Lorenz like flows

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Main goals

The main goal is to explain the results (Galatolo-P)

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Remark. Theorems A and B hold for a more general class of flows, defined axiomatically.

Definitions

Recall:

the local dimension of a μ at $x \in M$ is

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The hitting time $\tau_r(x, x_0)$ is the time needed for the orbit of a point x to enter for the first time in a ball $B_r(x_0)$ centered at x_0, with small radius r.

Plan of the talks

- Motivation

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- Hitting and recurrence time
- Proof of Theorems A and B.

Lorenz attractor

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Lorenz equations:

$$X(x,y,z) = \begin{cases} \dot{x} = -10 \cdot x + 10 \cdot y \\ \dot{y} = 28 \cdot x - y - x \cdot z \\ \dot{z} = -8/3 \cdot z + x \cdot y\,. \end{cases}$$

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The classical parameters are : $\alpha = 10, \; r = 28, \; b = 8/3.$

Easy facts about Lorenz's equations

- $(0,0,0)$ is an equilibrium and the eigenvalues of $DX^t((0,0,0,))$ are real numbers satisfying

$$-\lambda_{\mathbf{2}} > \lambda_{\mathbf{1}} > -\lambda_{\mathbf{3}} > \mathbf{0} \quad \text{so} \quad \lambda_1 + \lambda_3 > 0.$$

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- The above properties are robust.

Numerical integration

Numerical experiments by Lorenz showed: the solutions of the Lorenz's equations support a zero volume attractor that has a geometric structure like a butterfly.

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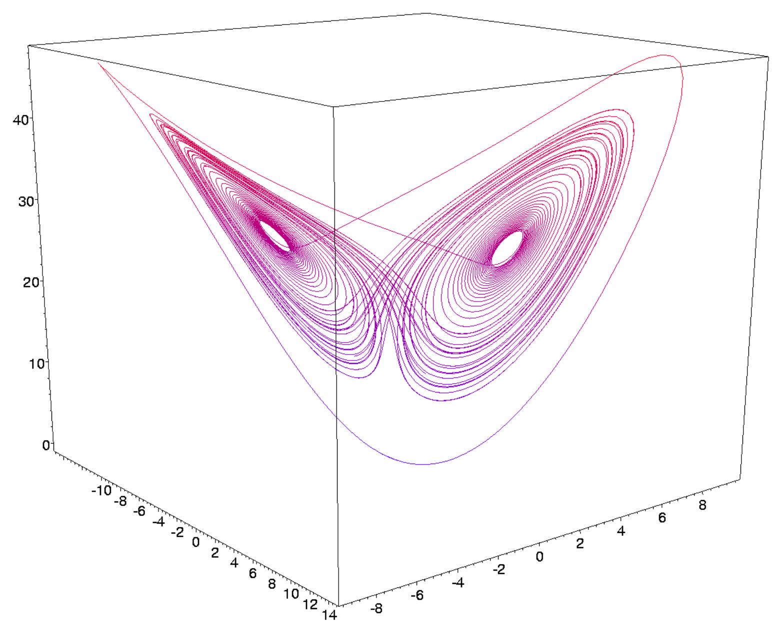

Attractor

An attractor is a bounded region in phase-space invariant under time evolution, such that the forward trajectories of most (positive probability) or, even, all nearby points converge to it.

An attractor is strange if trajectories converging to it are sensitive with respect to initial data: trajectories of any nearby points get apart under forward iteration by the flow.

Lorenz's conjecture

Based on his experiments, he conjectured the existence of a chaotic attractor with zero volume for the flow generated by the Lorenz's equations.

Chaotic : it has sensibility with respect to initial data: forward iteration of nearby points get far apart.

No explicity solutions

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- conceitual: the presence of an equilibrium accumulated by regular orbits prevents the Lorenz'attractor from being hyperbolic.
- numerical: solutions slow down through the passage near the equilibrium, which means unbounded return times and thus unbounded integration errors.

Hyperbolic sets

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1. E^s_Λ *is* (K, λ)-*contracting*, i.e.

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Geometrical models

The impossibility of solving the equations leads Afraimovich-Bykov-Shil'nikov and Guckenheimer-Williams, independently (in the seventies), to proposed a geometrical model for the behavior of X^t generated by the Lorenz' equations:

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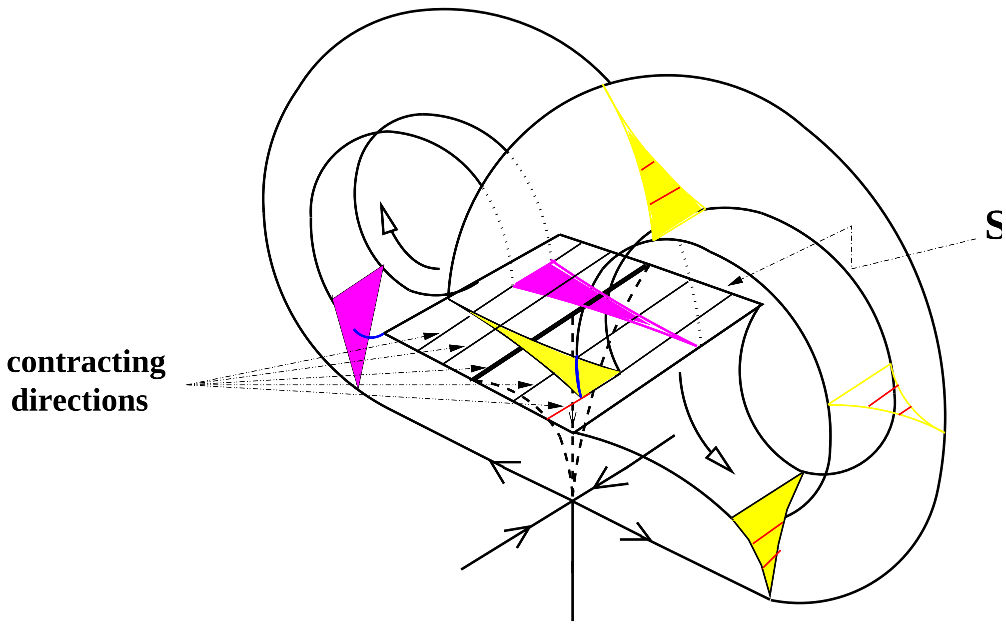

The equilibrium σ

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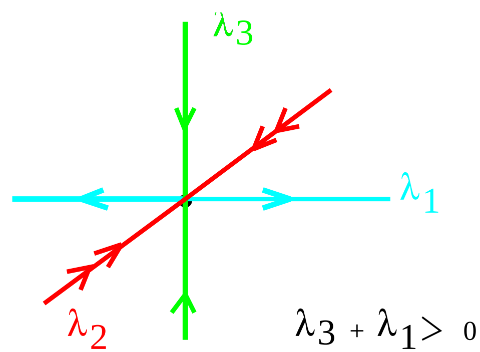

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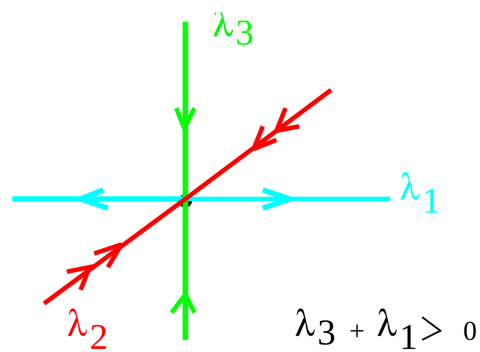

In this case $\frac{1}{2} < \alpha = -\frac{\lambda_3}{\lambda_1} < 1 < \beta = -\frac{\lambda_2}{\lambda_1}$.

Main hypothesis for a Geom. Model

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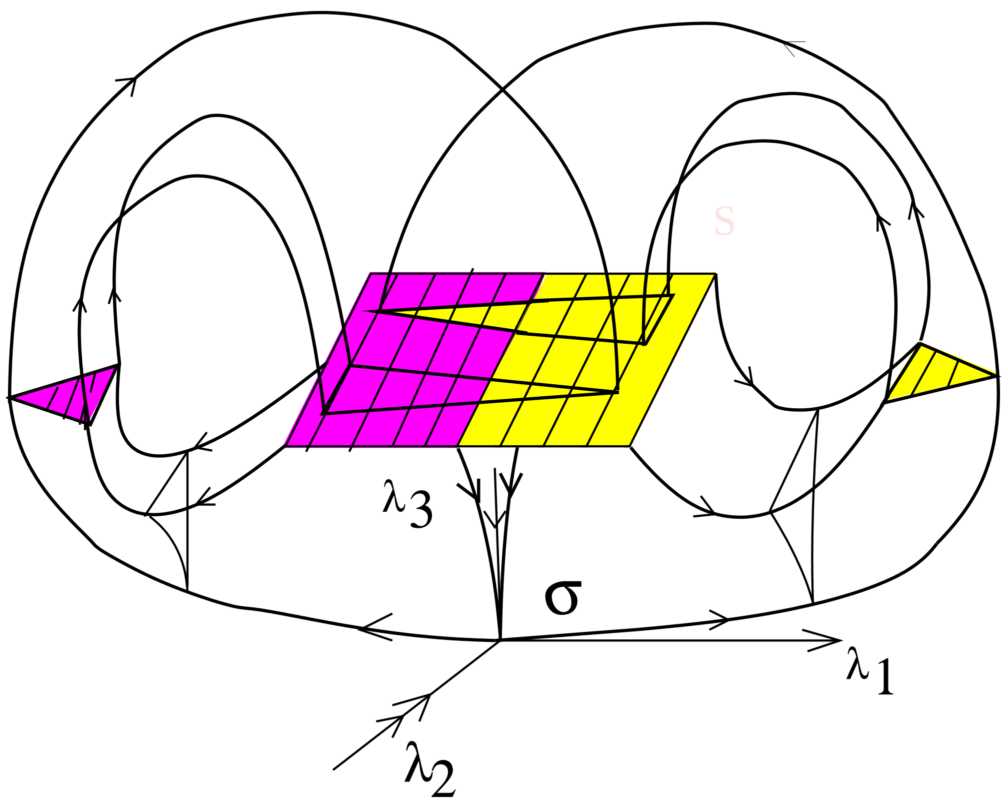

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Λ : is not hyperbolic: exists a singularity accumulated by regular orbits.

Λ is robust: can not be destroyed by small perturbations of the model.

How to construct a geo model

Start with a linear vector field whose orbits are as in the figure:

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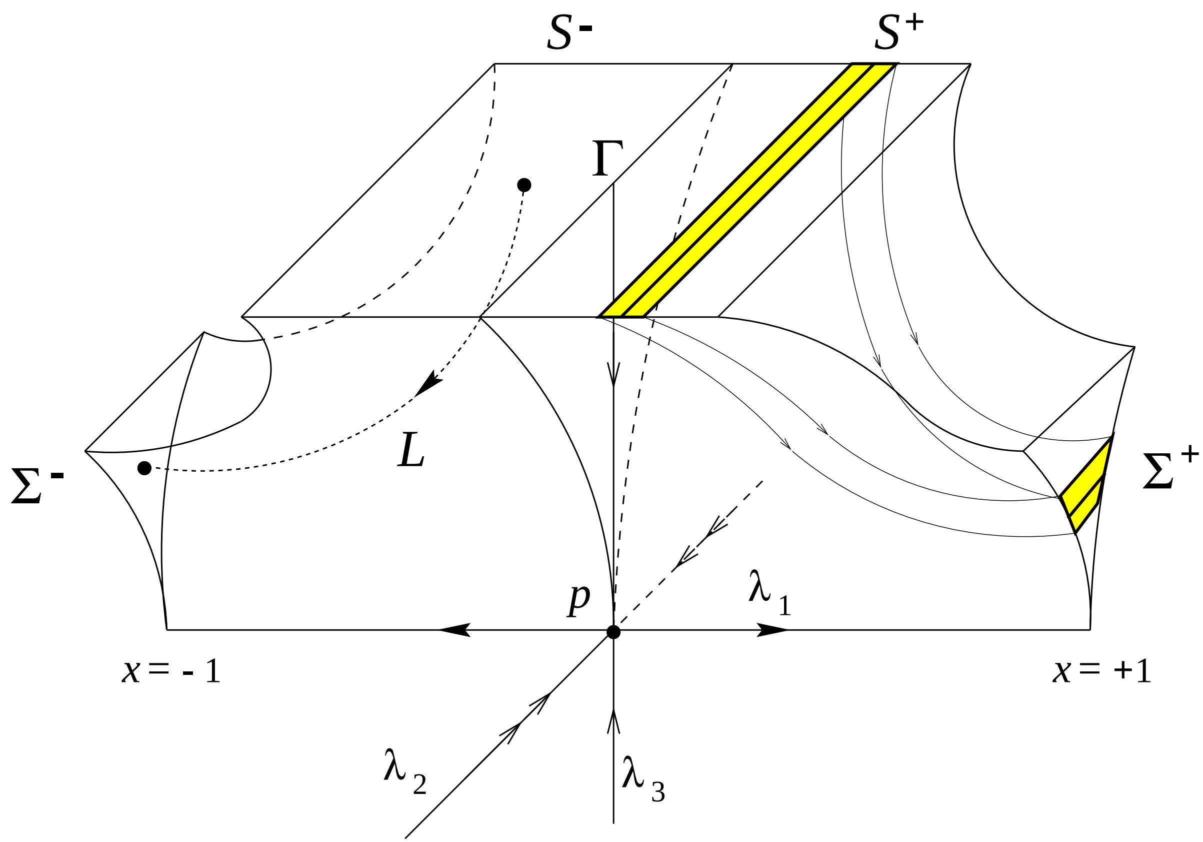

The model and first hit map F

Let L be the linear map, $L(S^*) = \Sigma^+ \cup \Sigma^-$.

$\Sigma^\pm$ should return to S through a flow described by a suitable composition of a rotation $R_\pm$, an expansion $E_{\pm\theta}$ and a translation $T_\pm$.

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Letting $F = E_\theta \circ R \circ T \circ L$, we have $F : S^\star \to S$ and since this composition preserves vertical lines of S we have that $\mathcal{F}^s(S) = \cup\{x = x_0\}$ are invariant by F and F is not defined on $\Gamma = S \cap W^s(\sigma)$.

An expression for F

$$F(x,y) = \big(f(x), g(x,y)\big) \quad \text{with} \quad f, \ g \text{ as:}$$

$$f(x) = \begin{cases} f_1(x^\alpha) & x < 0 \\ f_0(x^\alpha) & x > 0 \end{cases}$$

where $f_i = (-1)^i \theta \cdot x + b_i, i \in \{0,1\}$

$$g(x,y) = \begin{cases} g_1(x^\alpha, y \cdot x^\beta) & x < 0 \\ g_0(x^\alpha, y \cdot x^\beta) & x > 0, \end{cases}$$

$g_1 | I^- \times I \to I$ and $g_0 | I^+ \times I \to I$ are affine maps.

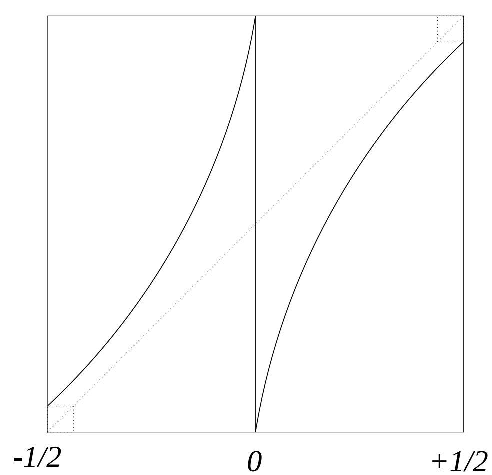
Properties of f

Set $I[-1/2, 1/2]$. The main properties of $f : I \setminus \{0\} \to I$:

(f1) $f(0_-) = 1/2$ and $f(0^+) = -1/2$

(f2) f is differentiable on $[-1/2, 1/2] \setminus \{0\}$, and $|f'(x)| > \sqrt{2}$

(f3) $f'(0^-) = +\infty$ and $f'(0^+) = -\infty$.



Consequences of (f1)–(f3)

Lemma(f is leo) Let $f : I \setminus \{0\} \to I$ satisfying (f1)-(f3). Then f is locally eventually onto: for any open $J, \ 0 \notin J$, $\exists$ an interval $J_0 \subset J$ and n s. t. $f^n \mid J_0 = f(I)$.

Proof that f is leo

Proof. Pick $J = J_0 \subset I$ and let $\eta = \inf |f'|$.

(a) $0 \notin J_0 \to \ell(f(J_0)) > \eta \cdot \ell(J_0)$.

(b) if $0 \notin f(J_0)$, put $J_1 = f^2(J_0)$. Then $\ell(J_1) > \eta^2 \cdot \ell(J_0)$.

(c) if $0 \in f(J_0)$ then $f^2(J_0) = I^- \cup I^+$ with

$$\ell(I^+) > \frac{\ell(f^2(J_0))}{2} > \eta^2 \cdot \frac{\ell(J_0)}{2} > \eta \cdot \ell(J_0)$$

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In the last case, replace J_0 by I^+ and re-start. As $\ell(I) < \infty$ and $\eta > 1$ iterations of the biggest connected component of $f^2(J_0)$ ends after finitely many steps with the interval $f(I)$.

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Each branch of f_{Lo} is the composition of an affine map with x^α then it is a convex function. Hence, the derivative f'_{Lo} is monotonic on each branch, implying that $(f'_{lo})^{-1}$ is also monotonic. On the other hand, $(f'_{Lo})^{-1}$ is bounded because $f'_{lo} > 1$. Thus $(f'_{Lo})^{-1}$ is monotonic and bounded and hence is BV.

Statistical properties of f_{Lo}

Theorem.(statistical properties) f admits a unique SBR measure μ_f.
Moreover $d\mu_f/dm$ is a BV function,
and f has exponential decay of correlations for L^1 and BV observables: for each n and observables f, g it holds:

$$|\int g(F^n(x))f(x)dm - \int g(x)d\mu \int f(x)dm| \leq C \cdot \|g\|_{L_1} \cdot \|f\| \cdot e^{-\lambda n}$$

Properties of the map $g(x, y)$

By construction g is piecewise C^2.

(a) For all $(x, y) \in \Sigma^*, x > 0$, we have $\partial_y g(x, y) = x^\beta$. As $\beta > 1$, $|x| \leq 1/2$, there is $0 < \lambda < 1$ such that

$$|\partial_y g| < \lambda.$$

The same bound works for $x < 0$.

(b) For all $(x, y) \in \Sigma^*, x \neq 0$, we have $\partial_x g(x, y) = \beta \cdot x^{\beta - \alpha}$. As $\beta - \alpha > 0$ and $|x| \leq 1/2$, we get

$$|\partial_x g| < \infty.$$

F preserves the vertical lines

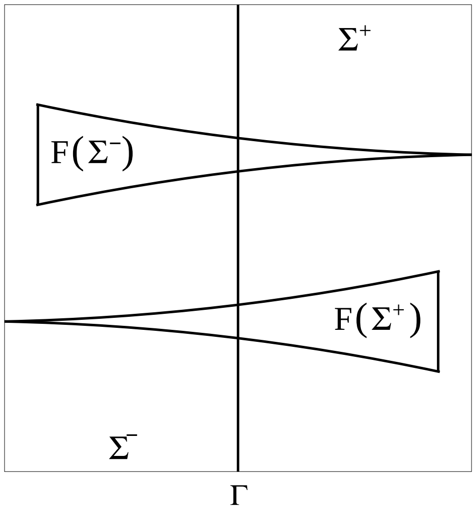
Item (a) above implies that $F(x, y) = (f(x), g(x, y))$ is uniformly contracting on the leaves of $\mathcal{F}^s$:

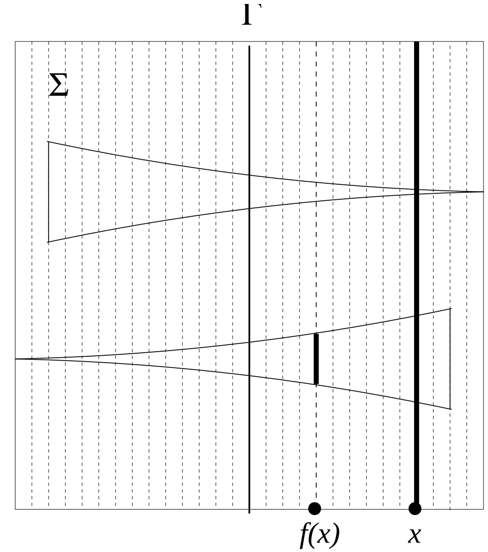
there are constants $\lambda < 1$ and $C > 0$ such that

$(\star\star)$ if γ is a leaf of $\mathcal{F}^s$ and $x, y \in \gamma$ then

$$\operatorname{dist}\big(F^n(x), F^n(y)\big) \leq \lambda^n \cdot C \cdot \operatorname{dist}(x, y).$$

Image $F(S)$





P-H with volume expanding E^{cu}

Λ is partially-hiperbolic if $T_\Lambda M = E^s \oplus E^{cu}$ such that

- E^s is uniformly contracting,
- $E^s \oplus E^{cu}$ is a dominated splitting: there are $0 < \lambda < 1$, $c > 0$, and $T_0 > 0$ such that

$$\|DY^T \mid E^s_p\| \cdot \|DY^{-T} \mid E^{cu}_{Y^T(p)}\| < c \cdot \lambda^T.$$

- E^{cu} is volume expanding, that is, for $x \in \Lambda$ and $t \in R$

  $J^c_t(x)$: absolute value of the determinant of $DX_t(x)/E^c_x : E^c_x \to E^c_{X_t(x)}$.

$$J^c_t(x) \geq Ke^{\lambda t}, \forall x \in \Lambda\ t \geq 0$$

Geo. model is P-H

The splitting of $\mathbb{R}^3$: $E = \mathbb{R} \times \{(0,0\}$ and $F = \{0\} \times \mathbb{R}^2$, is preserved by DX : $DX^t_w \cdot E = E$ and $DX^t_w \cdot F = F$ for all t and every point w in an orbit inside the trapping ellipsoid.

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1. $\|DX_w^t \mid E\| = e^{\lambda_2 t}$;
2. $\|DX_w^t \mid E\| = e^{(\lambda_2 - \lambda_3)t} \cdot m\big(DX^t \mid F\big)$,

where $m\big(DX^t \mid F\big)$ is the minimum norm of the linear map. Since $\lambda_2 < 0$ we see that E is uniformly contracting, this is a stable direction.

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Observe also that since $\lambda_1 + \lambda_3 > 0$ we have that $|\det DX^t \mid F| = e^{(\lambda_1+\lambda_3)t}$ and so the flow expands volume along the F direction.

P-H outside neig. origin

If the orbit of w passes outside the linear region k times from Σ to S lasting $s_1 + \cdots + s_k$ from time 0 to time t, then $t > s_1 + \cdots + s_k$ and $\exists b > 0$ bounding the derivatives of DX^t from 0 to t_0 and so

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$$\|DX^t_w \mid E\| \leq$$

$$e^{bk+\lambda_2(t-s_1-\cdots-s_k)} = \exp\left\{\lambda_2 t\big(1 - \frac{bk}{\lambda_2 t} - \frac{s_1 + \cdots + s_k}{t}\big)\right\},$$

and the last expression in brackets is bounded. We see that E is (K, λ_2)-contracting for some $K > 0$.

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Thus the maximal positively invariant set in the trapping ellipsoid is partially hyperbolic and the flow expands volume along a bi-dimensional invariant direction.
The geometrical model is the most significant example of a new class of attractors: singular-hyperbolic attractors.

Singular-hyperbolic attractor

An attractor Λ for X^t is singular-hyperbolic if

- all singularities contained in Λ are hyperbolic
- *p*arcially hiperbolic with central direction **volume expanding**.

The solution for Lorenz conjecture

- (Tucker): The original Lorenz equations do exhibit a robust strange attractor that is partially hyperbolic with volume expanding central direction.

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In another words, the flow given by the equations

$$X(x,y,z) = \begin{cases} \dot{x} = -10 \cdot x + 10 \cdot y \\ \dot{y} = 28x - y - x \cdot z \\ \dot{z} = -\frac{8}{3}z + x \cdot y \,, \end{cases}$$

presents a chaotic attractor. It can be seen that this attractor is singular-hyperbolic.

End of the first lecture

Many thanks.

We shall continue tomorrow, at 9 AM.