of $\mathrm{Homeo}(S^1)_0$ which acts freely (no non-trivial element has a fixed point). Then $\mathcal{G}$ is Abelian.

Theorem (Conley-Zehnder, Matsumoto)

Suppose

$$f \in \mathrm{Homeo}_\omega(\mathrm{T}^2)_0$$

is a commutator (ω is Lebesgue measure). Then f has (at least three) fixed points.

Corollary

Suppose $\mathcal{G}$ is a subgroup of $\mathrm{Homeo}_\omega(\mathrm{T}^2)_0$ which acts freely. Then $\mathcal{G}$ is Abelian.

Nilpotent Groups

Definition

A group $\mathbb{N}$ *is called* nilpotent *provided when we define*

$$\mathbb{N}_0 = \mathbb{N}, \ \mathbb{N}_i = [\mathbb{N}, \mathbb{N}_{i-1}],$$

there is an $n \geq 1$ *such that* $\mathbb{N}_n = \{e\}$. *Note if* $n = 1$ *it is Abelian.*

Theorem (Plante - Thurston [15])

Let N *be a nilpotent subgroup of* $\mathrm{Diff}^2(S^1)_0$. *Then* N *must be Abelian.*

Theorem (Farb - F)

Every finitely-generated, torsion-free nilpotent group is isomorphic to a subgroup of $\mathrm{Diff}^1(S^1)_0$.

Theorem (F - Handel[5])

Let $\mathbb{N}$ be a nilpotent subgroup of $\mathrm{Diff}^1_\mu(S)_0$ with μ a probability measure with supp$(\mu) = S$. If $S \neq S^2$ then $\mathbb{N}$ is Abelian, if $S = S^2$ then $\mathbb{N}$ is Abelian or has an index 2 Abelian subgroup.

Proof: (For the case $genus(S) > 1$) Suppose

$$\mathbb{N} = \mathbb{N}_1 \supset \cdots \supset \mathbb{N}_m \supset \{1\}$$

is the lower central series of $\mathbb{N}$. then $\mathbb{N}_m$ is in the center of $\mathbb{N}$. If $m > 1$ there is a non-trivial $f \in \mathbb{N}_m$ and elements g, h with $f = [g, h]$. No non-trivial element of $\mathrm{Diff}^1(S)_0$ has finite order since S has genus > 1. So g, h generate a Heisenberg group and f is distorted. Our theorem says supp$(\mu) \subset \mathrm{Fix}(f)$, but supp$(\mu) = S$ so $f = id$. This is a contradiction unless $m = 1$ and $\mathbb{N}$ is abelian. □

Two Commuting Diffeomorphisms of S^2

Theorem (Handel (1992) [10])

Let $\mathcal{G}$ be a subgroup of $\mathrm{Diff}^1(S^2)_0$ generated by two commuting diffeomorphisms. Then there is a subgroup $\mathcal{G}_0$ of $\mathcal{G}$ of index at most two and a point $x \in S^2$ such that $g(x) = x$ for all g in $\mathcal{G}_0$.

Fixed Points for Abelian Actions

Theorem (F, Handel, Parwani [7])

Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{Diff}^1(S^2)_0$. Then there is a subgroup $\mathcal{G}_0$ of $\mathcal{G}$ of index at most two and a point $x \in S^2$ such that $g(x) = x$ for all g in $\mathcal{G}_0$.

Theorem (F, Handel, Parwani [7])

Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{Diff}^1(\mathbb{R}^2)_0$ with the property that there is a compact $\mathcal{G}$ invariant subset of $\mathbb{R}^2$. Then there is a point $x \in \mathbb{R}^2$ such that $g(x) = x$ for all g in $\mathcal{G}$.

Abelian Actions: Genus ≥ 2

Theorem (F, Handel, Parwani [8])

Suppose S is a closed oriented surface of genus at least two and that $\mathcal{F}$ is an abelian subgroup of $\mathrm{Diff}_0(S)$ Then the set of global fixed points, $\mathrm{Fix}(\mathcal{F})$ is non-empty.

Theorem (F, Handel, Parwani [8])

Suppose S is a closed oriented surface of genus at least two and that $\mathcal{F}$ is an abelian subgroup of $\mathrm{Diff}(S)$. Then $\mathcal{F}$ has a finite index subgroup $\mathcal{F}_0$ such that $\mathrm{Fix}(\mathcal{F}_0)$ is non-empty.

A Fixed point Theorem

Theorem (F, Handel [6])

Let $\mathcal{G}$ be a subgroup of $\mathrm{Homeo}(D^2)$ and let f be an element of the center of $\mathcal{G}$. Suppose $\mathrm{Fix}(f) \cap \partial D^2$ consists of a finite set with more than two elements each of which is either an attracting or repelling fixed point for $f : D \to D$. Let $\mathcal{G}_0 \subset \mathcal{G}$ denote the finite index subgroup whose elements pointwise fix $\mathrm{Fix}(f) \cap \partial D^2$. Then $\mathrm{Fix}(\mathcal{G}_0) \cap int(D)$ is non-empty.

The Lifting Problem

Definition

The mapping class group $\mathrm{MCG}(S)$ *of a surface S with genus g is the group of isotopy classes of orientation preserving homeomorphisms of S.*

- $\mathrm{MCG}(S^2) \cong \{1\}$
- $\mathrm{MCG}(T^2) \cong SL(2, \mathbb{Z})$

The Lifting Problem

There is a natural homomorphism

$$\mathrm{Homeo}(S) \to \mathrm{MCG}(S).$$

Definition

A lift *of a subgroup* Γ *of* $\mathrm{MCG}(S)$ *is a homomorphism* $\Phi : \Gamma \to \mathrm{Homeo}(S)$ *such that the composition*

$$\Gamma \to \mathrm{Homeo}(S) \to \mathrm{MCG}(S)$$

is the inclusion.

The Lifting Problem

Question

Which subgroups of $\mathrm{MCG}(S)$ *lift to* $\mathrm{Homeo}(S)[\mathrm{Diff}(S)]$*?*

$\mathrm{MCG}(T^2)$ lifts to $\mathrm{Diff}(T^2)$ so assume that $g \geq 2$.

- Any free group or any free abelian group
- Any finite group [Kerckhoff]
- $\mathrm{MCG}(S)$ does not lift to $\mathrm{Diff}(S)$ for $g \geq 5$ [Morita]
- $\mathrm{MCG}(S)$ does not lift to $\mathrm{Homeo}(S)$ for $g \geq 6$ [Markovic]

An elementary proof of Morita's Theorem.

Theorem (F, Handel [6])

If S has genus $g \geq 3$ then $\mathrm{MCG}(S)$ does not lift to $\mathrm{Diff}(S)$.

Strategy of Proof

Let $S = M \# T^2$, where M has genus $g - 1 \geq 2$.
If there is a lift Φ of $\mathrm{MCG}(S)$ to $\mathrm{Diff}(S)$ we will show there is are infinitely many global fixed point for $\Phi(\mathrm{MCG}(M, \partial M))$. This leads to a contradiction.

Thurston's stability theorem again

Theorem (Thurston)

Suppose $\mathcal{G}$ is a finitely generated group,

$$\phi : \mathcal{G} \to \mathrm{Diff}^1(M^n)$$

is a homomorphism and there is $x_0 \in M$ such that for all $g \in \mathcal{G}$

$$\phi(g)(x_0) = x_0 \text{ and } D\phi(g)(x_0) = I.$$

Then either ϕ is trivial or there is a non-trivial homomorphism from $\mathcal{G}$ to $\mathbb{R}$.

Theorem (Korkmaz (see [11]))

If the genus g of S is ≥ 2 there is no non-trivial homomorphism to $\mathbb{R}$ from $\mathrm{MCG}(S)$ or from $\mathrm{MCG}(S, \partial S)$ if ∂S is connected.

Lemma

Let $f : X \to X$ be a homeomorphism of a locally compact metric space with a global attracting point x_0 i.e., suppose in the Hausdorf topology

$$\lim_{n \to \infty} f^n(Y) = \{x_0\}$$

for any compact subset Y of X. If $g : X \to X$ is a homeomorphism which commutes with f then there exists $m > 0$ such that $h = f^m g$ satisfies

$$\lim_{n \to \infty} h^n(Y) = \{x_0\}$$

for any compact subset Y of X.

A Fixed point Theorem

Theorem (F, Handel [6])

Let $\mathcal{G}$ be a subgroup of $\mathrm{Homeo}(D^2)$ and let f be an element of the center of $\mathcal{G}$. Suppose $\mathrm{Fix}(f) \cap \partial D^2$ consists of a finite set with more than two elements each of which is either an attracting or repelling fixed point for $f : D \to D$. Let $\mathcal{G}_0 \subset \mathcal{G}$ denote the finite index subgroup whose elements pointwise fix $\mathrm{Fix}(f) \cap \partial D^2$. Then $\mathrm{Fix}(\mathcal{G}_0) \cap int(D)$ is non-empty.

$\mathrm{MCG}(M)$ acting on S^1.

Theorem (Parwani [16])

Let M be a connected orientable surface with finitely many punctures, finitely many boundary components, and genus at least 6. Then any C^1 action of the mapping class group $\mathrm{MCG}(M)$ on the circle S^1 is trivial.

Let $M = M_1 \# M_2$, where each M_i has genus $g \geq 3$. Also let $\mathcal{G}_i = \mathrm{MCG}(M_i, \partial M_i)$ (each of which we consider as a subgroup of $\mathrm{MCG}(M)$).

Then we apply the following theorem.

Theorem (Parwani [16])

Let $\mathcal{H}$ and $\mathcal{G}$ be two finitely generated groups such that $H_1(\mathcal{G},\mathbb{Z}) = H_1(\mathcal{H},\mathbb{Z}) = 0$. Then for any C^1 action of $\mathcal{H} \times \mathcal{G}$ on the circle, either $\mathcal{H} \times id$ acts trivially or $id \times \mathcal{G}$ acts trivially.

Theorem (Deroin, Kleptsyn and Navas [2])

Let $\mathcal{G}$ be a countable group with an orientation preserving C^1 action on the circle. If there is no $\mathcal{G}$-invariant probability measure for the action, then there exists an element $g \in \mathcal{G}$ whose fixed point set is non-empty and finite.

References I

[1] **D. Calegari and M. Freedman**
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