Lorenz like flows-Third Lecture

Maria José Pacifico

pacifico@im.ufrj.br

IM-UFRJ

Rio de Janeiro - Brasil

Main goals

The main goal is to explain the results (Galatolo-P)

Theorem A. (decay of correlation for the Poincaré map) *Let F be the first return map associated to a geometrical Lorenz flow. The unique SRB measure μ_F of F has exponential decay of correlation with respect to Lipschitz observables.*

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Theorem B. (logarithm law for the hitting time) *For each regular x_0 s.t. the local dimension $d_{\mu_X}(x_0)$ is defined it holds*

$$\lim_{r\to 0}\frac{\log \tau_r(x,x_0)}{-\log r} = d_{\mu_X}(x_0) - 1 \quad \text{a.e. starting point } x.$$

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Remark. Theorems A and B hold for a more general class of flows, defined axiomatically.

Definitions

Recall:

- the local dimension of a μ at $x \in M$ is

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r}.$$

In this case $\mu(B_r(x)) \sim r^{d_\mu(x)}$.

Definitions

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In this case $\mu(B_r(x)) \sim r^{d_\mu(x)}$.

• the hitting time $\tau_r(x, x_0)$ is the time needed for the orbit of a point x to enter for the first time in a ball $B_r(x_0)$ centered at x_0, with small radius r.

Plan of the talks

- Motivation: Lorenz flow (First lecture)

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- Proof of Theorems A and B.

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Method for a geometrical Lorenz flow:

- f has μ_f which induces μ_F for F which, on its turn, induces μ_X for the flow.

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- Λ is a singular-attractor for a flow if

(a) all singularities in Λ are hyperbolic

(b) Λ is partially hyperbolic, E^{cu} volume expanding.

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- cover the domain of the flow by a finite number of adapted cross-sections $\cup_{0\le i\le n}\Sigma_i$
- the return map $R: \cup_{0\le i\le n}\Sigma_i \to \cup_{0\le i\le n}\Sigma_i$, preserves $\mathcal{F}^s$ and for big enoug return time is hyperbolic.

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- family μ_γ, $\gamma \in \mathcal{F}^s$ induces μ_F which in its turn induces μ_X
- $\exists\ f : I \to I$ s. t. $|f|^{-1}$ is α-BV and so it has statistical properties.

Recall: Second lecture III

- f is generalized bounded variation $\sim \alpha$-BV if

$$\sup_{a=a_0<a_1<\cdots<a_n=b} \left(\sum_{j=1}^{n} \left| f(a_i) - f(a_{i-1}) \right|^{1/\alpha} \right)^{\alpha} < \infty,$$

the supremum is taken over all finite partition of $I = [a, b]$.

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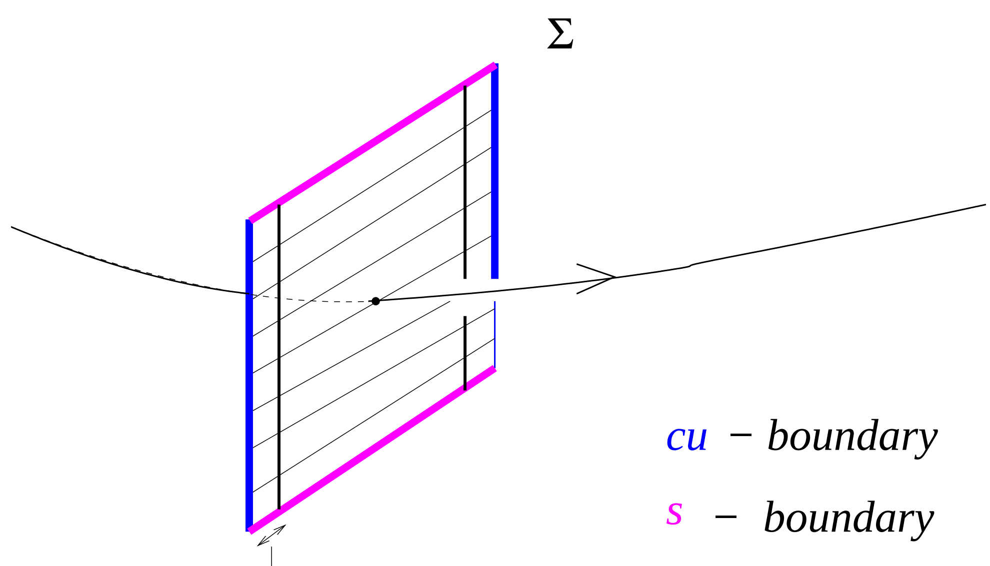

Now we start

Third lecture :

Wasserstein-Kantorovich distance and properties

Wasserstein-Kantorovich distance

Given two probabilities on M, μ_1 and μ_2, the

Wasserstein-Kantorovich distance is defined by

$$W_1(\mu_1, \mu_2) = \sup_{g \in Lip_1(M)} \left(\left| \int_M g d\mu_1 - \int_M g d\mu_2 \right| \right)$$

$Lip_1(M)$: the space of 1-Lipschitz maps on M.

W-K distance versus coupling

It is worth to remark the connection between the W-K distance, the notion of coupling and the optimal transport problems.

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Suppose μ_1 and μ_2 are two probability measures on $[0,1]$. Let $\mathcal{P}(\mu_1, \mu_2)$ be the space of all Borel probability measures P on $[0,1] \times [0,1]$ having marginals μ_1 and μ_2, i.e. $\mu_1(*) = P(* \times [0,1])$ and $\mu_2(*) = P([0,1] \times *)$.

The Kantorovich functional

Consider the (Kantorovich) functional:

$$\mathcal{A}(\mu_1, \mu_2) = \inf_{P \in \mathcal{P}} \int |x - y| dP(x, y)$$

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This functional can be interpreted as the minimal cost needed to transport an initial mass distribution μ_1 to a final distribution μ_2 over all the possible transportation plans, represented by the elements of $\mathcal{P}(\mu_1, \mu_2)$ where the cost to transport mass from the position x to the position y is given by $|x - y|$.

W-K distance versus K-functional

A classical result by Kantorovich and Rubinstein implies that in our case (where the space we consider is $[0, 1]$ with the distance $d(x, y) = |x - y|$)

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$$\mathcal{A}(\mu_1, \mu_2) = W_1(\mu_1, \mu_2).$$

Decay versus W-K distance

Proposition 1. (decay in function of distance) Let $\mu_1 \ll \mu$ and $d\mu_1 = f(x)d\mu$. Then, for $g \in Lip_1(M)$ we have

$$|\int g(F^n(x)) \cdot f(x)d\mu - \int f(x)d\mu \cdot \int g(x)d\mu| \leq$$

$$L(g) \cdot \|f\|_1 \cdot W_1((F^*)^n(\mu_1), \mu).$$

W-K distance versus decay

Proposition 2. (distance in function of decay) Assume that for each $f \in L^1(\mu)$ and $g \in Lip_1(M)$ it holds:

$$|\int g(F^n(x)) \cdot f(x) d\mu - \int f(x) d\mu \cdot \int g(x) d\mu| \leq$$

$$C \cdot \|g\|_{Lip_1(M)} \cdot \|f\|_{L^1(\mu)} \cdot \Phi(n).$$

Then, taking $d\mu_1 = f(x) d\mu$ with $\int f(x) d\mu = 1$ we get

$$W_1((F^*)^n(\mu_1), \mu) \leq 2 \cdot C \cdot \Phi(n)$$

W-K distance and disintegration

Proposition 3. Let μ^1 and μ^2 be invariant measures for (F, Σ) satisfying

- $\mu^1(A) = \int \mu^1_\gamma(A \cap \gamma) d\mu^1_\gamma$,

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(2) $\sup_{\|g\|_\infty} \left| \int g d\mu^1_\gamma - \int g d\mu^2_\gamma \right| \le \delta$.

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Then $\qquad W_1(\mu^1, \mu^2) \leq \epsilon + \delta.$

W-K distance versus stable foliation

Property $(\star\star)$ Let $\gamma \in \mathcal{F}^s$, and two probability measures μ, ν on it. Then

$$W_1(F^*(\mu), F^*(\nu)) \leq \lambda W_1(\mu, \nu) \quad (\star\star).$$

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Proof As F uniformly contracts each leaf we get that if g is 1-Lipschitz on $F(\gamma)$ then $g(F(*))$ is λ-Lipschitz on γ. This implies that

$$|\int_{F(\gamma)} g \, d(F^*\mu) - \int_{F(\gamma)} g \, d(F^*\nu)| = |\int_\gamma g \circ F \, d\mu - \int_\gamma g \circ F \, d\nu|$$

$$\leq \lambda \cdot W_1(\mu, \nu)$$

finishing the proof.

Fastly decay for F

Let $\mu^1 \ll \mu$, μ the SBR measure such that $d\mu_1 = f(x)d\mu$. Then, for each Borel set A we have

$$\mu_1(A) = \int_I \int_{A\cap\gamma} f(x)d\mu_\gamma d\mu_y.$$

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Thus we are in the setting of Proposition 3 above, in another words, the SBR measure for F disintegrates.

(Σ, F, μ) is fastly mixing

Theorem . The geometric Lorenz system (Σ, F, μ) is fastly mixing for Lipshitz and L^1 observables.

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To prove this theorem we shall use that

(*) μ is regular enough that for each ℓ-Lipschitz function $f : \Sigma \to \mathbb{R}$ the projection $\pi_x^*(f\mu)$ has bounded variation density $\overline{f}$ (which can also be expressed as $\overline{f}(x) = \int f(x,y)\, d\mu|_{\gamma_x}$), with

$$var(\overline{f}) \leq K\ell$$

where K does not depend on f.

Strategy

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and then apply Proposition 2 to deduce exponentially decay of correlations.

Proof-1

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Since
$|\int g\ d(T^{*n}(\nu_x)) - \int g\ d\mu_x| = |\int g(T^n(x))\overline{f}(x)dm - \int g(x)d\mu_x|$,
the fact that T has exponential decay implies

$$|\int gd(T^{*n}(\nu_x)) - \int gd\mu_x| \leq \|g\|_{L_1} \cdot \|\overline{f}\|_{BV} \cdot C \cdot e^{-\lambda n}.$$

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Proof-2

Thus

$$\sup_{\|g\|_\infty \leq 1} \left| \int g dT^{*n}(\nu_x) - \int g d\mu_x \right| \leq \|\overline{f}\|_{BV} \cdot C \cdot e^{-\lambda n} \leq$$

$$(K+1) \cdot \ell \cdot C \cdot e^{-\lambda n}.$$

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so item (2) at Proposition 3 is satisfied with exponential bound depending on the Lipschitz constant ℓ of f.

Proof-3

Let $\nu^n = F^{*n}\nu$ as before. Since F sends vertical leaves into vertical ones then there is a family of probability measures ν_γ^n on vertical leaves such that

$$(F^{*n}\nu)(g) = \int_{\gamma \in I} \int_\gamma g(*) d\nu_\gamma^n d((T^{*n}(\nu_x))).$$

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To satisfy item (1) at Proposition 3 and hence conclude the statement we only have to prove that there are C_2, λ_2 s.t.

$$\forall \gamma \in \mathcal{F}^s, \quad W_1(\nu_\gamma^n, \mu_\gamma) \leq C_2 \cdot e^{-\lambda_2 n}.$$

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$$W_1(F^*(\nu_\gamma), F^*(\rho_\gamma)) \leq \lambda \cdot W_1(\nu_\gamma, \rho_\gamma).$$

Now let $F^{-1}(\gamma) = \gamma_1 \cup \gamma_2$ and apply the above inequality to estimate the W-K distance of iterates of the measure on the leaves.

Proof-5

After one iteration of F^* on ν and μ the "new" measures $\nu^1_\gamma = (F^*(\nu))_\gamma$ and μ_γ (which is equal to $(F^*(\mu))_\gamma$ because μ is invariant) on the leaf γ will be a convex combination of the images of the "old" measures on γ_1 and γ_2

$$\nu^1_\gamma = a \cdot F^*(\nu_{\gamma_1}) + b \cdot F^*(\nu_{\gamma_2}),$$

$$\mu_\gamma = a \cdot F^*(\mu_{\gamma_1}) + b \cdot F^*(\mu_{\gamma_2})$$

with $a + b = 1, a, b \geq 0$ (the second equality is again because μ is invariant).

Proof-6

By the triangle inequality and the property of W-K distance with convex combinations, we have:

$$W_1(\nu_\gamma^1, \mu_\gamma) \leq a \cdot W_1(F^*(\nu_{\gamma_1}), F^*(\mu_{\gamma_1})) + b \cdot W_1(F^*(\nu_{\gamma_2}), F^*(\mu_{\gamma_2}))$$

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and by (*) (density is BV)

$$W_1(\nu_\gamma^1, \mu_\gamma) \leq \lambda(a \cdot W_1(\nu_{\gamma_1}, \mu_{\gamma_1}) + b \cdot W_1(\nu_{\gamma_2}, \mu_{\gamma_2}))$$

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$$W_1(\nu_\gamma^n, \mu_\gamma) < \lambda^n,$$

and the exponential bound on the distance of iterates on the leaves (item 1 of Proposition 3) is provided.

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This finishes the proof that (Σ, F, μ_F) is fastly mixing.

Finally

This finishes the third lecture.

We shall continue tomorrow, at 9 AM.