Lorenz like flows-Second Lecture

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Main goals

The main goal is to explain the results (Galatolo-P)

Theorem A. (decay of correlation for the Poincaré map) *Let* F *be the first return map associated to a geometrical Lorenz flow. The unique SRB measure* μ_F *of* F *has exponential decay of correlation with respect to Lipschitz observables.*

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Theorem B. (logarithm law for the hitting time) *For each regular x_0 s.t. the local dimension $d_{\mu_X}(x_0)$ is defined it holds*

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Remark. Theorems A and B hold for a more general class of flows, defined axiomatically.

Definitions

Recall:

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- Hitting and recurrence time
- Proof of Theorems A and B.

Recall: First lecture

- Motivation : Lorenz' equations

$$X(x,y,z) = \begin{cases} \dot{x} = -10 \cdot x + 10 \cdot y \\ \dot{y} = 28 \cdot x - y - x \cdot z \\ \dot{z} = -8/3 \cdot z + x \cdot y\,. \end{cases}$$

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Now we start

Second lecture :

F and the flow X have SRB measure μ_F and μ_X

Physical measures

An invariant probability μ is *physical* for the flow X_t, $t \in \mathbb{R}$ if the set $B(\mu)$ of points $z \in M$ satisfying

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi\big(X_t(z)\big)\, dt = \int \varphi \, d\mu$$

for all continuous $\varphi : M \to \mathbb{R}$ has positive Lebesgue measure.

$B(\mu)$: the ***basin*** of μ.

SRB meas. for Lorenz geo. flow-1

Piecewise expanding maps admits a unique invariant probability measure μ_f which is absolutely continuous with respect to Lebesgue measure m.

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Piecewise expanding maps admits a unique invariant probability measure μ_f which is absolutely continuous with respect to Lebesgue measure m.

From μ_f we may construct a SRB measure μ_F, for the first return map F through the following general procedure.

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Since μ_f is defined on the interval I which can be identified to the space of leaves of the contracting foliation $\mathcal{F}^s$,

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we may also think of it as a measure on the σ-algebra of Borel subsets of Σ which are union of entire leaves of $\mathcal{F}^s$.

SRB meas. for Lorenz geo. flow-3

Using the fact that F is uniformly contracting on leaves of $\mathcal{F}^s$ we conclude that the sequence

$$F^{*n}(\mu_f), \quad n \geq 1,$$

of push-forward of μ_f under F is weak*-Cauchy:

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Using the fact that F is uniformly contracting on leaves of $\mathcal{F}^s$ we conclude that the sequence

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of push-forward of μ_f under F is weak*-Cauchy: given any continuous $\psi : \Sigma \to \mathbb{R}$

$$\int \psi d(F^{n*}\mu_f) = \int (\psi \circ F^n) d\mu_f, \quad n \geq 1,$$

is a Cauchy sequence in $\mathbb{R}$.

SRB meas. for Lorenz geo. flow-4

Define μ_F to be the weak*-limit of the above sequence, that is,

$$\int \psi d\mu_F = \lim \int \psi d(F^{*n}\mu)$$

for each continuous ψ.

SRB meas. for Lorenz geo. flow-4

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Then μ_F is invariant under F, and it is an ergodic physical measure for F.

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Then μ_F is invariant under F, and it is an ergodic physical measure for F.
The last statement follows from the fact that μ_f is an ergodic physical measure for f, together with the fact that asymptotic time-averages of continuous functions $\psi : \Sigma \to \mathbb{R}$ are constant on the leaves of $\mathcal{F}^s$.

SRB meas. for Lorenz geo. flow-5

Given any point x whose orbit sooner or later will cross Σ we denote with $t(x)$ the first strictly positive time such that $X^{t(x)}(x) \in \Sigma$ (the *return time* of x to Σ). Denote by Σ^* the (full measure) subset of Σ where t is defined.

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Now we show how to construct an physical invariant measure for the flow, *when the return time is integrable*:

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Now we show how to construct an physical invariant measure for the flow, *when the return time is integrable*:

$$\int_{\Sigma^*} t d\mu_F < \infty.$$

SRB meas. for Lorenz geo. flow-5

Denote by $\sim$ the equivalence relation on $\Sigma \times \mathbb{R}$ given by $(w, t(w)) \sim (F(w), 0)$.

Let $N = (\Sigma^* \times \mathbb{R}) / \sim$ and $\nu = \pi_*(\mu_F \times dt)$, where $\pi : \Sigma^* \times \mathbb{R} \to N$ is the quotient map and dt is a Lebesgue measure in $\mathbb{R}$. We have that ν is a finite measure. Let $\phi : N \to \mathbb{R}^3$ be defined by $\phi(w, t) = X^t(w)$ and $\mu_X = \phi_* \nu$.

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$$\frac{1}{T}\int_0^T \psi(X^t(w))dt \to \int \psi d\mu_X \quad \text{as} \quad T \to \infty$$

for every continuous function $\psi : \mathbb{R}^3 \to \mathbb{R}$, and Lebesgue almost every point $w \in \phi(N)$.

SRB for Lorenz geo. flow-6

The Geometric Lorenz flow has integrable return time, hence the above construction for the invariant measure can be applied to it. In fact, there are $K, C > 0$ such that

$$-K^{-1} \cdot \log(d(x,\Gamma)) - C \leq t(x) \leq -K \cdot \log(d(x,\Gamma)) + C.$$

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Proposition The return time is integrable

$$t_0 = \int t d\mu_F < \infty.$$

Existence of a SRB measure

Thus we have:

Theorem The Lorenz geometric flow admits a SRB measure μ_X. Moreover, it can be verified that the support of μ_X is the whole attractor $\Lambda = \cap_{t \geq 0} X^t(U)$.
By construction μ_X admits a disintegration into a.c. conditional measures μ_γ along $\gamma \in \mathcal{F}^{cu}$ such that $\frac{d\mu_\gamma}{dm_\gamma}$ is uniformly bounded from above.

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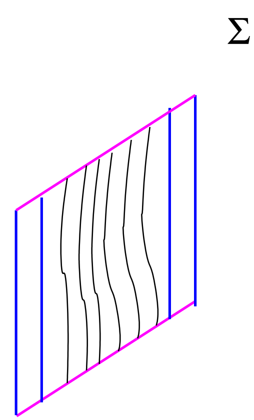

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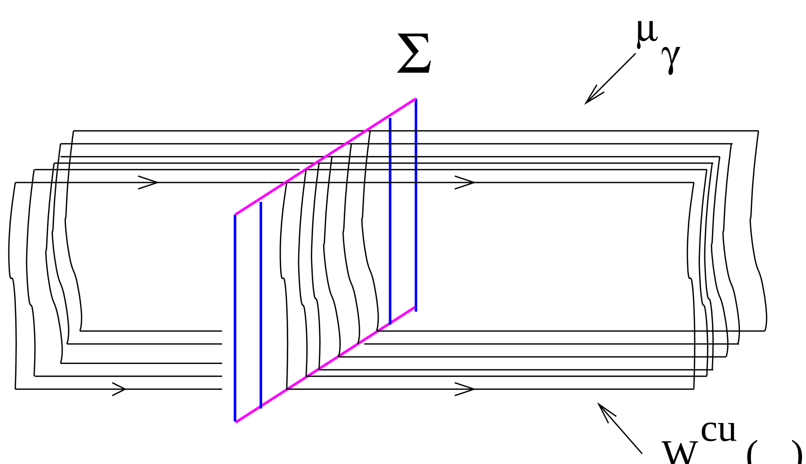

$$\mu(A) = \int \mu_\gamma(A)\, d\hat{\mu}(\gamma)$$

Local dimension for μ

$B_r(x)$: ball with radius r at $x \in \Lambda$.

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This notion was introduced by L-S Young (1982) and characterizes the local geometric structure of an invariant measure with respect to the metric in the phase space of the system.

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Theorem. (Galatolo-Pacifico) For μ-almost every x,

$$\lim_{\mathbf{r}\to\mathbf{0}} \frac{\log \tau_{\mathbf{r}}(\mathbf{x}, \mathbf{x_0})}{-\log \mathbf{r}} = \mathbf{d}_\mu(\mathbf{x_0}) - \mathbf{1}.$$

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Observe that the result above indicates once more the chaoticity of a Lorenz-like attractor: it shows that asymptotically, such attractors behave as an iid system.

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3. Let $x_0 \in \Sigma$ and $\tau_{r,\Sigma}(x, x_0)$ be the time needed to $\mathcal{O}_x$ enter for the first time in $B_r(x_0) \cap \Sigma = B_{r,\Sigma}$.

   Theorem.
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4. Theorem . $d_\mu(x) = d_{\mu_F}(x) + 1$.

Extention: sing-hyp attractors

Next we explain how to proceed in the case of

singular-hyperbolic attractors.

Singular-hyperbolic attractors - I

Consider a C^1 3-dimensional vector field X whose induced flow X_t admits a compact invariant subset Λ such that

- $\exists U$ open nbhd. of Λ satisfying $\Lambda = \cap_{t>0} X_t(U)$ (that is Λ is an *attracting set*).

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- $\exists z \in \Lambda$ s.t. $X(z) \neq 0$ (i.e. z is a *regular point* for X) and its orbit $\{X_t(z) : t > 0\}$ is dense in Λ (that is Λ is also an *attractor*).

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Moreover assume that

- Λ contains some (or several) non-degenerate (persistent) singularity σ of X (i.e. $X(\sigma) = 0$)

that is, Λ is a *singular-attractor:* an attracting set containing a dense orbit and singularities.

Singular-hyperbolic attractors - II

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- $\exists$ splitting $T_z M = E^s_z \oplus E^c_z$ with $\dim(E^s_z) = 1$ and $\dim(E^c_z) = 2$, and $\lambda \in (0,1)$ and $c > 0$ such that for all $x \in \Lambda$ and $t > 0$ we have

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— *domination:* $\|DX_t \mid E^s_x\| \|DX_{-t} \mid E^c_{X_t(x)}\| < c\lambda^t$

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- $\exists$ splitting $T_z M = E^s_z \oplus E^c_z$ with $\dim(E^s_z) = 1$ and $\dim(E^c_z) = 2$, and $\lambda \in (0,1)$ and $c > 0$ such that for all $x \in \Lambda$ and $t > 0$ we have

— *domination:* $\|DX_t \mid E^s_x\| \|DX_{-t} \mid E^c_{X_t(x)}\| < c\lambda^t$

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— *the volume along E^c sub-bundle is uniformly expanded:*

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Singular-hyperbolic attractors - III

Moreover the *singularities* are all *Lorenz-like*: $DX(\sigma)$ has *three distinct eigenvalues* $\lambda_1, \lambda_2, \lambda_3$ such that

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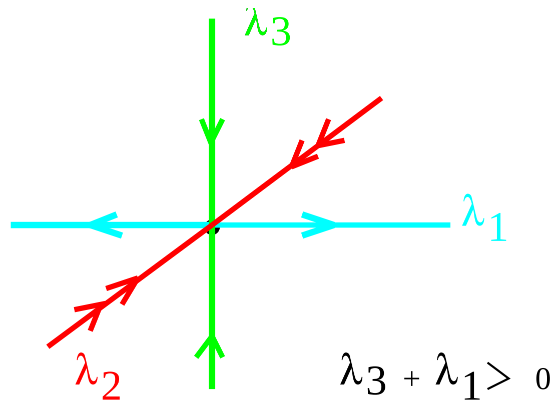

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- Λ is partially hyperbolic : $T_\lambda = E^s \oplus E^{cu}$,
- E^s 1-dimensional and uniformly contracting,
- E^{cu} 2-dimensional, contains the direction of the flow, and it is volume expanding.

Existence of a physical measure

Theorem B. *Let* Λ *be a singular-hyperbolic* attractor. *Then* Λ *supports a unique physical probability measure* μ *which is ergodic, hyperbolic and its ergodic basin covers a full Lebesgue measure subset of the topological basin of attraction, i.e.* $B(\mu) = W^s(\Lambda)$, $m \bmod 0$.

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The *hyperbolicity of* μ means that *along the* E^c *direction there exists a positive Lyapunov exponent along a measurable sub-bundle* $E^u \subsetneq E^c$ (the exponent along the flow direction is zero and along the E^s direction is negative).

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Theorem B is another statement of *sensitiveness of the dynamics* of X on Λ, since the presence of a positive Lyapunov exponent implies that orbits of infinitesimally close points tend to move apart from each other.

The physical measure is a Gibbs state

We say that μ *has an absolutely continuous disintegration along the center-unstable direction* if *for every given $x \in \Lambda$, each δ-adapted foliated neighborhood $\Pi_\delta(x)$ of x induces a disintegration $\{\mu_\gamma\}_{\gamma \in \Pi_\delta(x)}$ of $\mu \mid \hat{\Pi}_\delta(x)$, for all small enough $\delta > 0$, such that $\mu_\gamma \ll m_\gamma$ for $\hat{\mu}$-a.e. $\gamma \in \Pi_\delta(x)$* .

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Theorem C. *Let Λ be a singular-hyperbolic attractor for a C^2 three-dimensional flow. Then the physical measure μ supported in Λ has a disintegration into absolutely continuous conditional measures μ_γ along center-unstable surfaces $\gamma \in \Pi_\delta(x)$ such that $\frac{d\mu_\gamma}{dm_\gamma}$ is positive and uniformly bounded from above, for all δ-adapted foliated neighborhoods $\Pi_\delta(x)$ and every $\delta > 0$. Moreover $\operatorname{supp}(\mu) = \Lambda$.*

Entropy Formula for physical measure

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$$h_\mu(X_1) = \int \log \left| \det(DX_1 \mid E^{cu}) \right| d\mu = \int \log \|DX_1 \mid F_z\| \, d\mu(z).$$

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All these are valid for the *original Lorenz system* and for the *Geometric Lorenz attractors*.
This property is shared by every ***uniformly hyperbolic attractor***.

Method of proof to get SRB measure

The proofs are based on constructing a finite cover of the compact set Λ by flow-boxes through convenient cross-sections of the flow near Λ.

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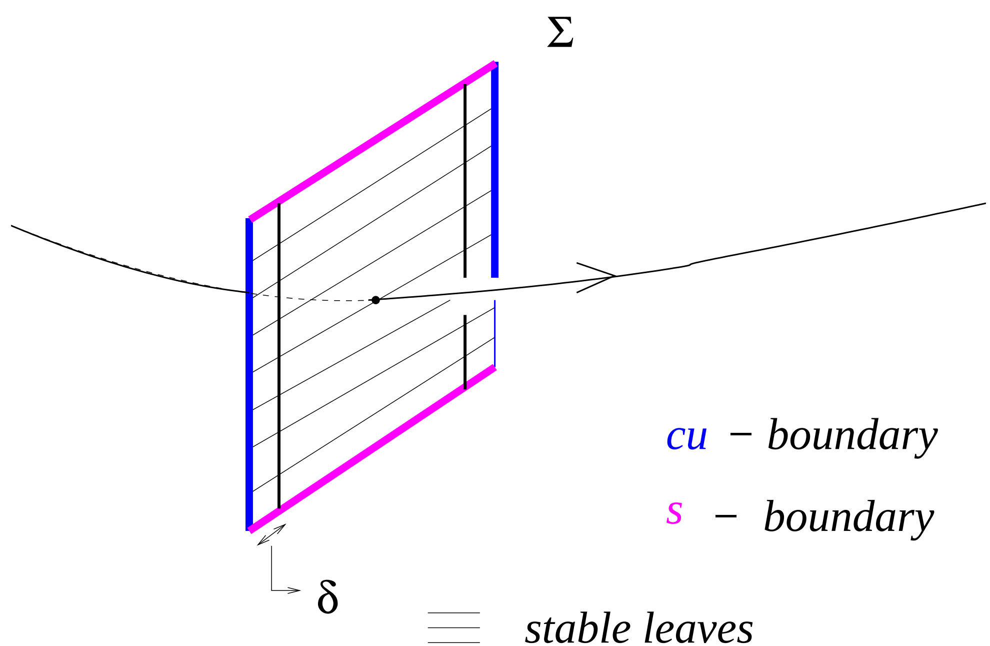
These cross-sections are *δ-adapted* for a small $\delta > 0$ as in the figure below.



Adapted cross-sections

Adapted cross-sections exist by the following property of singular-hyperbolic attractors.

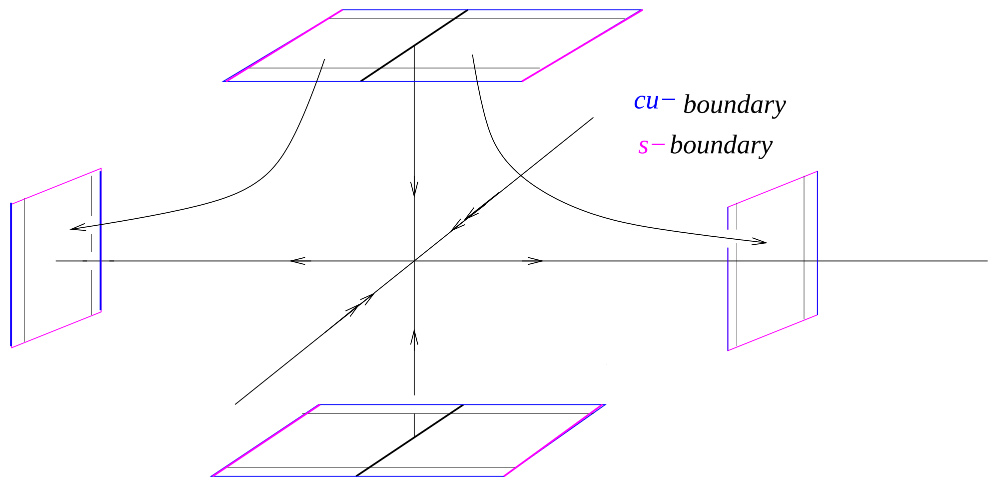
Lemma *Let* σ *be a singularity of a singular-hyperbolic attractor* Λ*. Then*

$$W^{ss}(\sigma) \cap \Lambda = \{\sigma\}.$$

Note: recall that σ is *Lorenz-like* and so it has a 1-dimensional W^u and a 2-dimensional W^s containing a 1-dimensional strong-stable manifold W^{ss}.

Adapted cross-sections near singularities

In a neighborhood of a singularity we consider the following ingoing and outgoing adapted cross-sections



The global Poincaré return map I

After having fixed a cover of Λ by such flow-boxes through adapted cross-sections, we consider the map R given by taking any point x in one cross-section and looking at the ***first return of*** $X_T(x)$ ***to some cross-section,*** for a fixed big value of $T > 0$.

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This value of $T > 0$ is chosen to *take advantage of the volume expanding property along the center-unstable direction*.

The global Poincaré return map II

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For a big T the return map admit an invariant (whenever R is defined) *uniformly expanding cone around the center-unstable direction restricted to the cross-sections*.

Moreover *the stable leaves inside each cross-section are send by the return map strictly inside the stable leaves in the image cross-section.* This is the key property in our arguments.

A 1-dimensional map

From this construction, one gets a transitive, α-Hölder, 1-dimensional expanding map $f : I \to I$ with ∞-many discontinuities and s. t. $1/|f'|$ has generalized bounded variation:

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Theorem. (Keller) Let $f : I \to I$ be C^1 piecewise expanding map such that $g = 1/|f'|$ is generalized BV. Then f has a finitely many absolutely continuous ergodic invariant measures.

End of the second lecture

Many thanks.

We shall continue tomorrow, at 9 AM.