Lorenz like flows-Last Lecture

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Main goals

The main goal is to explain the results (Galatolo-P)

Theorem A. (decay of correlation for the Poincaré map) *Let F be the first return map associated to a geometrical Lorenz flow. The unique SRB measure μ_F of F has exponential decay of correlation with respect to Lipschitz observables.*

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Theorem B. (logarithm law for the hitting time) *For each regular x_0 s.t. the local dimension $d_{\mu_X}(x_0)$ is defined it holds*

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Theorem B. (logarithm law for the hitting time) *For each regular x_0 s.t. the local dimension $d_{\mu_X}(x_0)$ is defined it holds*

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Remark. Theorems A and B hold for a more general class of flows, defined axiomatically.

Recall: First lecture

- Motivation : Lorenz' equations

$$X(x,y,z) = \begin{cases} \dot{x} = -10 \cdot x + 10 \cdot y \\ \dot{y} = 28 \cdot x - y - x \cdot z \\ \dot{z} = -8/3 \cdot z + x \cdot y\,. \end{cases}$$

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- f has μ_f which induces μ_F for F which, on its turn, induces μ_X for the flow.

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- Let $\mathcal{C}(f,g)$ be the correlation function:

$$\mathcal{C}(f,g) = \left| \int g(F^n(x)) f(x) dm - \int g(x) d\mu \int f(x) dm \right|$$

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(4) $W_1(F^*(\mu),F^*(\nu)) \leq \lambda \cdot W_1(\mu,\nu).$

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Since
$|\int g\ d(T^{*n}(\nu_x)) - \int g\ d\mu_x| = |\int g(T^n(x))\overline{f}(x)dm - \int g(x)d\mu_x|$,
and T has exponential decay implies

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Proof-2

Thus

$$\sup_{\|g\|_\infty \le 1} \left| \int g dT^{*n}(\nu_x) - \int g d\mu_x \right| \le \|\overline{f}\|_{BV} \cdot C \cdot e^{-\lambda n} \le$$

$$(K+1) \cdot \ell \cdot C \cdot e^{-\lambda n}.$$

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so item (2) at Proposition 3 is satisfied with exponential bound depending on the Lipschitz constant ℓ of f.

Proof-3

Let $\nu^n = F^{*n}\nu$ as before. Since F sends vertical leaves into vertical ones then there is a family of probability measures ν_γ^n on vertical leaves such that

$$(F^{*n}\nu)(g) = \int_{\gamma\in I}\int_\gamma g(*)d\nu_\gamma^n d((T^{*n}(\nu_x))).$$

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To satisfy item (1) at Proposition 3 and hence conclude the statement we only have to prove that there are C_2, λ_2 s.t.

$$\forall \gamma \in \mathcal{F}^s, \quad W_1(\nu^n_\gamma, \mu_\gamma) \leq C_2 \cdot e^{-\lambda_2 n}.$$

This is done by induction on n and using the preperties of $W-K$ distance.

Now we start

Final lecture :

Proof of Theorems A and B

Hitting time

Let $x, x_0 \in \mathbb{R}^3$ and

$$\tau_r^{X^t}(x, x_0) = \inf\{t \geq 0 | X^t(x) \in B_r(x_0)\}$$

be the time needed for the X-orbit of a point x to enter for the first time in a ball $B_r(x_0)$. The number $\tau_r^{X^t}(x, x_0)$ is the hitting time associated to the flow X^t and $B_r(x_0)$.

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If $x, x_0 \in \Sigma$ and $B_r^\Sigma(x_0) = B_r(x_0) \cap \Sigma$, we define

$$\tau_r^\Sigma(x, x_0) = \min\{n \in \mathbb{N}^+; F^n(x) \in B_r^\Sigma(x_0)\} :$$

the hitting time associated to the discrete system F.

Hitting time: flow and section

Given x, $t(x) > 0$ is the first time s. t. $X^{t(x)}(x) \in \Sigma$ (the return time of x to Σ). Relation between ${\tau_r}^X(x, x_0)$ and $\tau_r^\Sigma(x, x_0)$:

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Proposition If $\int_\Sigma t(x)\, d\mu_F < \infty$, then, $\exists\, K \geq 0$ and $A \subset \Sigma$, $\mu_F(A) = 1$ s. t. for each $x_0 \in \Sigma$, $x \in A$

$$c(x, r) \cdot \tau_{Kr}^\Sigma(x, x_0) \cdot \int_\Sigma t(x)\, d\mu_F \leq$$

$$\tau_r^{X^t}(x, x_0) \leq c(x, r) \cdot \tau_r^\Sigma(x, x_0) \cdot \int_\Sigma t(x)\, d\mu_F$$

with $c(x, r) \to 1$ as $r \to 0$.

Proof of Proposition-1

Proof. Assume that $x, x_0 \in \Sigma$, $x \neq x_0$ and $r \leq d(x, x_0)$. Since the flow cannot hit the section near x_0 without entering in a small ball of the space centered at x_0 before, then $\tau_r^{\Sigma}(x, x_0)$ and $\tau_r^{X^t}(x, x_0)$ are related by

$$\tau_r^{X^t}(x, x_0) \leq \sum_{i=0}^{\tau_r^{\Sigma}(x,x_0)} t(F^i(x)).$$

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Since the section is transversal to the flow, $\exists\ K$ s. t.

$$\tau_r^{X^t}(x, x_0) \geq \left[\sum_{i=0}^{\tau_{Kr}^{\Sigma}(x,x_0)} t(F^i(x)) \right].$$

The last inequality

The last inequality follows by the fact that if the flow at some time crosses the ball centered at x_0 then after a time $e(r)$ it will cross the section at a distance less than $K \cdot r$, K depending on the angle between the flow and the section.

Birkhoff sum

The above sums are Birkhoff sums of the observable t on the F-orbit of x and μ_F is ergodic. Then there is a full measure set $A \subset \Sigma$ (and $x_0 \notin A$) such that for $x \in A$,

$$\frac{1}{n}\sum_{i=0}^{n} t(F^i(x)) \longrightarrow \int_\Sigma t(x)\, d\mu_F, \quad \text{as} \quad n \to \infty$$

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Hence, for $x \in A$,

$$\frac{1}{\tau_r^\Sigma(x, x_0)} \sum_{i=0}^{\tau_r^\Sigma(x,x_0)} t(F^i(x)) \longrightarrow \int_\Sigma t(x)\, d\mu_F, \quad \text{as} \quad n \to \infty$$

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Thus we get that for each $x \in A$

$$\sum_{i=0}^{\tau_r^{\Sigma}(x,x_0)} t(F^i(x)) = c(x,r) \cdot \tau_r^{\Sigma}(x,x_0) \cdot \int_{\Sigma} t(x)\, d\mu_F \qquad (1)$$

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with $c(x,r) \to 1$ as $r \to 0$.

Combining Equations above we finish the proof of the proposition relating the discret with continuous hitting time.

Consequence

Let π be the projection on Σ defined before. The above statement implies the following

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Proposition There is a full measure set $B \subset \mathbb{R}^3$ s.t. if $x_0 \in \mathbb{R}^3$ is regular and $x \in B$ it holds

$$\lim_{r \to 0} \frac{\log \tau_r^{X^t}(x, x_0)}{-\log r} = \lim_{r \to 0} \frac{\log \tau_r^{\Sigma}(\pi(x), \pi(x_0))}{-\log r}.$$

Proof of the Proposition

Proof If $x_0, x \in \Sigma$ and $x \in A$ then

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If $x_0 \in \mathbb{R}^3$ is regular, X^t induces a bilipschitz homeo from a neigh. of $\pi(x_0) \in \Sigma$ to a neigh. of x_0. So $\exists\ K \geq 1$ s.t.

$$\tau_{K^{-1}r}^{X}(x, \pi(x_0)) + C \leq \tau_r^{X}(x, x_0) \leq \tau_{Kr}^{X}(x, \pi(x_0)) + C$$

where C is the time needed to go from $\pi(x_0)$ to x_0 by the flow. This is also true for $x \in B = \pi^{-1}(A)$. Extracting logarithms and taking the limits we get the required result.

Local dimension: section and flow

Theorem . Let $x \in \mathbb{R}^3$ and $\pi(x)$ be the projection on Σ given by $\pi(x) = y$ if x is on the orbit of $y \in \Sigma$ and the orbit from y to x does not cross Σ (if $x \in \Sigma$ then $\pi(x) = x$). For all regular points $x \in \mathbb{R}^3$ it holds

$$d_{\mu_X}(x) = d_{\mu_F}(\pi(x)) + 1.$$

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Proof For product measures as $\mu_X = \mu_F \times dt$, where dt is the Lebesgue measure at the line, the formula is trivially verified. By construction, $\mu_X = \phi_*(d\mu_F \times dt)$, where $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is a local bi-Lipschitz map at each regular point. Since the local dimension is invariant by local bi-Lipschitz maps, it follows the required inequality.

A logarithm law for the hitting time

Recall that if (Y, T, μ) is a measure preserving (discrete time) dynamical system, (X, T, μ) has super-polynomial decay of correlations with respect to Lipschitz observables if

$$\left| \int \varphi \circ T^n \psi \cdot d\mu - \int \varphi \cdot d\mu \cdot \int \psi \cdot d\mu \right| \leq \|\varphi\| \cdot \|\psi\| \cdot \theta_n,$$

$\lim_n \theta_n \cdot n^p = 0 \, \forall \, p > 0$ and $\| \cdot \|$:Lipschitz norm.

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$\lim_n \theta_n \cdot n^p = 0 \, \forall \, p > 0$ and $\| \cdot \|$:Lipschitz norm.
Theorem(Galatolo) Let (Y, T, μ) a measure preserving transformation having superpolynomial decay of correlations. If $d_\mu(x_0)$ is defined then for μ-almost $x \in Y$,

$$\lim_{r \to 0} \frac{\log \tau_r(x, x_0)}{-\log r} = d_\mu(x_0).$$

Log law hitting for geom Lorenz flow

Applying this to the 2-dimensional Lorenz system (Σ, F, μ_F) which has exponential decay of correlations, we conclude :

Log law hitting for geom Lorenz flow

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Theorem Let $F : \Sigma \to \Sigma$ be the Poincaré map associated to a geom. Lorenz flow. For $x_0 \in \Sigma$ s.t. $d_{\mu_F}(x_0)$ exists then for μ_F-almost $x \in \Sigma$.

$$\lim_{r \to 0} \frac{\log \tau_r^{\Sigma}(x, x_0)}{-\log r} = d_{\mu_F}(x_0).$$

Local dimension:section and flow-2

Since we have

$$\lim_{r\to 0} \frac{\log \tau_r(x, x_0)}{-\log r} = \lim_{r\to 0} \frac{\log \tau_{r,\Sigma}(x, x_0)}{-\log r} = d\mu_F(x_0)$$

Local dimension:section and flow-2

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$$\lim_{r\to 0} \frac{\log \tau_r(x, x_0)}{-\log r} = \lim_{r\to 0} \frac{\log \tau_{r,\Sigma}(x, x_0)}{-\log r} = d\mu_F(x_0)$$

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proving Theorem B.

Recurrence time

In the definition of hitting time, if you take $x_0 = x$, then the resulting expression is the recurrencce time, denoted by

$$\tau_r'(x) = \tau_r(x, x)$$

Using the next result by Saussol, we get a similar logarithm law for the recurrence time.

Saussol's result

(Y, T, μ): a measure preserving dynamical system, $h_\mu(T) > 0$ and T is s.t. $\exists$ a partition $\mathcal{A}$ into open sets s.t. for $A \in \mathcal{A}$, $T|_A$ is Lipschitz with constant $L_T(A)$.Suppose:

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(1) if $\mathcal{S}(\mathcal{A}) = \cup\{\partial A \in \mathcal{A}\}$ $\exists$ $c > 0,\ a > 0$ s.t.

$$\mu\left(\{x \in X : \mathrm{dist}(x, \mathcal{S}(\mathcal{A})) < \epsilon\}\right) < c \cdot \epsilon^a.$$

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Then

$$\liminf_{r\to 0}\frac{\log\tau_r(x,x)}{-\log r}=d_\mu^-(x)\ ,\text{ and }\limsup_{r\to 0}\frac{\log\tau_r(x,x)}{-\log r}=d_\mu^+(x)\ a.e.$$

Lorenz geo systems

Theorem The first return map (F, Σ, μ_F) of the geometric Lorenz system satisfies the hypothesis above.

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The partition $\mathcal{A} = \{A_i\}$, with

$$A_i = [\left(\frac{1}{i+2}, \frac{1}{i+1}\right) \cup \left(\frac{-1}{i+2}, \frac{-1}{i+1}\right)] \times \mathring{I},\ i \in \mathbb{N}^+$$

satisfies (1) and (2).

Still

The fact that μ_F has a bounded density marginal (the density will be denoted by f_0) on the x direction implies that the measure of the sets A_i can be estimated by

$$\mu(A_i) \leq \frac{4 \cdot \sup(f_0)}{i^2}.$$

Thus,

$$\sum_{A \in \mathcal{S}(\mathcal{A})} \log^+ L_F(A) \cdot \mu(A) = \sum_{A \in \mathcal{S}(\mathcal{A})} \log^+(K \cdot i^\beta) \cdot \frac{4 \cdot \sup(f_0)}{i^2} < \infty.$$

This finishes the proof.

Log law

Corollary For the geo. Lorenz system (F, Σ, μ_F) it holds

$$\liminf_{r\to 0} \frac{\log \tau_r^{\Sigma}(x,x)}{-\log r} = \underline{d}_{\mu_F}, \quad \limsup_{r\to 0} \frac{\log \tau_r^{\Sigma}(x,x)}{-\log r} = \overline{d}_{\mu_F}, \ \mu_F - a.e..$$

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Corollary For the geometric Lorenz flow it holds

$$\liminf_{r\to 0} \frac{\log \tau_r'(x)}{-\log r} = \underline{d}_{\mu_X} - 1, \quad \limsup_{r\to 0} \frac{\log \tau_r'(x)}{-\log r} = \overline{d}_{\mu_X} - 1, \ \mu_X - a.e.$$

where τ' is the recurrence time for the flow.

Main reference

We suggest to the interested reader the paper below and the references therein:

S. Galatolo and M. J. Pacifico, Lorenz like flows: exponential decay of correlations for the Poincaré map, logarithm law, quantitative recurrence, Ergodic Theory and Dynamical Systems, to appear

Finally

This is the end.

Many thanks to the organizers!!!!

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