LORENZ-LIKE ATTRACTORS

M. J. PACIFICO, S. GALATOLO

Abstract. In these notes we recall the construction of a class of flows called geometric Lorenz flows, we then analyze some of its statistical properties. We show that the Poincaré map associated to a geometric Lorenz flow has exponential decay of correlations with respect to Lipschitz observables. This implies that the hitting time associated to the flow satisfies a logarithm law. The hitting time $\tau_r(x, x_0)$ is the time needed for the orbit of a point x to enter for the first time in a ball $B_r(x_0)$ centered at x_0, with small radius r. As the radius of the ball decreases to 0 its asymptotic behavior is a power law whose exponent is related to the local dimension of the SRB measure at x_0: for each x_0 such that the local dimension $d_\mu(x_0)$ exists,

$$\lim_{r\to 0} \frac{\log \tau_r(x, x_0)}{-\log r} = d_\mu(x_0) - 1$$

holds for μ almost each x. The results described here are a particular case of the results in *Lorenz like flows: exponential decay of correlations for the Poincaré map, logarithm law, quantitative recurrence*, by S.Galatolo and M.J.Pacifico, [5], that deals with a class of flows defined axiomatically which contains the geometric Lorenz model.

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1. Introduction

It is well known that in a chaotic dynamics the pointwise, future behavior of an initial condition is unpredictable and even impossible to be described by using a finite quantity of information. On the other hand many of its statistical properties are rather regular and often described by suitable versions of classical theorems from probability theory: law of large numbers, central limit theorem, large deviations estimations, correlation decay, hitting times, various kind of quantitative recurrence and so on.

In order to anounce the results we shall discuss, recall that a geometric Lorenz flow has a global cross section Σ and a first return map $F : \Sigma \setminus \Gamma \to \Sigma$ associated, where Γ is a curve. In Section 5 we describe this model in detail.

The main results we shall prove here are:

First, folowing Viana [22], in Section 6 we prove that F has a unique SRB measure as well the geometric Lorenz flow. Then, in Section 7 we prove

Theorem A (decay of correlation for the Poincaré map) *Let F be the first return map associated to a geometrical Lorenz flow. The unique SRB measure μ_F of F has exponential decay of correlation with respect to Lipschitz observables.*

To announce the last result, recall that the *local dimension* of a measure μ at $x \in M$ is defined by

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r}$$

if this limit exists. In this case $\mu(B_r(x)) \sim r^{d_\mu(x)}$.

The hitting time $\tau_r(x, x_0)$ is the time needed for the orbit of a point x to enter for the first time in a ball $B_r(x_0)$ centered at x_0, with small radius r.

Using the above theorem and a result by Galatolo, [4], we then prove in Section 9

Theorem B (logarithm law for the flow) *For each regular x_0 such that the local dimension $d_{\mu_X}(x_0)$ is defined it holds*

$$\lim_{r \to 0} \frac{\log \tau_r(x, x_0)}{-\log r} = d_{\mu_X}(x_0) - 1 \tag{1}$$

for a.e. starting point x.

The results described here are a particular case of the results in *Lorenz like flows: exponential decay of correlations for the Poincaré map, logarithm law, quantitative recurrence*, by S.Galatolo and M.J.Pacifico, [5], that deals with a class of flows defined axiomatically which contains the geometric Lorenz model.

2. Fixing the notation and some preliminary results

Let us start fixing the notation and introduce definitions and results proved elsewhere.

Let M be a compact finite dimensional boundaryless manifold of dimension 3 and study the dynamics of the flow associated to a given smooth vector field X on M from the topological and measure-theoretic or ergodic point-of-view.

We fix on M some Riemannian metric which induces a distance dist on M and naturally defines an associated Riemannian volume form Leb which we call *Lebesgue measure* or simply *volume*, and always take Leb to be normalized: $\mathrm{Leb}(M)=1$.

We always assume that a C^r vector field X on M is given, $r \geqslant 1$, and consider the associated global *flow* $(X^t)_{t\in\mathbb{R}}$ (since X is defined on the whole of M, which is compact, X is bounded and X^t is defined for every $t\in\mathbb{R}$.) Recall that the flow $(X^t)_{t\in\mathbb{R}}$ is a family of C^r diffeomorphisms satisfying the following properties:

(1) $X^0 = Id : M \to M$ is the identity map of M;
(2) $X^{t+s} = X^t \circ X^s$ for all $t,s\in\mathbb{R}$,

and it is *generated by the vector field* X if

(3) $\frac{d}{dt}X^t(q)\big|_{t=t_0} = X\big(X_{t_0}(q)\big)$ for all $q\in M$ and $t_0\in\mathbb{R}$.

Note that reciprocally a given flow $(X^t)_{t\in\mathbb{R}}$ determines a unique vector field X whose associated flow is precisely $(X^t)_{t\in\mathbb{R}}$.

In what follows we denote by $\mathfrak{X}^r(M)$ the vector space of all C^r vector fields on M endowed with the C^r topology and by $\mathcal{F}^r(M)$ the space of all flows on M also with the C^r topology. Many times we usually denote

Given $X\in\mathfrak{X}^r(M)$ and $q\in M$, an orbit segment $\{X^t(q); a\le t\le b\}$ is denoted by $X^{[a,b]}(q)$. We denote by DX^t *the derivative of* X^t *with respect to the ambient variable* q *and when convenient we set* $D_qX^t = DX^t(q)$. Analogously, DX *is the derivative of the vector field* X with respect to the ambient variable q, and when convenient we write D_qX for the derivative DX at q, $DY(q)$.

An *equilibrium* or *singularity* for X is a point $\sigma\in M$ such that $X^t(\sigma)=\sigma$ for all $t\in\mathbb{R}$, i.e. a fixed point of all the flow maps, which corresponds to a zero of the associated vector field X: $X(\sigma)=0$. We denote by $S(X)$ the set of singularities (zeroes) of the vector field X. Every point $p\in M\setminus S(X)$, that is p satisfies $X(p)\neq 0$, is a *regular* point for X.

An *orbit* of X is a set $\mathcal{O}(q)=\mathcal{O}_X(q)=\{X^t(q): t\in\mathbb{R}\}$ for some $q\in M$. Hence $\sigma\in M$ is a singularity of X if, and only if, $\mathcal{O}_X(\sigma)=\{\sigma\}$. A *periodic orbit* of X is an orbit $\mathcal{O}=\mathcal{O}_X(p)$ such that $X^T(p)=p$ for some minimal $T>0$ (equivalently $\mathcal{O}_X(p)$ is compact and $\mathcal{O}_X(p)\neq\{p\}$). We denote by $\mathrm{Per}(X)$ the set of all periodic orbits of X.

A *critical element* of a given vector field X is either a singularity or a periodic orbit. The set $C(X)=S(X)\cup\mathrm{Per}(X)$ is the set of *critical elements* of X.

We say that $p\in M$ is *non-wandering* for X if for every $T>0$ and every neighborhood U of p there is $t>T$ such that $X^t(U)\cap U\neq\emptyset$. The set of non-wandering points of X is denoted by $\Omega(X)$. If $q\in M$, we define $\omega_X(q)$ as the set of accumulation points of the positive orbit $\{X^t(q): t\ge 0\}$ of q. We also define $\alpha_X(q)=\omega_{-X}$, where $-X$ is the time reversed vector field X, corresponding to the set of accumulation points of the negative orbit of q. It is immediate that $\omega_X(q)\cup\alpha_X(q)\subset\Omega(X)$ for every $q\in M$.

A subset Λ of M is *invariant* for X (or X-invariant) if $X^t(\Lambda)=\Lambda$, $\forall t\in\mathbb{R}$. We note that $\omega_X(q)$, $\alpha_X(q)$ and $\Omega(X)$ are X-invariant. For every compact invariant set Λ of X

we define the *stable set* of Λ

$$W^s_X(\Lambda)=\{q\in M:\omega_X(q)\subset\Lambda\},$$

and also its *unstable set*

$$W^u_X(\Lambda)=\{q\in M:\alpha_X(q)\subset\Lambda\}.$$

A compact invariant set Λ is *transitive* if $\Lambda=\omega_X(q)$ for some $q\in\Lambda$, and *attracting* if $\Lambda=\cap_{t\geq 0}X^t(U)$ for some neighborhood U of Λ satisfying $X^t(U)\subset U$, $\forall t>0$. An *attractor* of X is a transitive attracting set of X and a *repeller* is an attractor for $-X$. We say that Λ is a *proper* attractor or repeller if $\emptyset\neq\Lambda\neq M$.

A *sink* of X is a singularity of X which is also an attractor of X, it is a trivial attractor of X. A *source* of X is a trivial repeller of X, i.e. a singularity which is a attractor for $-X$.

A *singularity σ is hyperbolic* if the eigenvalues of $DX(\sigma)$, the derivative of the vector field at σ, have a real part different from zero. In particular sinks and sources are hyperbolic singularities, where all the eigenvalues of the former have negative real part and those of the latter have positive real part.

A *periodic orbit $\mathcal{O}_X(p)$ of X is hyperbolic* if the eigenvalues of $DX^T(p):T_pM\to T_pM$, the derivative of the diffeomorphism X^T, where $T>0$ is the period of p, are all different from 1.

3. Hyperbolic flows

Let $X\in\mathfrak{X}^r(M)$ be a flow on a compact manifold M. Denote by $m(T)=\inf_{\|v\|=1}\|T(v)\|$ the *minimum norm* of a linear operator T. A compact invariant set $\Lambda\subset M$ of X is *hyperbolic* if

(1) admits a continuous DX-invariant tangent bundle decomposition $T_\Lambda M=E^s_\Lambda\oplus E^X_\Lambda\oplus E^u_\Lambda$, that is we can write the tangent space TxM as a direct sum $E^s_x\oplus E^X_x\oplus E^u_x$, where E^X_x is the subspace in T_xM generated by $X(x)$, satisfying
   - $DX^t(x)\cdot E^i_x=E^i_{X^t(x)}$ for all $t\in\mathbb{R}$, $x\in\Lambda$ and $i=s,X,u$;

(2) there are constants $\lambda,K>0$ such that
   - E^s_Λ *is (K,λ)-contracting*, i.e. for all $x\in\Lambda$ and every $t\geqslant 0$
   $$\|DX^t(x)\mid E^s_x\|\leq K^{-1}e^{-\lambda t},$$
   - E^u_Λ *is (K,λ)-expanding*, i.e. for all $x\in\Lambda$ and every $t\geqslant 0$
   $$m(DX^t\mid E^u)\geq Ke^{\lambda t},$$

By the Invariant Manifold Theory [9] it follows that for every $p\in\Lambda$ the sets

$$W^{ss}_X(p)=\{q\in M:\operatorname{dist}(X_t(q),X_t(p))\xrightarrow[t\to\infty]{}0\}$$

and

$$W^{uu}_X(p)=\{q\in M:\operatorname{dist}(X_t(q),X_t(p))\xrightarrow[t\to-\infty]{}0\}$$

are invariant C^r-manifolds tangent to E^s_p and E^u_p respectively at p. Here dist is the *distance on M induced by some Riemannian norm.*

If $\mathcal{O}=\mathcal{O}_X(p)\subset\Lambda$ is an orbit of X one has that

$$W^s_X(\mathcal{O})=\cup_{t\in\mathbb{R}}W^{ss}_X(X^t(p))\quad\text{and}\quad W^u_X(\mathcal{O})=\cup_{t\in\mathbb{R}}W^{uu}_X(X^t(p))$$

are invariant C^r-manifolds tangent to $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ at p, respectively. We shall denote $W^s_X(p) = W^s_X(\mathcal{O}_X(p))$ and $W^u_X(p) = W^u_X(\mathcal{O}_X(p))$ for the sake of simplicity.

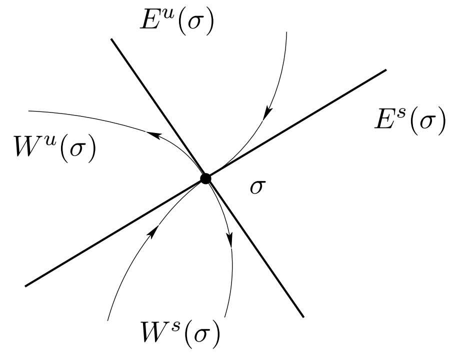

FIGURE 1. A saddle singularity σ for bi-dimensional flow.

A *singularity* (respectively *periodic orbit*) of X is *hyperbolic* if its orbit is a hyperbolic set of X. Note that $W^{ss}_X(\sigma) = W^s_X(\sigma)$ and $W^{uu}_X(\sigma) = W^u_X(\sigma)$ for every hyperbolic singularity σ of X. A sink and a source are both hyperbolic singularities. A *hyperbolic* singularity which is *neither* a sink *nor* a source is called a *saddle*.

A hyperbolic set Λ of X is called *basic* if it is transitive and *isolated*, that is $\Lambda = \cap_{t\in\mathbb{R}}\overline{X^t(U)}$ for some neighborhood U of H. It follows from the Shadowing Lemma [15] that every hyperbolic basic set of X either reduces to a singularity or else has no singularities and it is the closure of its periodic orbits.

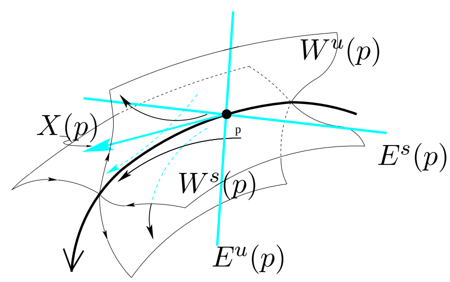

FIGURE 2. The flow near a hyperbolic saddle periodic orbit through p.

We say that X is *Axiom A* if the non-wandering set $\Omega(X)$ is both hyperbolic and the closure of its periodic orbits and singularities. The *Spectral Decomposition Theorem* asserts that if X is Axiom A, then there is a disjoint decomposition $\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_k$, where each Λ_i is a hyperbolic basic set of X, $i = 1, \cdots, k$.

A *cycle* of a Axiom A vector field X is a sub-collection $\{\Lambda_{i_0}, \cdots, \Lambda_{i_k}\}$ of $\{\Lambda_1, \cdots, \Lambda_n\}$ such that $i_0 = i_k$ and $W^u_X(\Lambda_{i_j}) \cap W^s_X(\Lambda_{i_{j+1}}) \neq \emptyset$, $\forall 0 \leq j \leq k-1$.

Hyperbolic sets and singularities. The continuity of the DX-invariant splitting on the tangent space of a uniformly hyperbolic set Λ is a consequence of the uniform expansion and contraction estimates (see e.g. [16]). This means that if $x_n \in \Lambda$ is a sequence of points converging to $x \in \Lambda$, and we consider orthonormal basis $\{e^n_i\}_{i=1,\ldots,\dim E^s(x_n)}$

of $E^s(x_n)$, $\{f_i^n\}_{i=1,\dots,\dim E^u(x_n)}$ of $E^u(x_n)$ and $X(x_n)$ of $E^X(x_n)$, then these vectors converge to a basis of $E^s(x), E^u(x)$ and $E^X(x)$ respectively. In particular the dimension of the subspaces in the hyperbolic splitting is constant if Λ is transitive.

This shows that a uniformly hyperbolic basic set Λ cannot contain singularities, except if Λ is itself a singularity. Indeed, if $\sigma \in \Lambda$ is a singularity then it is hyperbolic but the dimension of the central sub-bundle is zero since the flow is zero at σ. Therefore the dimensions of either the stable or the unstable direction at σ and those of a transitive regular orbit in Λ do not match.

In other words *an invariant subset* Λ *containing a singularity accumulated by regular orbits cannot be uniformly hyperbolic.*

4. Three dimensional chaotic attractors

In 1963 the meteorologist Edward Lorenz published in the Journal of Atmospheric Sciences [11] an example of a parametrized polynomial system of differential equations

$$\begin{aligned} \dot{x} &= a(y-x) & a &= 10 \\ \dot{y} &= rx - y - xz & r &= 28 \\ \dot{z} &= xy - bz & b &= 8/3 \end{aligned} \tag{2}$$

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecast. Later Lorenz [12] together with other experimental researches showed that the equations of motions of a certain laboratory water wheel are given by (2). Hence equations (2) can be deduced directly in order to model a physical phenomenon instead of as an approximation to a partial differential equation.

Numerical simulations for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a stranger attractor, called the *Lorenz attractor*. However Lorenz's equations proved to be very resistant to rigorous mathematical analysis, and also presented very serious difficulties to rigorous numerical study.

A very successful approa ch was taken by Afraimovich, Bykov and Shil'nikov [1], and Guckenheimer, Williams [6], independently: they constructed the so-called *geometric Lorenz models* for the behavior observed by Lorenz. These models are flows in 3-dimensions for which one can rigorously prove the existence of an attractor that contains an equilibrium point of the flow, together with regular solutions. The accumulation of regular orbits near a singularity prevents such sets to be hyperbolic. Moreover, for almost every pair of nearby initial conditions, the corresponding solutions move away from each other exponentially fast as they converge to the attractor, that is, the attractor is *sensitive to initial conditions*: this unpredictability is a characteristic of *chaos*. Most remarkably, this attractor is robust: it can not be destroyed by any small perturbation of the original flow.

Another approach was through rigorous numerical analysis. In this way, it could be proved, by [7, 8, 13, 14], that the equations (2) exhibit a suspended Smale Horseshoe. In particular, they have infinitely many closed solutions, that is, the attractor contains infinitely many periodic orbits. However, proving the existence of a strange attractor as in the geometric models is an even harder task, because one cannot avoid the main numerical difficulty posed by Lorenz's equations, which arises from the very presence

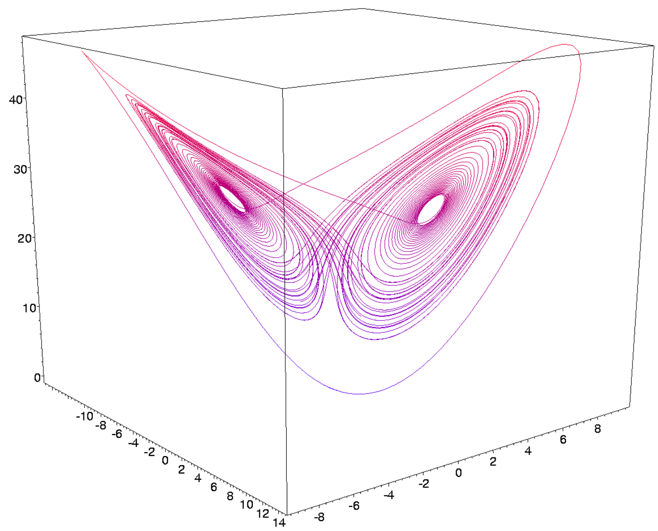

FIGURE 3. Lorenz strange attractor

of an equilibrium point: solutions slow down as they pass near the origin, which means unbounded return times and, thus, unbounded integration errors.

In the year 2000 this was finally settled by Warwick Tucker who gave a mathematical proof of the existence of the Lorenz attractor, see [19, 20, 21]. The algorithm developed by Tucker incorporates two kinds of ingredients: a numerical integrator, used to compute good approximations of trajectories of the flow far from the equilibrium point sitting at the origin, together with quantitative results from normal form theory, that make it possible to handle trajectories close to the origin.

The consequences of the sensitiveness to initial conditions on a (albeit simplified) model of the atmosphere were far-reaching: assuming that the weather behaves according to this model, then long-range weather forecasting is impossible.

For an historical account of the impact of the Lorenz paper [11] on Dynamical Systems and an overview of the proof by Tucker see [23].

5. Geometric Lorenz model

In this section we will construct the so-called Geometric Lorenz system. For this we proceed as follows.

5.1. **Near the equilibrium.** We first analyze the dynamics in a neighborhood of the singularity at the origin, and then we complete the flow, imitating the butterfly shape of the original Lorenz flow (see Figure 3 and compare with Figure 5).

In the original Lorenz system the origin $p = 0 = (0, 0, 0)$ is an equilibrium of saddle type for the vector field defined by equations (2) with real eigenvalues λ_i, $i \leq 3$ satisfying

$$0 < \frac{\lambda_1}{2} \leq -\lambda_3 < \lambda_1 < -\lambda_2 \tag{3}$$

(in the classical Lorenz system $\lambda_1 \approx 11.83$, $\lambda_2 \approx -22.83$, $\lambda_3 = -8/3$).

If certain nonresonance conditions are satisfied (see [18]) this vector field is smoothly linearizable in a neighborhood of the origin. To construct a model which is similar to the original Lorenz one we start with a linear system $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$, with λ_i, $1 \le i \le 3$ satisfying relation (3). This vector field will be considered in the cube $[-1,1]^3$ containing the origin.

For this linear flow, the trajectories are given by

$$X^t(x_0, y_0, z_0) = (x_0 e^{\lambda_1 t}, y_0 e^{\lambda_2 t}, z_0 e^{\lambda_3 t}), \tag{4}$$

where $(x_0, y_0, z_0) \in \mathbb{R}^3$ is an arbitrary initial point near $p = (0,0,0)$.

Consider $\Sigma = \{(x, y, 1) : |x| \le \frac{1}{2}, \quad |y| \le \frac{1}{2}\}$ and

$$\Sigma^- = \{(x,y,1) \in \Sigma : x < 0\}, \qquad \Sigma^+ = \{(x,y,1) \in \Sigma : x > 0\} \quad \text{and}$$
$$\Sigma^* = \Sigma^- \cup \Sigma^+ = \Sigma \setminus \Gamma, \qquad \text{where} \quad \Gamma = \{(x,y,1) \in \Sigma : x = 0\}.$$

Σ is a transverse section to the linear flow and every trajectory crosses Σ in the direction of the negative z axis.

Consider also $\tilde{\Sigma} = \{(x,y,z) : |x| = 1\} = \tilde{\Sigma}^- \cup \tilde{\Sigma}^+$ with $\tilde{\Sigma}^\pm = \{(x,y,z) : x = \pm 1\}$. For each $(x_0, y_0, 1) \in \Sigma^*$ the time t such that $X^t(x_0, y_0, 1) \in \tilde{\Sigma}$ is given by

$$t(x_0) = -\frac{1}{\lambda_1} \log |x_0| \tag{5}$$

which depends on $x_0 \in \Sigma^*$ only and is such that $t(x_0) \to +\infty$ when $x_0 \to 0$.

Hence, using (5), we get (where $\mathrm{sgn}(x) = x/|x|$ for $x \ne 0$)

$$X^{t(x_0)}(x_0, y_0, 1) = \big(\mathrm{sgn}(x_0), y_0 e^{\lambda_2 \cdot t(x_0)}, e^{\lambda_3 \cdot t(x_0)}\big) = \big(\mathrm{sgn}(x_0), y_0 |x_0|^{-\frac{\lambda_2}{\lambda_1}}, |x_0|^{-\frac{\lambda_3}{\lambda_1}}\big).$$

Since $0 < \frac{\lambda_1}{2} < -\lambda_3 < \lambda_1 < -\lambda_2$, we have $\frac{1}{2} < \alpha = -\frac{\lambda_3}{\lambda_1} < 1 < \beta = -\frac{\lambda_2}{\lambda_1}$.

Consider $L : \Sigma^* \to \tilde{\Sigma}^\pm$ defined by

$$L(x, y, 1) = \big(\mathrm{sgn}(x), y|x|^\beta, |x|^\alpha\big). \tag{6}$$

It is easy to see that $L(\Sigma^\pm)$ has the shape of a cusp triangle without the vertex

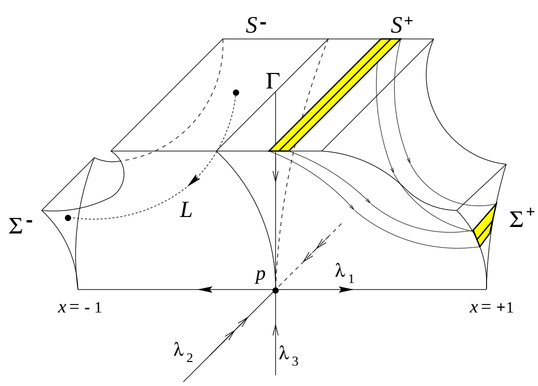

FIGURE 4. Behavior near the origin.

$(\pm 1, 0, 0)$. In fact the vertex $(\pm 1, 0, 0)$ are cusp points at the boundary of each of these sets. The fact that $0 < \alpha < 1 < \beta$ together with equation (6) imply that $L(\Sigma^{\pm})$ are uniformly compressed in the y-direction.

Clearly each segment $\Sigma^* \cap \{x = x_0\}$ is taken by L to another segment $\tilde{\Sigma}^{\pm} \cap \{z = z_0\}$ as sketched in Figure 4.

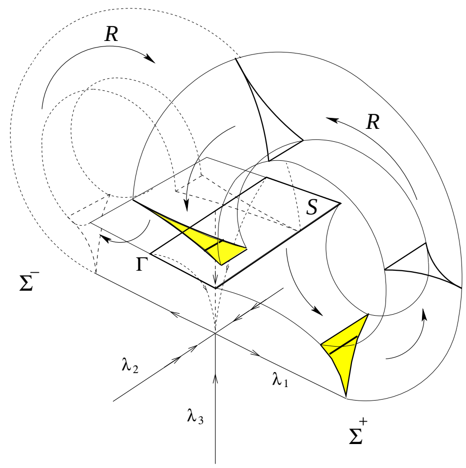

FIGURE 5. $T_{\pm} \circ R_{\pm}$ takes $\tilde{\Sigma}^{\pm}$ to Σ.

5.2. **The random turns around the origin.** To imitate the random turns of a regular orbit around the origin and obtain a butterfly shape for our flow, as it is in the original Lorenz flow depicted at Figure 3, we proceed as follows.

Recall that the equilibrium p at the origin is hyperbolic and so its stable $W^s(p)$ and unstable $W^u(p)$ manifolds are well defined, [16]. Observe that $W^u(p)$ has dimension one and so, it has two branches, $W^{u,\pm}(p)$, and $W^u(p) = W^{u,+}(p) \cup \{p\} \cup W^{u,-}(p)$.

The sets $L(\Sigma^{\pm})$ should return to the cross section Σ through a flow described by a suitable composition of a rotation $R_{\pm}$, an expansion $E_{\pm\theta}$ and a translation $T_{\pm}$.

The rotation $R_{\pm}$ has axis parallel to the y-direction. More precisely is such that $(x, y, z) \in \tilde{\Sigma}^{\pm}$, then

$$R_{\pm}(x, y, z) = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}. \tag{7}$$

The expansion occurs only along the x-direction, so, the matrix of E_θ is given by

$$E_{\pm\theta}(x, y, z) = \begin{pmatrix} \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

with $\theta \cdot (\frac{1}{2}^{\alpha}) < 1$ and $\theta \cdot \alpha \cdot 2^{1-\alpha} > 1$. The first condition is to ensure that the image of the resulting map is contained in Σ, the second condition makes a certain one dimensional induced map to be piecewise expanding. This point will be discussed below.

$T_{\pm} : \mathbb{R}^3 \to \mathbb{R}^3$ is chosen such that the unstable direction starting from the origin is sent to the boundary of Σ and the image of both $\tilde{\Sigma}^{\pm}$ are disjoint. These transformations

$R_\pm, E_{\pm\theta}, T_\pm$ take line segments $\tilde{\Sigma}^\pm \cap \{z = z_0\}$ into line segments $\Sigma \cap \{x = x_1\}$ as sketched in Figure 5, and so does the composition $T_\pm \circ E_{\pm\theta} \circ R_\pm$.

This composition of linear maps describes a vector field in a region outside $[-1,1]^3$ in the sense that one can use the above matrices to define a vector field V such that the time one map of the associated flow realizes $T_\pm \circ E_{\pm\theta} \circ R_\pm$ as a map $L(\Sigma^\pm) \to \Sigma$. This will not be explicit here, since the choice of the vector field is not really important for our purposes (provided the return time is integrable).

The above construction allow to describe for each $t \in \mathbb{R}$ the orbit $X^t(x)$ of each point $x \in \Sigma$: the orbit will start following the linear field until $\tilde{\Sigma}^\pm$ and then it will follow V coming back to Σ and so on. Let us denote with $\mathcal{B} = \{X^t(x), x \in \Sigma, t \in \mathbb{R}^+\}$ the set where this flow acts. The geometric Lorenz flow is then the couple $(\mathcal{B}, X^t)$ defined in this way.

The Poincaré first return map will be hence defined by $F : \Sigma^* \to \Sigma$ as

$$F(x,y) = \begin{cases} T_+ \circ E_{+\theta} \circ R_+ \circ L(x,y,1) & \text{for } x > 0 \\ T_- \circ E_{-\theta} \circ R_- \circ L(x,y,1) & \text{for } x < 0 \end{cases} \tag{9}$$

The combined effects of $T_\pm \circ R_\pm$ and L on lines implies that the foliation $\mathcal{F}^s$ of Σ given by the lines $\Sigma \cap \{x = x_0\}$ is invariant under the return map. In another words, we have

($\star$) *for any given leaf γ of $\mathcal{F}^s$, its image $F(\gamma)$ is contained in a leaf of $\mathcal{F}^s$.*

5.3. **An expression for the first return map and its differential.** Combining equations (6) with the effect of the rotation composed with the expansion and the translation, we obtain that F must have the form

$$F(x,y) = \big(f_{Lo}(x), g_{Lo}(x,y)\big) \tag{10}$$

where $f_{Lo} : I \setminus \{0\} \to I$ and $g_{Lo} : (I \setminus \{0\}) \times I \to I$ are given by

$$f_{Lo}(x) = \begin{cases} f_1(x^\alpha) & x < 0 \\ f_0(x^\alpha) & x > 0 \end{cases} \quad \text{with } f_i = (-1)^i \theta \cdot x + b_i, i \in \{0,1\}, \text{ and} \tag{11}$$

$$g_{Lo}(x,y) = \begin{cases} g_1(x^\alpha, y \cdot x^\beta) & x < 0 \\ g_0(x^\alpha, y \cdot x^\beta) & x > 0, \end{cases} \tag{12}$$

where $g_1|I^- \times I \to I$ and $g_0|I^+ \times I \to I$ are suitable affine maps. Here $I^- = (-1/2, 0)$, $I^+ = (0, 1/2)$.

Now, to find an expression for DF we proceed as follows. Recall $F = T_\pm \circ E_{\pm\theta} \circ R_\pm \circ L$, L is as in (6), $DR_\pm$ is as in (7). Given $q = (x,y) \in \Sigma^*$ with $x > 0$, we have

$$DL(x,y,1) = \begin{pmatrix} \beta \cdot y \cdot x^{\beta-1} & x^\beta \\ \alpha \cdot x^{\alpha-1} & 0 \end{pmatrix}.$$

Restricting the rotation and the other linear maps to $\tilde{\Sigma}^\pm$ and composing the resulting matrices we get

$$DF(x,y) = \begin{pmatrix} \theta \cdot \alpha \cdot x^{(\alpha-1)} & 0 \\ \beta \cdot y x^{(\beta-\alpha)} & x^\beta \end{pmatrix}. \tag{13}$$

The expression for DF at $q = (x,y)$ with $x < 0$ is similar.

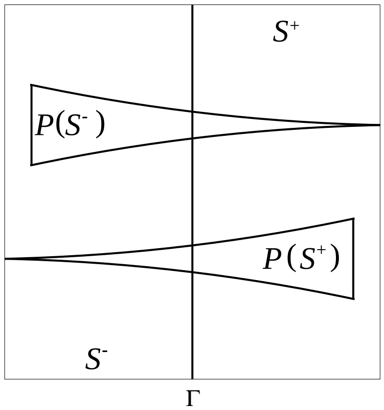

FIGURE 6. $F(\Sigma^*)$.

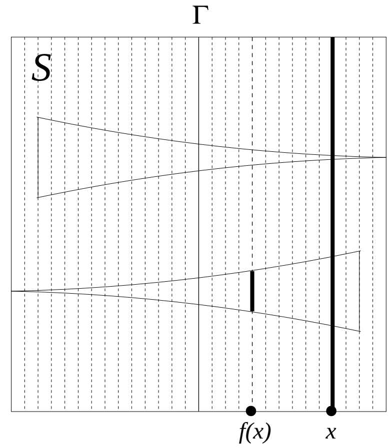

FIGURE 7. Projection on I.

5.4. **Properties of the map g_{Lo}.** Observe that by construction g_{Lo} in equation (9) is piecewise C^2. Moreover, equation (13) implies the following bounds on its partial derivatives :

(a) For all $(x, y) \in \Sigma^*, x > 0$, we have $\partial_y g_{Lo}(x, y) = x^\beta$. As $\beta > 1$, $|x| \leq 1/2$, there is $0 < \lambda < 1$ such that

$$|\partial_y g_{Lo}| < \lambda. \tag{14}$$

The same bound works for $x < 0$.

(b) For all $(x, y) \in \Sigma^*, x \neq 0$, we have $\partial_x g_{Lo}(x, y) = \beta \cdot x^{\beta-\alpha}$. As $\beta - \alpha > 0$ and $|x| \leq 1/2$, we get

$$|\partial_x g_{Lo}| < \infty. \tag{15}$$

Item (a) above implies that the map $F = (f_{Lo}, g_{Lo})$ is uniformly contracting on the leaves of the foliation $\mathcal{F}^s$: there is $C > 0$ such that, if γ is a leaf of $\mathcal{F}^s$ and $x, y \in \gamma$ then $\operatorname{dist}\big(F^n(x), F^n(y)\big) \leq \lambda^n \cdot C \cdot \operatorname{dist}(x, y)$ where λ can be chosen as the one given by equation (14).

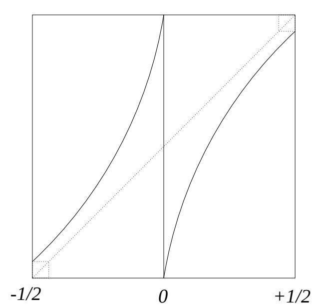

FIGURE 8. The Lorenz map f_{Lo}.

5.5. **Properties of the one-dimensional map f_{Lo}.** Now let us outline the main properties of f_{Lo}. We recall that we chosen θ such that $\theta \cdot \alpha \cdot 2^{1-\alpha} > 1$.

The following properties are easily implied from the construction of X^t:

(f1) By equation (11) and the way $T_\pm$ is defined, f_{Lo} is discontinuous at $x = 0$. The lateral limits $f_{Lo}(0^\pm)$ do exist, $f_{Lo}(0^\pm) = \pm\frac{1}{2}$,

(f2) f_{Lo} is C^2 on $I \setminus \{0\}$. By the choice of θ it holds $f'_{Lo}(1/2) > 1$. By the convexity properties of f_{Lo} we then obtain that

$$f'_{Lo}(x) > 1 \qquad \text{for all} \qquad x \in I \setminus \{0\}. \tag{16}$$

(f3) The limits of f'_{Lo} at $x = 0$ are $\lim_{x\to 0} f'_{Lo}(x) = +\infty$.

We obtain that f_{Lo} is a piecewise expanding map. Moreover f_{Lo} has a dense orbit, which in its turn implies that the closure of the maximal invariant set by f_{Lo} is the whole interval I.

Now recall that the *variation* $\operatorname{var}\phi$ of a function $\phi : [0,1] \to \mathbb{R}$ is defined by

$$\operatorname{var}\phi = \sup \sum_{i=1}^{n} |\phi(x_{i-1}) - \phi(x_i)|$$

where the supremum is taken over all finite partitions $0 = x_0 < x_1 < \cdots < x_n = 1$, $n \geq 1$, of $[0,1]$. The variation $\operatorname{var}_J \phi = \operatorname{var}(\phi|J)$ of ϕ over an arbitrary interval $J \subset [0,1]$ is defined by a similar expression, with the supremum taken over all the $x_0, x_1, \cdots, x_n \in J$, with $\inf J \leq x_0 < x_1 < \cdots < x_n \leq \sup J$. One says that ϕ has *bounded variation*, or ϕ is BV for short, if $\operatorname{var}\phi < \infty$.

The one dimensional map has the following property, which is important to obtain the existence of an SRB invariant measure and its statistical properties.

Lemma 5.1. *Let X^t a C^2 geometric Lorenz flow as before and f_{Lo} be the one-dimensional map associated to X^t. Then $\frac{1}{f'_{Lo}}$ is BV.*

We have seen that f_{Lo} is a topologically transitive piecewise expanding map with $\frac{1}{f'_{Lo}}$ BV. So, the following result holds:

Proposition 5.2. *([22], Prop.3.8) The one-dimensional f_{Lo} admits a unique invariant probability $\mu_{f_{Lo}}$ which is absolutely continuous with respect to Lebesgue measure m, it is ergodic and so a SRB measure for the map. Moreover $d\mu_{f_{Lo}}/dm$ is a BV function and in particular it is bounded. Furthermore f_{Lo} has exponential decay of correlations for L^1 and BV observables and any a.c.i.m. converges exponentially fast to the invariant measure: there are constants $C > 0$ and $\lambda > 0$, depending on the system such that for each n and observables f, g:*

$$\left| \int g(F^n(x)) f(x) dm - \int g(x) d\mu \int f(x) dm \right| \leq C \cdot \|g\|_{L_1} \cdot \|f\|_{BV} \cdot e^{-\lambda n}.$$

6. A physical measure for a geometric Lorenz flow

In this section, following [22] we construct a physical measure for a geometric Lorenz flow X^t, described in the previous section.

To simplify notation, from now on we denote the one-dimensional Lorenz-like map f_{Lo} by T. As seen before, Proposition 5.2) T admits a unique invariant probability measure μ_T which is absolutely continuous with respect to Lebesgue measure m.

From μ_T we may construct a SRB measure μ_F, for the first return map F through the following general procedure ([3, 22]). Since μ_T is defined on the interval I which can be identified to the space of leaves of the contracting foliation $\mathcal{F}^s$, we may also think of it as a measure on the σ-algebra of Borel subsets of Σ which are union of entire leaves of $\mathcal{F}^s$. Using the fact that F is uniformly contracting on leaves of $\mathcal{F}^s$ we conclude that the sequence

$$F^{*n}(\mu_T), \quad n \geq 1,$$

of push-forwards of μ_T under F is weak*-Cauchy: given any continuous $\psi : \Sigma \to \mathbb{R}$

$$\int \psi d(F^{n*}\mu_T) = \int (\psi \circ F^n) d\mu_T, \quad n \geq 1,$$

is a Cauchy sequence in $\mathbb{R}$, see [22, pp.173]. Define μ_F to be the weak*-limit of this sequence, that is,

$$\int \psi d\mu_F = \lim \int \psi d(F^{*n}\mu)$$

for each continuous ψ. Then μ_F is invariant under F, and it is an ergodic physical measure for F. The last statement follows from the fact that μ_T is an ergodic physical measure for T, together with the fact that asymptotic time-averages of continuous functions $\psi : \Sigma \to \mathbb{R}$ are constant on the leaves of $\mathcal{F}^s$.

Given any point x whose orbit sooner or later will cross Σ we denote with $t(x)$ the first strictly positive time such that $X^{t(x)}(x) \in \Sigma$ (the *return time* of x to Σ). Coherently with the Geometric Lorenz system, we will denote by Σ^* the (full measure, by the assumption 1 in the introduction) subset of Σ where t is defined.

Now we show how to construct an physical invariant measure for the flow, *when the return time is integrable*:

$$\int_{\Sigma^*} t d\mu_F < \infty. \tag{17}$$

Denote by $\sim$ the equivalence relation on $\Sigma \times \mathbb{R}$ given by $(w, t(w)) \sim (F(w), 0)$.

Let $N = (\Sigma^* \times \mathbb{R})/\sim$ and $\nu = \pi_*(\mu_F \times dt)$, where $\pi : \Sigma^* \times \mathbb{R} \to N$ is the quotient map and dt is a Lebesgue measure in $\mathbb{R}$. Equation (18) gives that ν is a finite measure. Let $\phi : N \to \mathbb{R}^3$ be defined by $\phi(w, t) = X^t(w)$. Let $\mu_X = \phi_*\nu$. The measure μ_X is a physical for the flow X^t:

$$\frac{1}{T}\int_0^T \psi(X^t(w))dt \to \int \psi d\mu_X \quad \text{as} \quad T \to \infty$$

for every continuous function $\psi : \mathbb{R}^3 \to \mathbb{R}$, and Lebesgue almost every point $w \in \phi(N)$.

We end the subsection remarking that the Geometric Lorenz flow has integrable return time, hence the above construction for the invariant measure can be applied to it. As before denote by $t : \Sigma \setminus \Gamma \to (0, \infty)$ the return time to Σ. Then, recalling Equation (5) there are $K, C > 0$ such that

$$-K^{-1}\log(d(x, \Gamma)) - C \leq t(x) \leq -K\log(d(x, \Gamma)) + C.$$

Combining this with the definition of μ_F and the remark made above that $d\mu_{f_{Lo}}/dm$ is a bounded function, we conclude that

Proposition 6.1. *The return time is integrable*

$$t_0 = \int t d\mu_F < \infty. \tag{18}$$

6.1. **Local dimension.** Let us recall the definition of local dimension and fix some notations for what follows.

Let (M, d) be a metric space and assume that μ is a Borel probability measure on M. Given $x \in M$, let $B_r(x) = \{y \in M; d(x, y) \leq r\}$ be the ball centered at x with radius r. The *local dimension* of μ at $x \in M$ is defined by

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}$$

if this limit exists. In this case $\mu(B_r(x)) \sim r^{d_\mu(x)}$.

This notion characterizes the local geometric structure of an invariant measure with respect to the metric in the phase space of the system see [17].

We can always define the *upper* and the *lower* local dimension at x as

$$\overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}, \qquad \underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}.$$

If $d^+(x) = d^-(x) = d$ almost everywhere the system is called *exact dimensional*. In this case many properties of dimension of a measure coincide. In particular, d is equal to the infimum Hausdorff dimension of full measure sets: $d = \inf\{\dim_H Z; \mu(Z) = 1\}$. This happens in a large class of systems, for example, in C^2 diffeomorphisms having non zero Lyapunov exponents almost everywhere, [17].

6.2. **Relation between local dimension for F and for X^t.** Let us establish a relation between d_{μ_F} and d_{μ_X} which will be used in the following.

Proposition 6.2. *Let $x \in \mathbb{R}^3$ and $\pi(x)$ be the projection on Σ given by $\pi(x) = y$ if x is on the orbit of $y \in \Sigma$ and the orbit from y to x does not cross Σ (if $x \in \Sigma$ then $\pi(x) = x$). For all regular points $x \in \mathbb{R}^3$*

$$\overline{d}_{\mu_X}(x) = \overline{d}_{\mu_F}(\pi(x)) + 1, \quad \underline{d}_{\mu_X}{}^-(x) = \underline{d}_{\mu_F}{}^-(\pi(x)) + 1. \tag{19}$$

Proof. First observe that for product measures as $\mu_X = \mu_F \times dt$, where dt is the Lebesgue measure at the line, the formula is trivially verified. But, by construction $\mu_X = \phi_*(d\mu_F \times dt)$, where $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is a local bi-Lipschitz map at each regular point. Since the local dimension is invariant by local bi-Lipschitz maps, it follows the required equation (19). $\square$

7. Decay of correlations for two dimensional Lorenz maps

In this section we estimate the decay of correlations for a class of Lorenz like maps containing the first return map of the geometric Lorenz system described above.

The main result in this section is the following:

Theorem A (decay of correlation for the Poincaré map) *The unique SRB measure μ_F of F has exponential decay of correlation with respect to Lipschitz observables.*

The proof of this theorem will be done by estimating the speed of approaching of iterates of suitable measures (corresponding to Lipschitz observables) to the invariant measure. For this purpose we will consider the space of measures on Σ as a metric space, endowed with the *Wasserstein-Kantorovich distance*, whose basic properties we are going to describe.

Notations. Let us introduce some notations: we will consider the sup distance on $\Sigma = [-\frac{1}{2}, \frac{1}{2}]^2$, so that the diameter, $diam(\Sigma) = 1$. This choice is not essential, but will avoid the presence of many multiplicative constants in the following making notations cleaner.

As before, the square Σ will be foliate by stable, vertical leaves. We will denote the leaf with x coordinate by γ_x or, with a small abuse of notation, when no confusion is possible we will denote both the leaf and its coordinate with γ.

Let $f\mu$ be the measure μ_1 such that $d\mu_1 = fd\mu$. Moreover, let us sometime for short denote the integral by $\mu(f) = \int fd\mu$. Let μ a measure on Σ. In the following, such measures on Σ will be often disintegrated in the following way: for each Borel set A

$$\mu(A) = \int_{\gamma \in I} \mu_\gamma(A \cap \gamma) d\mu_x \tag{20}$$

with μ_γ being probability measures on the leaves γ and μ_x is the marginal on the x axis which will be an absolutely continuous probability measure. We will also denote by ϕ_x its density.

Let us consider the projection π_y on the y coordinate. Let us denote the "restriction" of μ on the leaf γ by

$$\mu|_\gamma = \pi_y^*(\phi_x(\gamma)\mu_\gamma).$$

This is a measure on I and it is not normalized. We remark that $\mu|_\gamma(I) = \phi_x(\gamma)$. If Y is a metric space, we denote by $PM(Y)$ the set of Borel probability measures on Y. Let us finally denote by $L(g)$ be the best Lipschitz constant of g : $L(g) = \sup_{x,y} \frac{|g(x)-g(y)|}{|x-y|}$ and set $\|g\|_{lip} = \|g\|_\infty + L(g)$.

7.1. **The Wasserstein-Kantorovich distance.** Let us consider a bounded metric space Y and let us consider the following notion of distance between measures: given two probability measures μ_1 and μ_2 on Y

$$W_1(\mu_1, \mu_2) = \sup_{g \in 1lip(Y)} (|\int_Y g\ d\mu_1 - \int_Y g\ d\mu_2|)$$

where $1lip(Y)$ is the space of 1-Lipschitz functions on Y. We remark that adding a constant to the test function g does not change the above difference $\int g\ d\mu_1 - \int g\ d\mu_2$. The above defined W_1 has moreover the following basic properties, [2]:

Proposition 7.1. *(Ambrosio L., Gigli N., Savarè,) The following properties hold*

(1) *W_1 is a distance and if Y is separable and complete, then $PM(Y)$ with this distance is a separable and complete metric space.*

(2) *A sequence is convergent for the W_1 metrics if and only if it is convergent for the weak topology.*

It is worth to remark the connection between the above defined distance, the notion of coupling and the optimal transport problems.

Suppose μ_1 and μ_2 are two probability measures on $[0,1]$. Let $\mathcal{P}(\mu_1,\mu_2)$ be the space of all Borel probability measures P on $[0,1]\times[0,1]$ having marginals μ_1 and μ_2, i.e. $\mu_1(*) = P(*\times[0,1])$ and $\mu_2(*) = P([0,1]\times *)$.

Let us consider the (Kantorovich) functional:

$$\mathcal{A}(\mu_1,\mu_2) = \inf_{P\in\mathcal{P}} \int |x-y| dP(x,y) \tag{21}$$

this functional can be interpreted as the minimal cost needed to transport an initial mass distribution μ_1 to a final distribution μ_2 over all the possible *transportation plans*, represented by the elements of $\mathcal{P}(\mu_1,\mu_2)$ where the cost to transport mass from the position x to the position y is given by $|x-y|$.

A classical result by Kantorovich and Rubinstein implies that in our case (where the space we consider is $[0,1]$ with the distance $d(x,y)=|x-y|$)

$$\mathcal{A}(\mu_1,\mu_2) = W_1(\mu_1,\mu_2). \tag{22}$$

We refer the paper [10] and the book [2] for more on this subject, and to [2].

Remark 7.2. (distance and convex combinations) If $a+b=1, a\geq 0, b\geq 0$ then

$$W_1(a\mu_1+b\mu_2, a\mu_3+b\mu_4) \leq a\cdot W_1(\mu_1,\mu_3) + b\cdot W_1(\mu_2,\mu_4). \tag{23}$$

Indeed

$$W_1(a\mu_1+b\mu_2, a\mu_3+b\mu_4) = \sup_{g\in 1lip(Y)} (|\int g\ d(a\cdot\mu_1+b\cdot\mu_2) - \int g\ d(a\cdot\mu_3+b\cdot\mu_4)|) =$$

$$= \sup_{g\in 1lip(Y)} (|a\cdot\int g\ d\mu_1 + b\cdot\int g\ d\mu_2 - a\cdot\int g\ d\mu_3 - b\cdot\int g\ d\mu_4|)$$

$$\leq \sup_{g\in 1lip(Y)} (|a\int g\ d\mu_1 - a\cdot\int g\ d\mu_3| + |b\cdot\int g\ d\mu_2 - b\cdot\int g\ d\mu_4|) =$$

$$\sup_{g\in 1lip(Y)} (a\cdot|\int g\ d\mu_1 - \int g\ d\mu_3| + b\cdot|\int g\ d\mu_2 - \int g\ d\mu_4|) \leq a\cdot W_1(\mu_1,\mu_3) + b\cdot W_1(\mu_2,\mu_4).$$

We also remark that the same kind of estimation can be done if the convex combination has more than 2 summands.

Remark 7.3. If g is ℓ-Lipschitz and μ_1,μ_2 are probability measures then

$$|\int_Y g\ d\mu_1 - \int_Y g\ d\mu_2| \leq \ell\cdot W_1(\mu_1,\mu_2).$$

In the next subsections we describe the properties of the Wassertein distance we are interested in.

7.2. **Wassertein distance and decay of correlations.** We give some general facts on the relation between W_1 distance and decay of correlations.

Let (Y, F, μ) be a dynamical system on a metric space with invariant probability measure μ. The transfer operator associated to F will be indicated with F^*.

Proposition 7.4 (decay as function of distance)**.** *Let $g \in lip(Y)$ and $f \in L^1(Y, \mu)$, $f \geq 0$. Let μ_1 be a probability measure which is absolutely continuous with respect to μ, and $d\mu_1 = \frac{f(x)}{||f||_{L^1}} d\mu$. Then*

$$|\int g(F^n(x))\ f(x)d\mu - \int f(x)d\mu \int g(x)d\mu| \leq L(g) \cdot ||f||_{L^1} \cdot W_1((F^*)^n(\mu_1), \mu). \quad (24)$$

Proof. Dividing by $L(g)$ we can suppose $g \in 1lip(Y)$. As $\int g(F(x)) \frac{f(x)}{||f||_{L^1}} d\mu = \int g(x) d(F^*(\mu_1))$ then the decay of correlations between f and g can be estimated in function of the distance between $(F^*)^n(\mu_1)$ and μ as:

$$L(g)||f||_{L^1}|\int g(F^n(x))\ \frac{f(x)}{||f||_{L^1}} d\mu - \int g(x)d\mu| = L(g)||f||_{L^1}|\int g(x)d(F^{*n}(\mu_1)) - \int g(x)d\mu|$$

$$\leq L(g)||f||_{L^1} \sup_{g \in 1lip(Y)} (|\int g\ d(F^{*n}(\mu_1)) - \int g\ d\mu|) = L(g)||f||_{L^1} W_1((F^*)^n(\mu_1), \mu).$$

□

Conversely,

Proposition 7.5 (distance as function of decay)**.** *If for each $f \in L^1(\mu)$, $f \geq 0$ and $g \in lip(Y)$ it holds*

$$|\int g(F^n(x))\ f(x)d\mu - \int f(x)d\mu \int g(x)d\mu| \leq C \cdot L(g) \cdot \|f\|_{L^1} \cdot \Phi(n)$$

then taking $d\mu_1 = \frac{f(x)}{||f||_{L^1}} d\mu$ it holds

$$W_1((F^*)^n(\mu_1), \mu) \leq C \cdot \Phi(n).$$

Proof. Consider $g \in 1lip$. Hence

$$\begin{aligned} \frac{C \cdot L(g)\|f\|_{L^1} \cdot \Phi(n)}{||f||_{L^1}} &\geq \frac{|\int g(F^n(x))\ f(x)d\mu - \int f(x)d\mu \int g(x)d\mu|}{||f||_{L^1}} = \\ &= |\int g(x)d(F^{*n}(\mu_1)) - \int g(x)d\mu| \end{aligned}$$

since this hold for each g hence $W_1(F^{*n}(\mu_1), \mu) \leq C \cdot \Phi(n)$. □

7.3. **Disintegration and Wasserstein distance.** We will consider maps having an invariant foliation, as we have seen in the Lorenz map. The invariant measure will then be disintegrated as in Equation (20) into a family of measures μ_γ on almost each stable leaf γ and an absolutely continuous measure μ_x on the unstable direction.

If μ^1 and μ^2 are two disintegrated measures as above, their W_1 distance can be estimated in function of some distance between their respective marginals on the x axis and measures on the leaves:

Proposition 7.6. *Let μ^1, μ^2 be measures on Σ as above, such that for each Borel set A*

- $\mu^1(A) = \int_{\gamma\in I} \mu^1_\gamma(A\cap\gamma)d\mu^1_x$
- $\mu^2(A) = \int_{\gamma\in I} \mu^2_\gamma(A\cap\gamma)d\mu^2_x$

with μ^i_x absolutely continuous with respect to the Lebesgue measure, moreover let us suppose

(1) *for almost each vertical leaf γ, $W_1(\mu^1_\gamma, \mu^2_\gamma) \le \varepsilon$ and*
(2) $\sup_{\|h\|_\infty \le 1} |\int h d\mu^1_x - \int h d\mu^2_x| \le \delta$

then $W_1(\mu^1, \mu^2) \le \varepsilon + \delta$.

Proof. Considering the W_1 distance and disintegrating μ^1 and μ^2:

$$W_1(\mu^1,\mu^2) \le \sup_{g\in 1lip} |\mu^1(g) - \mu^2(g)| = \tag{25}$$

$$= \sup_{g\in 1lip} |\int_{\gamma\in I}\int_\gamma g(*)d\mu^1_\gamma d\mu^1_x - \int_{\gamma\in I}\int_\gamma g(*)d\mu^2_\gamma d\mu^2_x|.$$

Adding and subtracting $\int\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x$ the last expression is equivalent to

$$\sup_{g\in 1lip} |\int_I\int_\gamma g(*)d\mu^1_\gamma d\mu^1_x - \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x +$$
$$+ \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x - \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^2_x|.$$

This becomes

$$\sup_{g\in 1lip} |\int_I(\int_\gamma g(*)d\mu^1_\gamma - g(*)d\mu^2_\gamma)d\mu^1_x + \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x - \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^2_x| \le$$
$$\le \sup_{g\in 1lip} |\int_I \varepsilon d\mu^1_x + \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x - \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^2_x| \le$$
$$\le \varepsilon + |\int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x - \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^2_x| \tag{26}$$

Since $g\in 1lip$ and $diam(\Sigma) = 1$ (on the square we consider the sup distance), then by adding a constant to g (which does not change $\int g d\mu^1_\gamma - \int g d\mu^2_\gamma$) we can suppose without loss of generality that $g \le 1$ and then for almost each γ it holds $h(\gamma) = |\int_\gamma g(*)d\mu^2_\gamma| \le 1$. Hence, by assumption (2) the statement is proved. □

7.4. **Exponential decay of correlations.** Now we are ready to prove Theorem A from the begining of this section. Recall (see Proposition 5.2) that for a piecewise expanding map of the interval T, there are constants $C > 0$ and $\lambda > 0$, depending on the system such that, if g and f are respectively L^1 and BV (bounded variation) observables on I for each n it holds:

$$|\int g(T^n(x))f(x)dm - \int g(x)d\mu \int f(x)dm| \le C\cdot \|g\|_{L_1}\cdot\|f\|_{BV}\cdot e^{-\lambda n} \tag{27}$$

(recall that m is the Lebesgue measure above). This will be used in the proof of the following theorem

Theorem 7.7. *Let $F : \Sigma \to \Sigma$ a Borel function such that $F(x, y) = (T(x), G(x, y))$. Let μ be an invariant measure for F with marginal μ_x on the x-axis (which is invariant for $T : I \to I$). Let us suppose that*

(1) *(T, μ_x) satisfies the above equation 27 and $T^{-1}(x)$ is finite for each $x \in I$.*
(2) *F is a contraction on each vertical leaf: G is λ-Lipschitz in y with $\lambda < 1$ for each x .*
(3) *μ is regular enough that for each ℓ-Lipschitz function $f : \Sigma \to \mathbb{R}$ the projection $\pi_x^*(f\mu)$ has bounded variation density $\overline{f}$ [1], with*
$$var(\overline{f}) \leq K\ell \tag{28}$$
where K is not depending on f.

Then (F, μ) has exponential decay of correlation (with respect to Lipschitz and L^1 observables as in Equation 24).

We already saw that the first two points in the above proposition are satisfied by the first return map of the Geometric Lorenz system. The third point is also satisfyed by the Geometrical Lorenz model. The interested reader can check the proof in the paper by Galatolo-Pacifico mentioned in the Introduction. In fact, there it is proved that point (3) above holds for a more general class of flows, containing the Lorenz geometric flow.

Before the proof of Theorem 7.7 we make the following remark which is a simple but important fact implied by the uniform contraction on stable leaves

Remark 7.8. Under the above assumptions, let us consider a leaf γ and two probability measures μ, ν on it. Then
$$W_1(F^*(\mu), F^*(\nu)) \leq \lambda W_1(\mu, \nu).$$

Proof. This is because the map is uniformly contracting on each leaf. If g is 1-Lipschitz on $F(\gamma)$ then $g(F(*))$ is λ-Lipschitz on γ. This implies that
$$|\int_{F(\gamma)} g\ d(F^*\mu) - \int_{F(\gamma)} g\ d(F^*\nu)| = |\int_\gamma g \circ F\ d\mu - \int_\gamma g \circ F\ d\nu| \leq \lambda W_1(\mu, \nu)$$
finishing the proof. □

Proof. (of Theorem 7.7) Let us consider $\nu = f\mu$ with $f \geq 0$ being $\ell-$Lipschitz and $\int f d\mu = 1$ (remark that this implies $\ell \geq 1$). The strategy is to use Proposition 7.6 and find exponentially decreasing bounds for ε and δ so that we can estimate the Wasserstein distance between μ and iterates of $f\mu$ and then apply Proposition 7.4 to deduce decay of correlations from the distance. Let us consider the leaf γ_x with coordinate x. The density $\overline{f}$, by item 3 has bounded variation and $||\overline{f}||_{BV} \leq K\ell + 1 \leq (K + 1)\ell$. Let $\nu_x = \overline{f}m$ the measure on the x-axis with density $\overline{f}$ (as before m is the Lebesgue measure). Let us consider the base map T. Let $g \in L^1([-\frac{1}{2}, \frac{1}{2}])$. Since $|\int g\ d(T^{*n}(\nu_x)) - \int g\ d\mu_x| = |\int g(T^n(x))\overline{f}(x)dm - \int g(x)d\mu_x|$, by equation (27)
$$|\int gd(T^{*n}(\nu_x)) - \int gd\mu_x| \leq \|g\|_{L_1} \cdot \|\overline{f}\|_{BV} \cdot C \cdot e^{-\lambda n},$$

[1] which can also be expressed as $\overline{f}(x) = \int f(x, y)\ d\mu|_{\gamma_x}$.

implying that $\sup_{\|g\|_\infty \leq 1} |\int g dT^{*n}(\nu_x) - \int g d\mu_x| \leq \|\overline{f}\|_{BV} \cdot C \cdot e^{-\lambda n} \leq (K+1)\ell C \cdot e^{-\lambda n}$ and hence we see that item (2) at Proposition 7.6 is satisfied with an exponential bound depending on the Lipschitz constant ℓ of f.

Let us consider $\nu^n = F^{*n}\nu$ again. Since, as said before the map F sends vertical leaves into vertical ones then there is a family of probability measures ν_γ^n on vertical leaves such that

$$(F^{*n}\nu)(g) = \int_{\gamma \in I} \int_\gamma g(*) d\nu_\gamma^n d((T^{*n}(\nu_x))).$$

To satisfy item (1) at Proposition 7.6 and hence conclude the statement we only have to prove that there are C_2, λ_2 s.t.

$$\forall \gamma \quad W_1(\nu_\gamma^n, \mu_\gamma) \leq C_2 \cdot e^{-\lambda_2 n}$$

this is because of uniform contraction on stable leaves.

Indeed, by remark 7.8, if ν_γ and ρ_γ are the two probability measures on the leaf γ then the measures $F^*(\nu_\gamma), F^*(\rho_\gamma)$ on the contracting leaf $F(\gamma)$ are such that

$$W_1(F^*(\nu_\gamma), F^*(\rho_\gamma)) \leq \lambda \cdot W_1(\nu_\gamma, \rho_\gamma).$$

Now let us consider $F^{-1}(\gamma) = \gamma_1 \cup \gamma_2 ... \cup \gamma_k$ and apply the above inequality to estimate the distance of iterates of the measure on the leaves. For simplicity let us show the case where the pre-image of a leaf consists of two leaves as it happen in the Geometric Lorenz system, the case where the pre-image consists of more leaves is analogous: let hence $F^{-1}(\gamma) = \gamma_1 \cup \gamma_2$, after one iteration of F^* on ν and μ the "new" measures $\nu_\gamma^1 = (F^*(\nu))_\gamma$ and μ_γ (which is equal to $(F^*(\mu))_\gamma$ because μ is invariant) on the leaf γ will be a convex combination of the images of the "old" measures on γ_1 and γ_2

$$\nu_\gamma^1 = a \cdot F^*(\nu_{\gamma_1}) + b \cdot F^*(\nu_{\gamma_2}),$$

$$\mu_\gamma = a \cdot F^*(\mu_{\gamma_1}) + b \cdot F^*(\mu_{\gamma_2}) \tag{29}$$

with $a + b = 1, a, b \geq 0$ (the second equality is again because μ is invariant). By the triangle inequality (remark 7.2)

$$W_1(\nu_\gamma^1, \mu_\gamma) \leq a \cdot W_1(F^*(\nu_{\gamma_1}), F^*(\mu_{\gamma_1})) + b \cdot W_1(F^*(\nu_{\gamma_2}), F^*(\mu_{\gamma_2}))$$

and by remark 7.8

$$W_1(\nu_\gamma^1, \mu_\gamma) \leq \lambda(a \cdot W_1(\nu_{\gamma_1}, \mu_{\gamma_1}) + b \cdot W_1(\nu_{\gamma_2}, \mu_{\gamma_2}))$$

hence

$$W_1(\nu_\gamma^1, \mu_\gamma) \leq \lambda \sup_\gamma (W_1(\nu_\gamma, \mu_\gamma)).$$

The same can be done in the case when the pre-image $F^{-1}(\gamma) = \gamma_1$ is only one leaf or more than two, hence by induction $W_1(\nu_\gamma^n, \mu_\gamma) < \lambda^n$, and the exponential bound on the distance of iterates on the leaves (item 1 of Proposition 7.6) is provided. $\square$

8. Hitting time: flow and section

We now consider again a Lorenz like flow, with integrable return time, i.e. a flow X^t having a transversal section Σ whose first return map satisfies the assumptions of Theorem 7.7 and the return time is integrable, as before. As before $F : \Sigma \setminus \Gamma \to \Sigma$ is the first return map associated.

Let $x, x_0 \in \mathbb{R}^3$ and

$$\tau_r^{X^t}(x, x_0) = \inf\{t \geq 0 | X^t(x) \in B_r(x_0)\}$$

be the time needed for the X-orbit of a point x to enter for the *first time* in a ball $B_r(x_0)$. The number $\tau_r^{X^t}(x, x_0)$ is the *hitting time associated to* the flow X^t and $B_r(x_0)$.

If $x, x_0 \in \Sigma$ and $B_r^\Sigma(x_0) = B_r(x_0) \cap \Sigma$, we define

$$\tau_r^\Sigma(x, x_0) = \min\{n \in \mathbb{N}^+; F^n(x) \in B_r^\Sigma(x_0)\} :$$

the *hitting time associated to* the discrete system F.

Given any x we recall that we denoted with $t(x)$ the first strictly positive time, such that $X^{t(x)}(x) \in \Sigma$ (the *return time* of x to Σ). A relation between ${\tau_r}^X(x, x_0)$ and $\tau_r^\Sigma(x, x_0)$ is given by

Proposition 8.1. *Under the above assumptions, if $\int_\Sigma t(x)\ d\mu_F < \infty$, then, there is $K \geq 0$ and a set $A \subset \Sigma$ having full μ_F measure such that for each $x_0 \in \Sigma$, $x \in A$*

$$c(x, r) \cdot \tau_{Kr}^\Sigma(x, x_0) \cdot \int_\Sigma t(x)\ d\mu_F \leq \tau_r^{X^t}(x, x_0) \leq c(x, r) \cdot \tau_r^\Sigma(x, x_0) \cdot \int_\Sigma t(x)\ d\mu_F \quad (30)$$

with $c(x, r) \to 1$ as $r \to 0$.

Proof. Let us assume that $x, x_0 \in \Sigma$, $x \neq x_0$ and $r \leq d(x, x_0)$. Since the flow cannot hit the section near x_0 without entering in a small ball of the space centered at x_0 before, then $\tau_r^\Sigma(x, x_0)$ and $\tau_r^{X^t}(x, x_0)$ are related by

$$\tau_r^{X^t}(x, x_0) \leq \sum_{i=0}^{\tau_r^\Sigma(x,x_0)} t(F^i(x)). \quad (31)$$

Moreover, since the section is supposed to be transversal to the flow, there is a K such that

$$\tau_r^{X^t}(x, x_0) \geq \left[\sum_{i=0}^{\tau_{Kr}^\Sigma(x,x_0)} t(F^i(x))\right]. \quad (32)$$

The last inequality follows by the fact that if the flow at some time crosses the ball centered at x_0 then after a time $e(r)$ it will cross the section at a distance less than Kr, where K depends on the angle between the flow and the section (when r is small approximate locally the flow by a constant one).

The above sums are Birkhoff sums of the observable t on the F-orbit of x and μ_F is ergodic. Then there is a full measure set $A \subset \Sigma$ (and $x_0 \notin A$) such that

$$\frac{1}{n}\sum_{i=0}^{n} t(F^i(x)) \longrightarrow \int_\Sigma t(x)\ d\mu_F, \quad \text{as} \quad n \to \infty$$

for $x \in A$. Hence

$$\frac{1}{\tau_r^{\Sigma}(x,x_0)} \sum_{i=0}^{\tau_r^{\Sigma}(x,x_0)} t(F^i(x)) \longrightarrow \int_{\Sigma} t(x)\ d\mu_F, \quad \text{as} \quad n \to \infty$$

for $x \in A$. Thus we get that for each $x \in A$

$$\sum_{i=0}^{\tau_r^{\Sigma}(x,x_0)} t(F^i(x)) = c(x,r) \cdot \tau_r^{\Sigma}(x,x_0) \cdot \int_{\Sigma} t(x)\ d\mu_F \tag{33}$$

with $c(x,r) \to 1$ as $r \to 0$. Combining Equations (31,32) and (33) we get (30). $\square$

Let π be the projection on Σ defined in Proposition 6.2. The above statement implies the following

Proposition 8.2. *There is a full measure set $B \subset \mathbb{R}^3$ (for the flow invariant measure) such that if $x_0 \in \mathbb{R}^3$ is regular and $x \in B$ it holds (provided the limits exist)*

$$\lim_{r\to 0} \frac{\log \tau_r^{X^t}(x,x_0)}{-\log r} = \lim_{r\to 0} \frac{\log \tau_r^{\Sigma}(\pi(x),\pi(x_0))}{-\log r}. \tag{34}$$

Proof. The above Proposition implies that if $x_0, x \in \Sigma$ and $x \in A$ then

$$\lim_{r\to 0} \frac{\log \tau_r^{X^t}(x,x_0)}{-\log r} = \lim_{r\to 0} \frac{\log \tau_r^{\Sigma}(x,x_0)}{-\log r}. \tag{35}$$

If $x_0 \in \mathbb{R}^3$ is a regular point, the flow X induces a bilipschitz homeomophism from a neighborhood of $\pi(x_0) \in \Sigma$ to a neighborhood of x_0.

Hence there is $K \geq 1$ such that

$$\tau_{K^{-1}r}^X(x,\pi(x_0)) + Const \leq \tau_r^X(x,x_0) \leq \tau_{Kr}^X(x,\pi(x_0)) + Const$$

where $Const$ represents the time which is needed to go from $\pi(x_0)$ to x_0 by the flow. This is also true for each $x \in B = \pi^{-1}(A)$. Extracting logarithms and taking the limits we get the required result. $\square$

We recall that (see Section 6) the assumption $\int_{\Sigma} t(x)\ d\mu_F < \infty$ is verified for the geometric Lorenz flow. Hence these results applies for this example.

9. A logarithm law for the hitting time

In this section we give the main result for the behavior of the hitting time on Lorenz like flows. First let us recall a result on discrete time systems.

Let (Y,T,μ) be a measure preserving (discrete time) dynamical system. We say that (X,T,μ) has super-polynomial decay of correlations with respect to Lipschitz observables if

$$\left| \int \varphi \circ T^n \psi \cdot d\mu - \int \varphi \cdot d\mu \cdot \int \psi \cdot d\mu \right| \leq \|\varphi\| \cdot \|\psi\| \cdot \theta_n,$$

where $\lim_n \theta_n \cdot n^p = 0$ for all $p > 0$ and $\|\cdot\|$ is the Lipschitz norm.

Galatolo proved the following fact for discrete time systems:

Theorem 9.1. *Let (Y, T, μ) a measure preserving transformation having superpolynomial decay of correlations as above. For each $x_0 \in Y$ such that $d_\mu(x_0)$ is defined*

$$\lim_{r \to 0} \frac{\log \tau_r(x, x_0)}{-\log r} = d_\mu(x_0)$$

for μ-almost each $x \in Y$.

Applying this to the 2-dimensional system (Σ, F, μ_F) (which satisfies the assumptions of Theorem 7.7 since ans hence has exponential decay of correlations). We conclude the following

Corollary 9.2. *Let $F : \Sigma \to \Sigma$ be a map with an invariant measure μ_F satisfying the assumptions of Theorem 7.7. For each $x_0 \in \Sigma$ such that $d_{\mu_F}(x_0)$ exists then*

$$\lim_{r \to 0} \frac{\log \tau_r^\Sigma(x, x_0)}{-\log r} = d_{\mu_F}(x_0).$$

for μ_F-almost $x \in \Sigma$.

Now, if we consider a flow having such a map as its Poincaré section and integrable return time, we can construct as in Section 6 an SRB invariant measure μ_X for the flow. By Proposition 8.2, Corollary 9.2 and Proposition 6.2 we can estimate the hitting time to balls for the flow by the corresponding estimation for the Poincaré map and we get our main result, which corresponds to Theorem B in the introduction:

Theorem 9.3. *If X^t is a Lorenz like flow, that is a flow having a transversal section, with a Poincaré map satisfying the assumptions of proposition 7.7 and integrable return time, then for each regular $x_0 \in \mathbb{R}^3$ such that $d_{\mu_X}(x_0)$ exists, it holds*

$$\lim_{r \to 0} \frac{\log \tau_r^{X^t}(x, x_0)}{-\log r} = d_{\mu_X}(x_0) - 1$$

for μ_X-almost each $x \in \mathbb{R}^3$.

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S. Galatolo, Dipartimento di Matematica Applicata via Buonarroti 1 Pisa
E-mail address: `s.galatolo@docenti.ing.unipi.it`
URL: `http://www2.ing.unipi.it/~d80288/`

Maria José Pacifico, Instituto de Matemática, Universidade Federal do Rio de Janeiro, C. P. 68.530, 21.945-970 Rio de Janeiro, Brazil
E-mail address: `pacifico@im.ufrj.br`