Essential hyperbolicity versus homoclinic bifurcations

Global dynamics beyond uniform hyperbolicity, Beijing 2009

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Generic dynamics

Consider:

- M: compact boundaryless manifold,
- $\mathrm{Diff}(M)$.

Goal: *understand the dynamics of "most" $f \in \mathrm{Diff}(M)$.*
"Most": at least a dense part.

Our viewpoint: describe a *generic* subset of $\mathrm{Diff}^1(M)$.
Generic (Baire): a countable intersection of open and dense subsets.

Hyperbolic diffeomorphisms: definition

Definition

$f \in \mathrm{Diff}(M)$ is *hyperbolic* if there exists $K_0, \ldots, K_d \subset M$ s.t.:

- each K_i is a hyperbolic invariant compact set

$$T_K M = E^s \oplus E^u,$$

- for any $x \in M \setminus (\bigcup_i K_i)$, there exists $U \subset M$ open such that

$$f(\overline{U}) \subset U \text{ and } x \in U \setminus f(\overline{U}).$$

Hyperbolic diffeomorphisms: properties

Good properties of hyperbolic diffeomorphisms:
Ω-stability, coding, physical measures,...

The set $hyp(M) \subset \mathrm{Diff}^r(M)$ of hyperbolic dynamics is

- open,

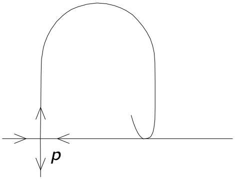
and:

- *dense*, when $\dim(M) = 1$, $r \geq 1$ (*Peixoto*),
- *not dense*,

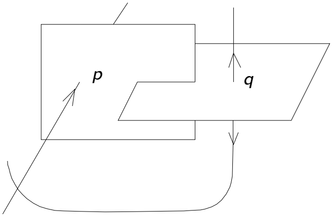
  when $\dim(M) = 2$, $r \geq 2$ (*Newhouse*)
  or when $\dim(M) > 2$ and $r \geq 1$ (*Abraham-Smale*),
- *dense??*, when $\dim(M) = 2$, $r = 1$ (*Smale'conjecture* = yes).

Obstructions to hyperbolicity

Homoclinic tangency associated to a hyperbolic periodic point p.



Heterodimensional cycle associated to two hyperbolic periodic points p, q such that $\dim(E^s(p)) \neq \dim(E^s(q))$.



Palis' conjecture

Describe of the dynamics in $\mathrm{Diff}(M)$ by *phenomena/mechanisms*.

Conjecture (Palis)

Any $f \in \mathrm{Diff}(M)$ can be approximated by a hyperbolic diffeomorphism or by a diffeomorphism exhibiting a homoclinic bifurcation (tangency or cycle).

This holds when $\dim(M) = 1$. In higher dimensions, there are progresses for $\mathrm{Diff}^1(M)$.

Theorem (Pujals-Sambarino)

The Palis conjecture holds for C^1-diffeomorphisms of surfaces.

Remark (Bonatti-Díaz). For the C^1-topology, it could be enough to consider only the heterodimensional cycles.

Essential hyperbolicity far from homoclinic bifurcations

Theorem (Pujals, C-)

Any generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is essentially hyperbolic.*

Definition

$f \in \mathrm{Diff}(M)$ is *essentially hyperbolic* if there exists $K_1, \dots, K_s$ s.t.:

- each K_i is a hyperbolic attractor,
- the union of the basins of the K_i is (open and) dense in M.

Remarks.

- The set of these diffeomorphisms is not open apriori.
- There was a previous result by Pujals about attractors in dimension 3.

Partial hyperbolicity far from homoclinic bifurcations

Theorem 1 (C-)

Any generic diffeomorphism $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is partially hyperbolic.*

More precizely, there exists $K_0, \dots, K_d \subset M$ such that:

- each K_i is a *partially hyperbolic* invariant compact set $T_K M = E^s \oplus E^u$ or $E^s \oplus E^c \oplus_< E^u$ or $E^s \oplus E^c_1 \oplus E^c_2 \oplus E^u$, and E^c, E^c_1, E^c_2 are one-dimensional.
- for any $x \in M \setminus (\bigcup_i K_i)$, there exists $U \subset M$ open such that

$$f(\overline{U}) \subset U \text{ and } x \in U \setminus f(\overline{U}).$$

Extremal bundles

Theorem 2 (Pujals, Sambarino, C-)

For any

- *generic* $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$,
- *partially hyperbolic transitive set* K,

the extremal bundles E^s, E^u *on* K *are non-degenerated, or* K *is a sink/source.*

Program of the lectures

Goal. *Any generic* $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is essentially hyperbolic.*

Part 1. Topological hyperbolicity
Obtain the existence of a finite number of "attractors" that are "topologically hyperbolic" and have dense basin.

- *Lecture 1.* How Theorems 1 & 2 are used to prove the essential topological hyperbolicity?
- *Lecture 2.* Theorem 1 (partial hyperbolicity).
- *Lecture 3.* Theorem 2 (extremal bundles).

Part 2. From topological to uniform hyperbolicity

- *Lectures 4,5,6.*

I- Decomposition of the dynamics: the chain-recurrence classes

The *chain-recurrent set* $\mathcal{R}(f)$: the set of $x \in M$ s.t. for any $\varepsilon > 0$, there exists a ε-pseudo-orbit $x = x_0, x_1, \dots, x_n = x$, $n \geq 1$.

The *chain-recurrence classes*: the equivalence classes of the relation "for any $\varepsilon > 0$, there is a periodic ε-pseudo-orbit containing x, y".

- This gives a partition of $\mathcal{R}(f)$ into compact invariant subsets.

Theorem (Bonatti, C-)

For $f \in \mathrm{Diff}^1(M)$ generic, any chain-recurrence class which contains a periodic point p coincides with the homoclinic class of p:

$$H(p) = \overline{W^s(O(p)) \pitchfork W^u(O(p))}.$$

The other chain-recurrence classes are called *aperiodic classes*.

I- Decomposition of the dynamics: the quasi-attractors

A *quasi-attractor* is a chain-recurrence class having a basis of neighborhoods U which satisfy $f(\overline{U}) \subset U$.

- There always exist quasi-attractors.

Theorem (Morales, Pacifico, Bonatti, C-)

For a generic $f \in \mathrm{Diff}^1(M)$, the basins of the quasi-attractors of f are dense in M.

- In order to prove the main theorem we have to prove that the quasi-attractors are hyperbolic and finite.

II- Weak hyperbolicity of the quasi-attractors

One uses:

Theorem 1
Any generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is partially hyperbolic.*

More precizely:

- Each aperiodic class K has a partially hyperbolic structure $T_K M = E^s \oplus E^c \oplus E^u$ with $\dim(E^c) = 1$.
  The dynamics in the central is neutral.

- Each homoclinic class $H(p)$ has a partially hyperbolic structure $T_{H(p)} M = E^{cs} \oplus E^{cu} = (E^s \oplus E_1^c) \oplus (E_2^c \oplus E^u)$ with $\dim(E_i^c) = 0$ or 1.
  The stable dimension of p coincides with $\dim(E^{cs})$.

II- Weak hyperbolicity of the quasi-attractors

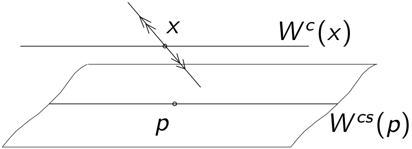
Corollary

For a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$, *each quasi-attractor is a homoclinic class* $H(p)$.

Proof. Consider

- an aperiodic class and $x \in K$ point x in an aperiodic class K,
- a periodic point p close to x.

Then, $W^{uu}(x)$ meets the center-stable plaque of p.



Since each quasi-attractors contain its strong unstable manifolds, K is not a quasi-attractor. □

II- Weak hyperbolicity of the quasi-attractors

Corollary

For a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$, *each non-hyperbolic quasi-attractor* $H(p)$ *is a partially hyperbolic:*

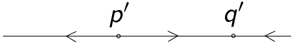
$$T_{H(p)} = E^s \oplus E^c \oplus E^u \ \textit{with} \ \dim(E^c) = 1.$$

E^c *is "center-stable": the stable dimension of* p *is* $\dim(E^s \oplus E^c)$.

Proof. Consider $H(p)$ with a "center-unstable" bundle E^c.

- There exists periodic $p' \in H(p)$ with short unstable manifolds.

$W^c(p')$



- Since $H(p)$ is a quasi-attractor, it contains q'.
- p' and q' have different stable dimension. By perturbation, one gets a heterodimensional cycle between p' and q'. □

III- Finiteness of the quasi-attractors

Corollary

For a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$, *the union of the non-trivial quasi-attractors is closed.*

Proof. Consider a collection of non-trivial quasi-attractors:

$$A_n \underset{Hausdroff}{\longrightarrow} \Lambda.$$

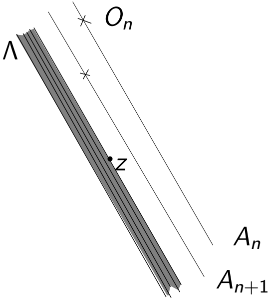
Then, Λ has a partially hyperbolic structure.

- The A_n are saturated by their strong unstable manifolds
  $\Rightarrow \Lambda$ is saturated by the invariant manifolds tangent to E^u
  $\Rightarrow \Lambda$ is a non-trivial homoclinic class $H(p)$.
- If the unstable dimension of p equals $\dim(E^u)$, then, $H(p)$ contains $W^u(p) \Rightarrow H(p)$ is a quasi attractor (we are done).
  Otherwise $\Lambda \subset H(p)$ *has a partially hyperbolic structure* $E^{cs} \oplus E^c \oplus E^u$ *and* E^c *is center-unstable.*

III- Finiteness of the quasi-attractors

Consider a sequence of quasi-attractors $A_n \longrightarrow \Lambda \subset H(p)$ *and a splitting* $T_{H(p)}M = E^{cs} \oplus E^c \oplus E^u$ *with* E^c *center-unstable.*

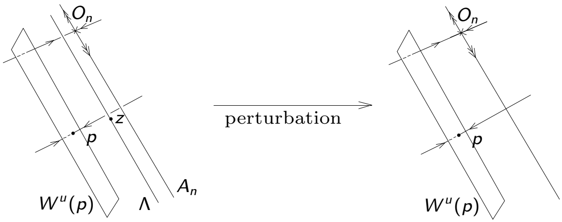
- Consider $z \in \Lambda$. By expansivity, each A_n contains a periodic orbit O_n which avoids a neighborhood of z.
- For the A_n, E^c is center-stable. Otherwise Λ is saturated by plaques tangent to $E^c \oplus E^u$. One concludes as before.
- Consequently, O_n has a point whose stable manifold tangent to E^{cs} is uniform. $\Rightarrow$ Robustly $W^u(p)$ intersects $W^s(O_n)$.



III- Finiteness of the quasi-attractors

Conclusion.

- Since the A_n converge towards Λ, the unstable manifold of O_n meets the neighborhoods of z.
- The stable manifold of p meets the neighborhoods of $z \in H(p)$.
- The connecting lemma allows to create a connection between $W^u(O_n)$ and $W^s(p)$.
- The connection between $W^s(O_n)$ and $W^u(p)$ is preserved.
  $\Rightarrow$ One gets a heterodimensional cycle by perturbation.



III- Finiteness of the quasi-attractors

Proposition

For a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$, *the number of sinks is finite.*

Proof. Consider a sequence of sinks $O_n \underset{Hausdorff}{\longrightarrow} \Lambda$.

- Λ is contained in a chain-recurrence class.
- By Theorem 1, it is partially hyperbolic.
- By Theorem 2, E^u is non trivial. □

Proof of the essential hyperbolicity

Consider a generic $f \in \text{Diff}^1(M) \setminus \overline{\text{Tangency} \cup \text{Cycle}}$.

- The union of the basin of the quasi-attractors is dense (residual) in M.
- From theorem 1:
  - the quasi-attractors are homoclinic classes;
  - their central bundle (when it exists) has dimension 1 and is center-stable;
  - there are only finitely many non-trivial quasi-attractors.
- From theorems 1 and 2, there are only finitely many sinks.

$\Rightarrow$ **one has obtained the essential topological hyperbolicity.**

Partial hyperbolicity far from homoclinic bifurcations

Partial hyperbolicity far from homoclinic bifurcations

Conjecture (Palis)
Any generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is hyperbolic.*

Theorem 1
Any generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is partially hyperbolic.*

More precizely, each (chain-)transitive invariant compact set K of f has a partially hyperbolic structure of one of the following types:

- $T_K M = E^s \oplus_< E^u$,
- $T_K M = E^s \oplus_< E^c \oplus_< E^u$ with $\dim(E^c) = 1$,
- $T_K M = E^s \oplus_< E^c_1 \oplus_< E^c_2 \oplus_< E^u$ with $\dim(E^c_1), \dim(E^c_2) = 1$.

($\oplus_<$ means that the sum is *dominated*.)

Program of the lectures

Goal. *Any generic* $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is essentially hyperbolic.*

Part 1. Topological hyperbolicity
Obtain the existence of a finite number of "attractors" that are "topologically hyperbolic" and have dense basin.

- *Lecture 1.* How Theorems 1 & 2 are used to prove the essential topological hyperbolicity?
- *Lecture 2.* Theorem 1 (partial hyperbolicity).
- *Lecture 3.* Theorem 2 (extremal bundles).

Part 2. From topological to uniform hyperbolicity

- *Lectures 4,5,6.*

How to use "far from heterodimensional cycles"?

In the last lecture, we have seen:

- The non-trivial dynamics splits into the (disjoint, compact, invariant) chain-recurrence classes.
- Generically, any chain-recurrence class that contains a hyperbolic periodic point is a homoclinic class

$$H(p) = \overline{W^s(O(p)) \pitchfork W^u(O(p))}.$$

(= closure of the hyperbolic periodic orbits O homoclincally related to p: $W^s(O) \pitchfork W^u(p)$ and $W^s(p) \pitchfork W^u(O)$ are $\neq \emptyset$.)

Proposition

For a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Cycle}}$, all the periodic points in a same homoclinic class have the same stable dimension.

How to use "far from homoclinic tangencies"?

Theorem (Wen)

Consider $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency}}$ *and a sequence of hyperbolic periodic orbits* (O_n) *with the same stable dimension* d_s*.*
Then $\Lambda = \overline{\cup_n O_n}$ *has a splitting* $T_\Lambda M = E \oplus_< F$ *with* $\dim(E) = d_s$*.*

- This allows to build dominated splittings.

Corollary (Wen)

If the O_n *have a weak Lyapunov exponent (i.e.* ~ 0*), there is a corresponding splitting* $T_\Lambda M = E' \oplus_< E^c \oplus_< F'$ *with* $\dim(E^c) = 1$*.*

- A periodic orbit has at most one weak exponent.

Decomposition of non-uniform bundles

Consider a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency}}$ and an invariant compact set Λ with a splitting $T_\Lambda M = E \oplus_< F$.

Proposition

If E is not uniformly contracted then one of the following holds:

- $\Lambda \subset H(p)$ *for some periodic* p *with* $\dim(E^s(p)) < \dim(E)$.
- $\Lambda \subset H(p)$ *for some periodic* p *with* $\dim(E^s(p)) = \dim(E)$. $H(p)$ *contains periodic orbits with a weak stable exponent.*
- Λ *contains* K *partially hyperbolic:* $T_K M = E^s \oplus_< E^c \oplus_< E^u$, *with* $\dim(E^c) = 1$, $\dim(E^s) < \dim(E)$. *Any measure on* K *has a zero Lyapunov exponent along* E^c.

- In the two first cases, the bundle E splits $E = E' \oplus_< E^c$.

Decomposition of non-uniform bundles: proof.

Consider a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency}}$,
an invariant compact set Λ with a splitting $T_\Lambda M = E \oplus_< F$.
Assume that E is not uniformly contracted.

- There exists an ergodic measure μ with a non-negative Lyapunov exponent along E.
- Mañé's ergodic closing lemma $\Rightarrow$ μ is the limit of periodic orbits O_n with Lyapunov exponents close to those of μ.
- If μ is hyperbolic, the O_n are homoclinically related $\Rightarrow$ case 1.
- Otherwise μ has an exponent equal to zero. Let $K = \mathrm{Supp}(\mu)$. One has $T_K M = E' \oplus_< E^c \oplus_< F'$.
- Taking $\dim(E')$ minimal, the central exponent of any measure supported on K is ≤ 0.
- Taking K minimal for the inclusion, if some measure has a negative central exponent, Liao's selecting lemma $\Rightarrow$ case 2.
- Otherwise, all the central exponents are zero $\Rightarrow$ case 3.

Wen's local result

Any non-hyperbolic diffeomorphism has a non-hyperbolic chain-transitive set which is minimal for the inclusion.

Corollary (Wen)

For a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$, any minimally non-hyperbolic (chain-)transitive set Λ is partially hyperbolic.

Proof. Consider the finest splitting $T_\Lambda M = E_1 \oplus_< E_2 \oplus_< \cdots \oplus_< E_s$ and E_i is not uniformly contracted nor expanded.

- If Λ contains K partially hyperbolic, $\Lambda = K$ by minimality.
- Otherwise Λ is contained in a homoclinic class $H(p)$.
- Far from heterodimentional cycles $\Rightarrow$ all the periodic points in $H(p)$ have the same stable dimension d_s.
- If $\dim(E_1 \oplus \cdots \oplus E_i) \leq d_s$, then $\dim(E_i) = 1$ and $\dim(E_1 \oplus \cdots \oplus E_i) = d_s$.
- Otherwise $\dim(E_i) = 1$ and $\dim(E_1 \oplus \cdots \oplus E_{i-1}) = d_s$. □

From local to global: principle

Consider

- a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency}}$,
- a chain-recurrence class Λ with a splitting $E \oplus F$.

1. If E is not uniformly contracted,
   - either it splits as $E = E' \oplus_< E^c$,
   - or Λ contains K with $T_K M = E^s \oplus_< E^c \oplus_< E^u$, $\dim(E^c) = 1$ and $\dim(E^s) < \dim(E)$.

   In the second case,
   - One looks for periodic orbits that shadows Λ and spends most of its time close to K.
   - The splitting on K extends on Λ as $T_K M = E' \oplus_< E^c \oplus_< F$.
2. One repeats step 1 with the bundle E'.
3. One argues similarly with F.

(Topological) dynamics in the central direction

In order to go from local to global: one has to consider,

- a transitive set K,
- with a splitting $T_K M = E^s \oplus_< E^c \oplus_< E^u$, $\dim(E^c) = 1$.

The dynamics in the central direction can be lifted.

Proposition

There exists a local continuous dynamics $(K \times \mathbb{R}, h)$ and a projection $\pi \colon K \times \mathbb{R} \to M$ such that

- *$(K \times \mathbb{R}, h)$ is a skew product above (K, f),*
- *π semi-conjugates h to f and sends $K \times \{0\}$ on K,*
- *π sends the $\{x\} \times \mathbb{R}$ on a familly of central plaques.*

$(K \times \mathbb{R}, h)$ is called a *central model* for the central dynamics on K. It is in general not unique.

Classification of the dynamics in the central direction

Let $(K \times \mathbb{R}, h)$ be a central model. One of the following holds.

- Hyperbolic type: the chain-stable set of $K \times \{0\}$ contains small attracting neighborhoods.

- Neutral type: there are small attracting and small repelling neighborhoods of $K \times \{0\}$.

- Parabolic type: one side has small attracting neigborhoods, the other one has small repelling neighborhoods.
- Recurrent type: the intersection of the chain-stable and chain-unstable sets contains a segment $\{x\} \times [0, \varepsilon]$.

The type does not depend on the choice of a central model.

From local to global: one easy example

Consider a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency}}$ and

- K transitive with $T_K M = E^s \oplus_< E^c \oplus_< E^u$, $\dim(E^c) = 1$, s.t. any measure on K has central exponent equal to zero,
- Λ the chain-recurrence class containing K.

Proposition

If K has hyperbolic type, then Λ satisfies $T_\Lambda M = E \oplus_< E^c \oplus_< F$. It is a homoclinic class $H(p)$. The stable dimension of p is $\dim(E)$.

Proof. Assume K with hyperbolic repelling type.

- There are periodic orbits $O_n \underset{Hausdorff}{\longrightarrow} K$, with stable dimension $d_s = \dim(E^s)$ and homoclinically related.
- $\Lambda = H(O_n)$ for each n. There is a splitting $T_\Lambda M = E \oplus_< F_0$ with $\dim(E) = d_s$.
- The central exponents of O_n is weak $\Rightarrow$ $H(O_n)$ contains a dense set of weak periodic orbits. Hence $F_0 = E^c \oplus_< F$. □

Central dynamics: the different cases

Consider a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency}}$,
a chain-recurrence class Λ and a minimal set $K \subset \Lambda$ s.t.:

- $T_K M = E^s \oplus_< E^c \oplus_< E^u$, $\dim(E^c) = 1$,
- all the measure on K have a zero central Lyapunov exponent.

The central type of K is hyperbolic, recurrent, parabolic untwisted
$\Rightarrow$ Λ is a homoclinic class.
It contains periodic orbits whose central exponent is weak.

The central type of K is parabolic twisted
$\Rightarrow$ one can create a heterodimensional cycle by perturbation.

The central type of K is neutral and $K \subsetneq \Lambda$
$\Rightarrow$ one creates a cycle or Λ is a homoclinic class as before.

The central type is neutral and $K = \Lambda$
$\Rightarrow$ the class is aperiodic.

Proof of theorem 1

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Chain-hyperbolic classes

Consider an invariant compact set Λ with a dominated splitting $T_\Lambda M = E \oplus F$ such that.

Hyperbolicity of the extremal bundles

Dynamics far from homoclinic bifurcations

Consider a generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$.

Theorem 1

Any non-hyperbolic chain-recurrence class K is partially hyperbolic:

$$T_K M = E^s \oplus_< E^c \oplus_< E^u \text{ or } E^s \oplus_< E_1^c \oplus_< E_2^c \oplus_< E^u,$$

where E^c, E_1^c, E_2^c are one-dimensional bundles.

Theorem 2

The cases $E^s \oplus_< E^c$ and $E^s \oplus_< E_1^c \oplus_< E_2^c$ don't appear.

Corollary

f has only finitely many sinks.

Setting

Consider

- $f \in \mathrm{Diff}^1(M)$,
- Λ: an invariant compact set,
- $T_\Lambda M = E \oplus_< F$: a dominated splitting with $\dim(F) = 1$.

Under general assumptions we expect that

F is uniformly expanded unless Λ contains a sink.

Motivation: the 1D case

Theorem (Mañé)

Consider

- f: *a* C^2 *endomorphism of the circle,*
- Λ: *an invariant compact set.*

Assume furthermore that

- $f_{|\Lambda}$ *is not topologically conjugated to an irrational rotation,*
- *all the periodic points of* f *in* Λ *are hyperbolic.*

Then $Df_{|\Lambda}$ *is uniformly expanding unless* Λ *contains a sink.*

The surface case

Theorem (Pujals-Sambarino)

Consider

- *f: a C^2 surface diffeomorphism,*
- *Λ: an invariant compact set with a dominated splitting $T_\Lambda M = E \oplus_< F$, $\dim(F) = 1$.*

Assume furthermore that

- *Λ does not contain irrational curves,*
- *all the periodic points of f in Λ are hyperbolic.*

Then F is uniformly expanding unless Λ contains a sink.

Irrational curve: a simple closed curve γ, invariant by an iterate f^n such that $f^n_{|\gamma}$ is topologically conjugated to an irrational rotation.

The surface generic case

Corollary

Consider

- f*: a* C^1*-generic surface diffeomorphism,*
- Λ*: an invariant compact set with a dominated splitting* $T_\Lambda M = E \oplus_< F$, $\dim(F) = 1$.

Then Λ *is a hyperbolic set or contains a sink/source.*

The one-codimensional uniform bundle case

Theorem (Pujals-Sambarino)

Consider $f \in \mathrm{Diff}^2(M)$ *and* $H(p)$ *a homoclinic class such that:*

- $T_{H(p)}M = E^s \oplus_< F$*: a dominated splitting with* $\dim(F) = 1$,
- E^s *is uniformly contracted,*
- *all the periodic orbits in* $H(p)$ *are hyperbolic saddles,*
- $H(p)$ *does not contain irrational curves.*

Then, F *is uniformly expanded.*

Corollary

Consider $f \in \mathrm{Diff}^1(M)$ *generic and* $H(p)$, *invariant compact set s.t.:*

- $T_{H(p)}M = E^s \oplus_< F$*: a dominated splitting with* $\dim(F) = 1$,
- E^s *is uniformly contracted,*
- $H(p)$ *does not contain sinks.*

Then $H(p)$ *is a hyperbolic set.*

How to replace the uniform contraction on E?

Consider Λ with a splitting $T_\Lambda M = E \oplus F$.

By Hirsch-Pugh-Shub, there exists a *locally invariant plaque familly tangent to E*,
i.e. a continuous collection of C^1-plaques $(\mathcal{D}_x)_{x\in\Lambda}$ such that

- $\mathcal{D}_x$ is tangent to E_x at x,
- $f(\mathcal{D}_x)$ contains a uniform neighborhood of $f(x)$ in $\mathcal{D}_{f(x)}$.

The plaques are *trapped* if for each x, $\overline{f(\mathcal{D}_x)}$ is contained in the open plaque $\mathcal{D}_{f(x)}$.

- In this case, the plaques are essentially unique.

The bundle E is *thin trapped* if there exists trapped plaque families with arbitrarily small diameter.

The one-codimensional non-uniform bundle case

Theorem

Consider $f \in \mathrm{Diff}^2(M)$ and Λ a chain-recurrence class such that:

- $T_\Lambda M = E \oplus_< F$*: a dominated splitting with* $\dim(F) = 1$,
- E *is thin trapped,*
- Λ *is totally disconnected in the center-stable plaques,*
- *all the periodic orbits in* Λ *are hyperbolic saddles,*
- Λ *does not contain irrational curves.*

Then, F *is uniformly expanded.*

Summary of the different cases

If Λ has a dominated splitting $T_\Lambda M = E \oplus_< F$ with $\dim(F) = 1$, and if E satisfies one of these properties :

- $\dim(E) = 1$,
- E is uniformly contracted,
- E is thin trapped + Λ is totally disconnected along the plaques tangent to E.

then, F is uniformly contracted or Λ contains a sink.

Strategy

$f \in \mathrm{Diff}^2(M)$ and Λ with a splitting $E \oplus_< F$, $\dim(F) = 1$.
Λ does not contain irrational curves nor non-saddle periodic points.

Assuming that any proper invariant compact set $\Lambda' \subsetneq \Lambda$ is hyperbolic, we have to prove that Λ is hyperbolic.

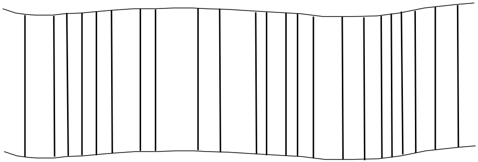
- Step 1: topological hyperbolicity. (Pujals-Sambarino)
  Each point $x \in \Lambda$ has a well defined one-dimensional unstable manifold $W^u(x)$ which is (topologically) contracted by f^{-1}.
- Step 2: existence of a markov box B. (Specific in each case)
- Step 3: uniform expansion along F. (Pujals-Sambarino)
  Obtained by inducing in B.

Markov boxes

Step 1 $\Rightarrow \exists$ thin trapped plaque families $\mathcal{D}^s, \mathcal{D}^u$ tangent to E, F.

A box B is a union of curves (J_x) that are

- contained in the plaques $\mathcal{D}^u$,
- bounded by two plaques of $\mathcal{D}^s$.



We assume furthermore that

- B has interior $\mathring{B}$ in Λ. $\triangleright$ allows to induce
- B is Markovian: for each $z \in \mathring{B} \cap f^{-n}(\mathring{B})$, one has
  - $f^n(J_z) \supset J_{f^n(z)}$. $\triangleright$ B sees the expansion along F
  - z is contained in a sub-box $B' \subset B$ that meets all the curves J_x and $f^n(B')$ is a union of curves of B.

    $\triangleright$ quotient the dynamics along center-unstable plaques

Construction of Markov boxes

E, F are thin trapped $+$ Λ transitive
$\Rightarrow$ there exists a periodic orbit O that shadows Λ.

Consider the one-codimensional plaques $\mathcal{D}^s_y$ for $y \in O$.
B is the region bounded by two such "consecutive" plaques.

- B is Markovian along the center-unstable curves.

E thin trapped $+$ Λ totally disconnected along the center-stable
$\Rightarrow$ one can choose open trapped plaques $\mathcal{D}^s$ such that:

- for each x, $\Lambda \cap \mathcal{D}^s_x$ is a compact subset Δ_x of $\mathcal{D}^s_x$,
- for each x, y, the sets Δ_x, Δ_y coincide or are disjoint.

- B is Markovian along the center-stable plaques.

How to get disconnectedness?

$H(p)$: a homoclinic class for a generic $f \in \mathrm{Diff}^1 \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$.
Goal: rule out the splitting $T_{H(p)}M = E^s \oplus_< E^c_1 \oplus_< E^c_2$.

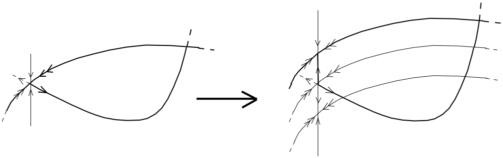
$H(p)$ contains q periodic with weak (stable) exponent along E^c_1.

Lemma

If q has a strong homoclinic intersection:

$$W^u(O(q)) \cap W^{ss}(O(q)) \neq \emptyset,$$

then, one can create a heterodimensional cycle by perturbation.



- For any $q \in H(p)$ periodic, one has $W^{ss}(q) \cap H(p) = \{q\}$.

A geometrical result on partially hyperbolic sets

Let $H(p)$ be a homoclinic class with a splitting

$$T_{H(p)}M = E^{cs} \oplus_< E^{cu} = (E^s \oplus_< E^c_1) \oplus_< E^c_2,$$

such that E^{cs}, E^{cu} are thin trapped for f, f^{-1} respectively.

Theorem (Pujals, C-)

If for any $q \in H(p)$ periodic, one has $W^{ss}(q) \cap H(p) = \{q\}$, then

- *either $H(p)$ is contained in an invariant submanifold tangent to $E^c_1 \oplus E^c_2$,*
- *or $H(p)$ is totally disconnected along the center-stable plaques.*

Codimensional dynamics

We use:

Theorem (Bonatti, C-)

Consider Λ with a splitting $E^s \oplus_< F$. Then,

- *either Λ is contained in an invariant submanifold tangent to F,*
- *or there exists $x \in \Lambda$ such that $W^{ss}(x) \cap \Lambda \setminus \{x\}$ is non-empty.*

- In our case, x is not periodic.

Program of the lectures

Goal. *Any generic* $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathrm{Tangency} \cup \mathrm{Cycle}}$ *is essentially hyperbolic.*

Part 1. Topological hyperbolicity

Obtain the existence of a finite number of "attractors" that are "topologically hyperbolic" and have dense basin.

- *Lecture 1.* How Theorems 1 & 2 are used to prove the essential topological hyperbolicity?
- *Lecture 2.* Theorem 1 (partial hyperbolicity).
- *Lecture 3.* Theorem 2 (extremal bundles).

Part 2. From topological to uniform hyperbolicity

- *Lectures 4,5,6.*

Uniform hyperbolicity of quasi-attractors

We need another result on the geometry of partially hyp. sets.

Theorem (Pujals,C-)

Consider $H(p)$ with $T_{H(p)}M = E^s \oplus_< E^c \oplus_< E^u$, $dim(E^s) = 1$ s.t.

- *$E^{cs} = E^s \oplus E^c$ is thin trapped,*
- *for each $x \in H(p)$, one has $W^u(x) \subset H(p)$.*

Then, there exists $g \in \text{Diff}^1(M)$ close to f such that

a) *either for any $x \in H(p_g)$ one has $W^{ss}(x) \cap H(p_g) = \{x\}$,*

b) *or there exists $q \in H(p_g)$ periodic with a strong connection.*

In case a), for f generic, $H(p)$ is contained in an invariant submanifold tangent to $E^c \oplus E^u \Rightarrow H(p)$ is hyperbolic.

In case b), if E^c is not uniformly contracted, one can create a heterodimensional cycle.