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THE SPECTRUM OF THE ALMOST MATHIEU OPERATOR

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Abstract. These notes are based on a series of six lectures, given during my stay at the CRC 701 in June/July 2008. The lecture series intended to give a survey of some of the results for the almost Mathieu operator that have been obtained since the early 1980's. Specifically, the metal-insulator transition is discussed in detail, along with its relation to the ten Martini problem via duality and reducibility.

1. Introduction and Overview

1.1. **The Operator and the Main Results.** We will study the almost Mathieu operator

$$[H_\omega^{\lambda,\alpha}\psi](n) = \psi(n+1) + \psi(n-1) + 2\lambda\cos(2\pi(\omega + n\alpha))\psi(n).$$

The potential $2\lambda\cos(2\pi(\omega + n\alpha))$ is periodic if $\lambda = 0$ or α is rational and hence we will only consider the case where $\lambda \neq 0$ and α is irrational. Furthermore, by periodicity of the cosine, we consider α and ω as elements of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Finally, it is also easy to see that $H_\omega^{\lambda,\alpha} = H_{\omega+\frac{1}{2}}^{-\lambda,\alpha}$. Thus, we will only consider: $\lambda > 0$, $\alpha \in \mathbb{T}$ irrational, $\omega \in \mathbb{T}$.

The two theorems below concern the metal-insulator transition and the ten Martini problem, that is, an exact description of the spectral type of the operator, exhibiting a remarkably strict transition at $\lambda = 1$, and the Cantor structure of the spectrum. They are stated in the generality in which they are currently known and summarize the results of many authors, obtained over the course of about three decades.

Theorem 1 (Metal-Insulator Transition)**.** (a) *If $\lambda < 1$, then for every α and every ω, the spectrum is purely absolutely continuous.*
(b) *If $\lambda = 1$, then for every α and all but countably many ω, the spectrum is purely singular continuous.*
(c) *If $\lambda > 1$, then for almost every α and almost every ω, the spectrum is pure point and the eigenfunctions decay exponentially.*
(d) *If $\lambda > 1$, then for generic α and every ω, the spectrum is purely singular continuous.*
(e) *If $\lambda > 1$, then for every α and generic ω, the spectrum is purely singular continuous.*

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Theorem 2 (Ten Martini Problem)**.** *The spectrum of $H_\omega^{\lambda,\alpha}$ is a Cantor set, that is, it is closed and it contains no isolated points and no intervals.*

We will present many of the main ideas that go into the proof of these theorems. Some of the statements above will be proved completely here, while the proof of others will only be sketched. In the next subsection, we state the results we discuss in more depth in subsequent sections.

1.2. **A Quick Guide to Proving the Main Results.** Consider the Hilbert space $L^2(\mathbb{T} \times \mathbb{Z})$ and the operator $H^{\lambda,\alpha} : L^2(\mathbb{T} \times \mathbb{Z}) \to L^2(\mathbb{T} \times \mathbb{Z})$ given by

$$[H^{\lambda,\alpha}\varphi](\omega, n) = \varphi(\omega, n+1) + \varphi(\omega, n-1) + 2\lambda \cos(2\pi(\omega + n\alpha))\varphi(\omega, n).$$

Introduce the duality transform $\mathcal{A} : L^2(\mathbb{T} \times \mathbb{Z}) \to L^2(\mathbb{T} \times \mathbb{Z})$, which is given by

$$[\mathcal{A}\varphi](\omega, n) = \sum_{m \in \mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega + n\alpha)m} e^{-2\pi i n\eta} \varphi(\eta, m)\, d\eta.$$

This definition assumes initially that φ is such that the sum in m converges, but note that in terms of the Fourier transform on $L^2(\mathbb{T} \times \mathbb{Z})$, we have $[\mathcal{A}\varphi](\omega, n) = \hat{\varphi}(n, \omega + n\alpha)$, which may be used to extend the definition to all of $L^2(\mathbb{T} \times \mathbb{Z})$ and shows that $\mathcal{A}$ is unitary.

Theorem 3 (Gordon-Jitomirskaya-Last-Simon 1997)**.** *Suppose $\lambda > 0$ and $\alpha \in \mathbb{T}$ is irrational.*
(a) *We have $H^{\lambda,\alpha}\mathcal{A} = \lambda \mathcal{A} H^{\lambda^{-1},\alpha}$.*
(b) *If $H_\omega^{\lambda,\alpha}$ has pure point spectrum for almost every $\omega \in \mathbb{T}$, then $H_\omega^{\lambda^{-1},\alpha}$ has purely absolutely continuous spectrum for almost every $\omega \in \mathbb{T}$.*
(c) *If $H_\omega^{\lambda,\alpha}$ has some point spectrum for almost every $\omega \in \mathbb{T}$, then $H_\omega^{\lambda^{-1},\alpha}$ has some absolutely continuous spectrum for almost every $\omega \in \mathbb{T}$.*

Definition 4. *Fix λ and α. For $E \in \mathbb{C}$, the Lyapunov exponent is given by*

$$\gamma(E) = \inf_{n \ge 1} \frac{1}{n} \int \log \|M_E(n, \omega)\|\, d\omega = \lim_{n \to \infty} \frac{1}{n} \log \|M_E(n, \omega)\| \text{ for } \mu\text{-almost every } \omega \in \Omega,$$

where

$$M_E(n, \omega) = T_E(n, \omega) \times \cdots \times T_E(1, \omega), \quad T_E(m, \omega) = \begin{pmatrix} E - 2\lambda \cos(2\pi(\omega + m\alpha)) & -1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 5 (Herman 1983)**.** *The Lyapunov exponent obeys $\gamma(E) \ge \log \lambda$.*

Definition 6. *An irrational number $\alpha \in \mathbb{T}$ is called Liouville if there is a sequence of rational numbers $\frac{p_k}{q_k}$ with $q_k \to \infty$ such that*

$$\left| \alpha - \frac{p_k}{q_k} \right| < k^{-q_k}.$$

Theorem 7 (Avron-Simon 1982)**.** *Suppose $\alpha \in \mathbb{T}$ is Liouville. Then, for every λ and ω, $H_\omega^{\lambda,\alpha}$ has purely continuous spectrum.*

Theorem 8 (Jitomirskaya-Simon 1994)**.** *For every λ and α, $H_\omega^{\lambda,\alpha}$ has purely continuous spectrum for generic ω.*

Definition 9. *An irrational number $\alpha \in \mathbb{T}$ is called Diophantine if there are constants $c = c(\alpha) > 0$ and $r = r(\alpha) > 1$ such that*

$$|\sin(2\pi n\alpha)| > \frac{c}{|n|^r} \quad \text{for every } n \in \mathbb{Z} \setminus \{0\}.$$

Given such an α, $\omega \in \mathbb{T}$ is called resonant if the relation

$$\left|\sin\left(2\pi\left(\omega + \frac{n}{2}\alpha\right)\right)\right| < \exp\left(-|n|^{\frac{1}{2r}}\right)$$

holds for infinitely many $n \in \mathbb{Z}$; otherwise ω is called non-resonant.

It is known that Lebesgue almost every α is Diophantine. Moreover, the set of resonant ω's is a dense G_δ set (as can be seen directly from the definition) of zero Lebesgue measure (by Borel-Cantelli).

Theorem 10 (Jitomirskaya 1999)**.** *Suppose $\lambda > 1$, $\alpha \in \mathbb{T}$ is Diophantine, and $\omega \in \mathbb{T}$ is non-resonant. Then, the almost Mathieu operator $H_\omega^{\lambda,\alpha}$ has pure point spectrum with exponentially decaying eigenfunctions.*

Theorem 11 (Puig 2004)**.** *Suppose α is Diophantine and $\lambda \neq 1$. Then, $\Sigma^{\lambda,\alpha}$ is a Cantor set.*

2. The Herman Estimate and Aubry Duality

2.1. **The Herman Estimate.** The spectral type of $H_\omega^{\lambda,\alpha}$ can be studied by looking at the solutions of the time-independent Schrödinger equation:

$$u(n+1) + u(n-1) + 2\lambda\cos(2\pi(\omega + n\alpha))u(n) = Eu(n).$$

Notice that u solves this equation if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M_E(n,\omega)\begin{pmatrix} u(1) \\ u(0) \end{pmatrix},$$

where the transfer matrix $M_E(n,\omega)$ is given (at least for $n \geq 1$) by

$$M_E(n,\omega) = T_E(n,\omega) \times \cdots \times T_E(1,\omega), \quad T_E(m,\omega) = \begin{pmatrix} E - 2\lambda\cos(2\pi(\omega + m\alpha)) & -1 \\ 1 & 0 \end{pmatrix}.$$

Here we leave the dependence on λ and α implicit as only ω will be varied in this section. Thus, decay or growth of solutions is closely related to growth (of the norm) of the transfer matrices. To measure the growth on an exponential scale, one introduces the Lyapunov exponent. Initially, consider an ω-average and define

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|M_E(n,\omega)\|\, d\omega.$$

Clearly, $\gamma(E) \geq 0$ since the matrices $M_E(n,\omega)$ have determinant one and hence norm at least one. The existence of the limit defining $\gamma(E)$ follows from Kingman's subadditive ergodic theorem. In fact, this theorem also shows that

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \log \|M_E(n,\omega)\|$$

for almost every $\omega \in \mathbb{T}$. For every such ω, Osceledec' theorem then shows that if $\gamma(E) > 0$, there is a one-dimensional subspace of $(u(1), u(0))^T$ for which the norm of $(u(n+1), u(n))^T$ decays like $e^{-\gamma(E)n}$ as $n \to \infty$ and all linearly independent initial conditions yield $e^{\gamma(E)n}$ growth.

Theorem 12 (Herman 1983)**.** *We have $\gamma(E) \geq \log\lambda$ for every E.*

Proof. Setting $w = e^{2\pi i\omega}$, we see that

$$2\lambda\cos(2\pi(\omega + m\alpha)) = \lambda\left(e^{2\pi i\alpha m}w + e^{-2\pi i\alpha m}w^{-1}\right).$$

Thus, the one-step transfer matrices have the form

$$T_E(m,\omega) = \begin{pmatrix} E - \lambda\left(e^{2\pi i\alpha m}w + e^{-2\pi i\alpha m}w^{-1}\right) & -1 \\ 1 & 0 \end{pmatrix}$$

If we define

$$N_n(w) = w^n M_E(n,\omega) = (wT_E(n,\omega))\cdots(wT_E(1,\omega)),$$

initially on $|w| = 1$, we see that N_n extends to an entire function and hence $w \mapsto \log\|N_n(w)\|$ is subharmonic. Thus,

$$\int_0^1 \log\|N_n(e^{2\pi i\omega})\|\,d\omega \ge \log\|N_n(0)\| = n\log\lambda.$$

Moreover, $\|N_n(e^{2\pi i\omega})\| = \|M_E(n,\omega)\|$. Thus,

$$\begin{aligned}\gamma(E) &= \lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}}\log\|M_E(n,\omega)\|\,d\omega\\ &= \lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}}\log\|N_n(e^{2\pi i\omega})\|\,d\omega\\ &\ge \log\lambda,\end{aligned}$$

as claimed. □

2.2. **Aubry Duality.** The Herman estimate suggests strongly that there are exponentially decaying solutions of the time-independent Schrödinger equation when $\lambda > 1$. Notice, however, that we can infer this only for almost every ω for any given E and, moreover, we have treated only exponential decay near $+\infty$. One may perform a similar analysis near $-\infty$, but even if one finds two initial conditions which yield exponential decays at $\pm\infty$, it is not clear whether they coincide and give rise to a solution u which decays at both ends and hence is an exponentially decaying eigenfunction corresponding to the eigenvalue E.

Nevertheless, it is indeed sometimes possible to show the existence of genuine eigenfunctions u, this will be discussed in Section 4. Aubry duality then constructs from such solutions corresponding solutions for the dual coupling constant λ^{-1} and the dual energy $\lambda^{-1}E$. The special form of these dual solutions will suggest that the spectral measures for λ^{-1} should have some absolutely continuous component.

Let us demonstrate how this works. The relevant equations are

$$u(n+1) + u(n-1) + 2\lambda\cos(2\pi(\omega + n\alpha))u(n) = Eu(n), \tag{1}$$

and the dual difference equation

$$\tilde{u}(n+1) + \tilde{u}(n-1) + 2\lambda^{-1}\cos(2\pi(\tilde{\omega} + n\alpha))\tilde{u}(n) = (\lambda^{-1}E)\tilde{u}(n). \tag{2}$$

Lemma 13. (a) *Suppose $u \in \ell^1(\mathbb{Z})$ is a solution of* (1). *Consider its Fourier transform*

$$\hat{u}(\theta) = \sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m\theta}.$$

Then, given any $\tilde{\omega} \in \mathbb{T}$, the sequence $\tilde{u}$ defined by

$$\tilde{u}(n) = \hat{u}(\tilde{\omega} + n\alpha)e^{2\pi i n\omega} \tag{3}$$

is a solution of (2).

(b) *Suppose $u \in \ell^2(\mathbb{Z})$ is a solution of* (1). *Then, for $\tilde\omega$ from a full-measure subset of $\mathbb{T}$, the sequence $\tilde u$ defined by* (3) *is a solution of* (2).

Proof. (a) If $u \in \ell^1(\mathbb{Z})$, $\hat u \in C(\mathbb{T})$ and we can evaluate it pointwise. We have

$$\begin{aligned}
&(\lambda^{-1}E)\tilde u(n)\\
&= (\lambda^{-1}E)\hat u(\tilde\omega + n\alpha)e^{2\pi i n\omega}\\
&= (\lambda^{-1}E)\sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m(\tilde\omega+n\alpha)}e^{2\pi i n\omega}\\
&= \lambda^{-1}\sum_{m\in\mathbb{Z}} (Eu(m))e^{2\pi i m(\tilde\omega+n\alpha)}e^{2\pi i n\omega}\\
&= \lambda^{-1}\sum_{m\in\mathbb{Z}} \left[u(m+1)+u(m-1)+2\lambda\cos(2\pi(\omega+m\alpha))u(m)\right]e^{2\pi i m(\tilde\omega+n\alpha)}e^{2\pi i n\omega}\\
&= \lambda^{-1}\sum_{m\in\mathbb{Z}} \left[u(m+1)+u(m-1)+\lambda\left(e^{2\pi i(\omega+m\alpha)}+e^{-2\pi i(\omega+m\alpha)}\right)u(m)\right]e^{2\pi i m\tilde\omega}e^{2\pi i n(\omega+m\alpha)}\\
&= \lambda^{-1}\sum_{m\in\mathbb{Z}} u(m+1)e^{2\pi i m\tilde\omega}e^{2\pi i n(\omega+m\alpha)} + \lambda^{-1}\sum_{m\in\mathbb{Z}} u(m-1)e^{2\pi i m\tilde\omega}e^{2\pi i n(\omega+m\alpha)}\\
&\quad + \sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m\tilde\omega}e^{2\pi i(n+1)(\omega+m\alpha)} + \sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m\tilde\omega}e^{2\pi i(n-1)(\omega+m\alpha)}\\
&= \lambda^{-1}\sum_{m\in\mathbb{Z}} u(m)e^{2\pi i(m-1)\tilde\omega}e^{2\pi i n(\omega+(m-1)\alpha)} + \lambda^{-1}\sum_{m\in\mathbb{Z}} u(m)e^{2\pi i(m+1)\tilde\omega}e^{2\pi i n(\omega+(m+1)\alpha)}\\
&\quad + \sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m\tilde\omega}e^{2\pi i(n+1)(\omega+m\alpha)} + \sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m\tilde\omega}e^{2\pi i(n-1)(\omega+m\alpha)}\\
&= \lambda^{-1}\left(e^{-2\pi i(\tilde\omega+n\alpha)}+e^{2\pi i(\tilde\omega+n\alpha)}\right)\sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m(\tilde\omega+n\alpha)}e^{2\pi i n\omega}\\
&\quad + \sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m(\tilde\omega+(n+1)\alpha)}e^{2\pi i(n+1)\omega} + \sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m(\tilde\omega+(n-1)\alpha)}e^{2\pi i(n-1)\omega}\\
&= 2\lambda^{-1}\cos(2\pi(\tilde\omega+n\alpha))\tilde u(n) + \tilde u(n+1) + \tilde u(n-1).
\end{aligned}$$

(b) If $u \in \ell^2(\mathbb{Z})$, $\hat u$ exists as an element of $L^2(\mathbb{T})$ and hence it is determined almost everywhere. Consider $\tilde\omega$ from the full measure set of elements for which all the quantities in the calculation above are determined. Then carry out the calculation to verify that $\tilde u$ is indeed a solution of (2) for the $\tilde\omega$ in question. □

By pursuing these relations further, it is possible to show the following:

Theorem 14 (Gordon-Jitomirskaya-Last-Simon 1997)**.** (a) *If $H_\omega^{\lambda,\alpha}$ has pure point spectrum for almost every $\omega \in \mathbb{T}$, then $H_\omega^{\lambda^{-1},\alpha}$ has purely absolutely continuous spectrum for almost every $\omega \in \mathbb{T}$.*
(b) *If $H_\omega^{\lambda,\alpha}$ has some point spectrum for almost every $\omega \in \mathbb{T}$, then $H_\omega^{\lambda^{-1},\alpha}$ has some absolutely continuous spectrum for almost every $\omega \in \mathbb{T}$.*

Another consequence of Aubry duality is a formula relating the spectra of $H_\omega^{\lambda,\alpha}$ and $H_\omega^{\lambda^{-1},\alpha}$. Note first that for α irrational, the spectrum of $H_\omega^{\lambda,\alpha}$ is independent of ω and may therefore be denoted by $\Sigma^{\lambda,\alpha}$. This follows from minimality of irrational rotations and strong operator convergence.

Theorem 15 (Avron-Simon 1983)**.** *We have $\Sigma^{\lambda,\alpha} = \lambda\Sigma^{\lambda^{-1},\alpha}$. In particular, $\Sigma^{\lambda,\alpha}$ is a Cantor set if and only if $\Sigma^{\lambda^{-1},\alpha}$ is a Cantor set.*

3. Exceptional Frequencies and Phases

3.1. Liouville Frequencies and the Gordon Method.

For the almost Mathieu operator $H_\omega^{\lambda,\alpha}$ with $\lambda > 0$, $\alpha \in \mathbb{T}$ irrational, and $\omega \in \mathbb{T}$, we investigate the spectral type by looking at the solutions of the time-independent Schrödinger equation:

$$u(n+1) + u(n-1) + 2\lambda\cos(2\pi(\omega + n\alpha))u(n) = Eu(n).$$

We have seen in Section 2 that for $\lambda > 1$, the Lyapunov exponent $\gamma(E)$ is uniformly bounded away from zero. By Osceledec' theorem, it follows that for every E, there is an E-dependent full measure set of ω's for which the equation has an exponentially decaying solution at $+\infty$. Similarly, there is also an exponentially decaying solution at $-\infty$. Since it is known that almost everywhere with respect to any spectral measure, there are polynomially bounded solutions, this suggests that we should expect that there are exponentially decaying eigenfunctions for spectrally almost every energy and hence localization. However, here we need to interchange the quantifiers and the application of Fubini then loses track of sets of Lebesgue measure zero. Thus, the argument is inconclusive and in fact wrong in general as we will see in this section.

The expected localization result fails when α is very well approximated by rational numbers.

Theorem 16 (Avron-Simon 1982)**.** *Suppose $\alpha \in \mathbb{T}$ is Liouville. Then, for every λ and ω, $H_\omega^{\lambda,\alpha}$ has purely continuous spectrum.*

The heart of the argument is the Cayley-Hamilton Theorem, which for $\mathrm{SL}(2,\mathbb{C})$ matrices M takes the form

$$M^2 - \mathrm{Tr}M \cdot M + I = 0. \tag{4}$$

Recall that the transfer matrices belong to $\mathrm{SL}(2,\mathbb{C})$. We will apply (4) to these matrices when there are suitable local repetitions. Explicitly, the following version of Gordon's Lemma implements this.

Lemma 17. *Suppose $V : \mathbb{Z} \to \mathbb{R}$ obeys $V(n+p) = V(n)$ for some $p \in \mathbb{Z}_+$ and $-p+1 \le n \le p$, $E \in \mathbb{R}$, and u solves*

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n).$$

Then, we have

$$\max\left\{ \left\| \begin{pmatrix} u(-p+1) \\ u(-p) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(p+1) \\ u(p) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(2p+1) \\ u(2p) \end{pmatrix} \right\| \right\} \ge \frac{1}{2} \left\| \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \right\|. \tag{5}$$

Proof. By assumption, we have

$$\begin{pmatrix} u(2p+1) \\ u(2p) \end{pmatrix} = M_E(2p) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} = M_E(p)^2 \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

and similarly

$$\begin{pmatrix} u(p+1) \\ u(p) \end{pmatrix} = M_E(p)^2 \begin{pmatrix} u(-p+1) \\ u(-p) \end{pmatrix}.$$

Moreover, (4) implies

$$M_E(p)^2 - \mathrm{Tr}M_E(p) \cdot M_E(p) + I = 0.$$

Consequently, we have

$$\begin{pmatrix} u(2p+1) \\ u(2p) \end{pmatrix} - \mathrm{Tr} M_E(p) \begin{pmatrix} u(p+1) \\ u(p) \end{pmatrix} + \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6}$$

and

$$\begin{pmatrix} u(p+1) \\ u(p) \end{pmatrix} - \mathrm{Tr} M_E(p) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} + \begin{pmatrix} u(-p+1) \\ u(-p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{7}$$

The assertion (5) follows from (6) when $|\mathrm{Tr} M_E(p)| \le 1$ and it follows from (7) when $|\mathrm{Tr} M_E(p)| > 1$. $\square$

The estimate (5) can of course be used to exclude the existence of decaying solutions. Notice that the energy E does not enter the argument. In particular, if the potential V has the required local periodicity for infinitely many values of p, we have the estimate (5) for infinitely many values of p. This in turn shows that no E can be an eigenvalue. It is clear that one can perturb about this situation a little bit and still deduce useful estimates. In light of this, the following definition is natural.

Definition 18. *A bounded potential $V : \mathbb{Z} \to \mathbb{R}$ is called a Gordon potential if there are positive integers $q_k \to \infty$ such that*

$$\forall\, C > 0 : \lim_{k \to \infty} \max_{1 \le n \le q_k} |V(n) - V(n \pm q_k)| C^{q_k} = 0. \tag{8}$$

Lemma 19. *Suppose V is a Gordon potential. Then, the operator $H = \Delta + V$ has purely continuous spectrum. More precisely, for every $E \in \mathbb{R}$ and every solution u of $Hu = Eu$, we have*

$$\limsup_{|n| \to \infty} \left\| \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} \right\| \ge \frac{1}{2} \left\| \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \right\|. \tag{9}$$

Proof. By assumption, there is a sequence $q_k \to \infty$ such that (8) holds. Given $E \in \mathbb{R}$, we consider a solution u of $Hu = Eu$ and, for every k, a solution u_k of

$$u_k(n+1) + u_k(n-1) + V_k(n) u_k(n) = E u_k(n)$$

with $u_k(1) = u(1)$ and $u_k(0) = u(0)$, where V_k is the q_k-periodic potential that coincides with V on the interval $1 \le n \le q_k$.

It follows from Lemma 17 that u_k satisfies the estimate

$$\max \left\{ \left\| \begin{pmatrix} u_k(-q_k+1) \\ u_k(-q_k) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u_k(q_k+1) \\ u_k(q_k) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u_k(2q_k+1) \\ u_k(2q_k) \end{pmatrix} \right\| \right\} \ge \frac{1}{2} \left\| \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \right\|.$$

Since V is very close to V_k on the relevant interval and u and u_k have the same initial conditions, we expect that they are close throughout the relevant interval and hence u obeys a similar estimate.

Let us make this observation explicit. Denote the transfer matrices associated with V_k by $M_{k,E}(n)$. We have

$$\begin{aligned} \max_{-q_k \le n \le 2q_k} \left\| \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} - \begin{pmatrix} u_k(n+1) \\ u_k(n) \end{pmatrix} \right\| &\le \max_{-q_k \le n \le 2q_k} \| M_E(n) - M_{k,E}(n) \| \left\| \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \right\| \\ &\le 2 q_k C^{q_k} \max_{-q_k \le n \le 2q_k} |V(n) - V_k(n)| \left\| \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \right\|, \end{aligned}$$

which goes to zero by (8). $\square$

Proof of Theorem 16. Let α be Liouville and denote by $\frac{p_k}{q_k}$ the associated rational approximants. Consider arbitrary $\omega \in \mathbb{T}$ and $C > 0$. Since $|\cos x - \cos y| \le |x - y|$, we have

$$\max_{1 \le n \le q_k} |2\lambda \cos(2\pi(\omega + n\alpha)) - 2\lambda \cos(2\pi(\omega + (n \pm q_k)\alpha))| C^{q_k} \le 4\lambda\pi \operatorname{dist}(q_k\alpha, \mathbb{Z})\, C^{q_k}$$

and it follows from the Liouville condition that the right-hand side goes to zero as $k \to \infty$. Thus, $V(n) = 2\lambda \cos(2\pi(\omega + n\alpha))$ is a Gordon potential and hence $H_\omega^{\lambda,\alpha}$ has no eigenvalues by Lemma 19. □

3.2. **Resonant Phases and the Jitomirskaya-Simon Method.** In the previous subsection we saw how local translation symmetries of the potential induce certain local translation symmetries of solutions. While the latter are quite weak, they suffice to exclude decay at infinity and hence square-summability. In this section we briefly discuss an analogous study based on reflection symmetries.

Theorem 20 (Jitomirskaya-Simon 1994)**.** *Given λ and α, there is a dense G_δ set of ω's for which $H_\omega^{\lambda,\alpha}$ has purely continuous spectrum.*

Definition 21. *A bounded potential $V : \mathbb{Z} \to \mathbb{R}$ is called a Jitomirskaya-Simon potential if there are*

$$B > 4 \log(3 + 2\|V\|_\infty)$$

and integers $m_k \to \infty$ such that for every k,

$$\sup_{n \in \mathbb{Z}} |V(2m_k - n) - V(n)| < e^{-Bm_k}.$$

Lemma 22. *If V is a Jitomirskaya-Simon potential and $E \in \mathbb{R}$, then*

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$

has no non-zero ℓ^2 solutions u.

Proof of Theorem 20. Since cos is an even function, there is a dense set of ω's, namely $\{k\alpha\}$, that have a center around which the associated potential is symmetric. Take balls of suitable radius to generate open sets which cover $\mathbb{T}$. The $\limsup$ of these balls is then a dense G_δ set. □

4. Localization at Supercritical Coupling

In this section we discuss the main steps in the proof of Theorem 10.

4.1. **Existence of Generalized Eigenfunctions.** We say that $E \in \mathbb{R}$ is a generalized eigenvalue of H if $Hu = Eu$ has a non-trivial solution u_E, called the corresponding generalized eigenfunction, satisfying

$$|u_E(n)| \le C(1 + |n|)^\delta \tag{10}$$

for suitable finite constants C and δ, and every $n \in \mathbb{Z}$.

Theorem 23. (a) *Every generalized eigenvalue of H belongs to $\sigma(H)$.*
(b) *Fix $\delta > \frac{1}{2}$ and some spectral measure μ. Then, for μ-almost every $E \in \mathbb{R}$, there exists a generalized eigenfunction satisfying* (10)*.*
(c) *The spectrum of H is given by the closure of the set of generalized eigenvalues of H.*

4.2. **Solutions and Green's Function.** For

$$[n_1, n_2] = \{n \in \mathbb{Z} : n_1 \le n \le n_2\},$$

denote by $H_{[n_1,n_2]}$ the restriction of H to this interval, that is,

$$H_{[n_1,n_2]} = P_{[n_1,n_2]} H P^*_{[n_1,n_2]},$$

where $P_{[n_1,n_2]} : \ell^2(\mathbb{Z}) \to \ell^2([n_1,n_2])$ is the canonical projection and $P^*_{[n_1,n_2]} : \ell^2([n_1,n_2]) \to \ell^2(\mathbb{Z})$ is the canonical embedding.

Moreover, for $E \notin \sigma(H_{[n_1,n_2]})$ and $n, m \in [n_1, n_2]$, let

$$G_{[n_1,n_2]}(n, m; E) = \langle \delta_n, \left(H_{[n_1,n_2]} - E\right)^{-1} \delta_m \rangle.$$

Then, the following formula holds.

Lemma 24. *Suppose $n \in [n_1, n_2] \subset \mathbb{Z}$ and u is a solution of the difference equation $Hu = Eu$. If $E \notin \sigma(H_{[n_1,n_2]})$ and $n, m \in [n_1, n_2]$, then*

$$u(n) = -G_{[n_1,n_2]}(n, n_1; E)u(n_1 - 1) - G_{[n_1,n_2]}(n, n_2; E)u(n_2 + 1).$$

Proof. Since u is a solution, we have

$$\begin{aligned} 0 &= P_{[n_1,n_2]}(H - E)u \\ &= P_{[n_1,n_2]}(H - E)P^*_{[n_1,n_2]}P_{[n_1,n_2]}u + P_{[n_1,n_2]}(H - E)P^*_{\mathbb{Z}\setminus[n_1,n_2]}P_{\mathbb{Z}\setminus[n_1,n_2]}u, \end{aligned}$$

which in turn implies

$$\left(H_{[n_1,n_2]} - E\right)\left(P_{[n_1,n_2]}u\right) = -\left(u(n_1 - 1)\delta_{n_1} + u(n_2 + 1)\delta_{n_2}\right).$$

Thus, with the inner product of $\ell^2([n_1, n_2])$, we find for $n \in [n_1, n_2]$,

$$\begin{aligned} u(n) &= \langle \delta_n, (P_{[n_1,n_2]}u) \rangle \\ &= \left\langle \delta_n, \left(H_{[n_1,n_2]} - E\right)^{-1} \left(H_{[n_1,n_2]} - E\right)\left(P_{[n_1,n_2]}u\right) \right\rangle \\ &= -\left\langle \delta_n, \left(H_{[n_1,n_2]} - E\right)^{-1} \left(u(n_1 - 1)\delta_{n_1} + u(n_2 + 1)\delta_{n_2}\right) \right\rangle \\ &= -u(n_1 - 1)\left\langle \delta_n, \left(H_{[n_1,n_2]} - E\right)^{-1} \delta_{n_1} \right\rangle - u(n_2 + 1)\left\langle \delta_n, \left(H_{[n_1,n_2]} - E\right)^{-1} \delta_{n_2} \right\rangle \end{aligned}$$

as claimed. □

4.3. **Green's Function and Determinants.** Let

$$P_k(\omega, E) = \det\left((H^{\lambda,\alpha}_\omega - E)_{[0,k-1]}\right).$$

By Cramer's Rule, we have for n_1, $n_2 = n_1 + k - 1$, and $n \in [n_1, n_2]$,

$$\begin{aligned} \left|G_{[n_1,n_2]}(n_1, n; E)\right| &= \left|\frac{P_{n_2-n}(\omega + (n+1)\alpha, E)}{P_k(\omega + n_1\alpha, E)}\right|, \\ \left|G_{[n_1,n_2]}(n, n_2; E)\right| &= \left|\frac{P_{n-n_1}(\omega + n_1\alpha, E)}{P_k(\omega + n_1\alpha, E)}\right|. \end{aligned}$$

4.4. **Determinants and Lyapunov Exponents.** We have

$$M_E(k,\omega) = \begin{pmatrix} P_k(\omega,E) & -P_{k-1}(\omega+\alpha,E) \\ P_{k-1}(\omega,E) & -P_{k-2}(\omega+\alpha,E) \end{pmatrix}. \tag{11}$$

as can be checked by considering the degree, leading coefficient and zeros of the entries of the transfer matrix, regarded as polynomials in E.

Lemma 25. *For every $E \in \mathbb{R}$ and $\varepsilon > 0$, there exists $k(E,\varepsilon)$ such that*

$$|P_k(\omega,E)| < \exp\left((\gamma(E)+\varepsilon)k\right)$$

for every $k > k(E,\varepsilon)$ and every $\omega \in \mathbb{T}$.

Proof. This is a consequence of (11) and the subadditive ergodic theorem for uniquely ergodic transformations. □

Definition 26. *Fix $E \in \mathbb{R}$ and $\gamma \in \mathbb{R}$. A point $n \in \mathbb{Z}$ is called (γ,k)-regular if there exists an interval $[n_1,n_2]$, containing n such that*

(i) $n_2 = n_1 + k - 1$,
(ii) $n \in [n_1,n_2]$,
(iii) $|n - n_i| > \dfrac{k}{5}$,
(iv) $\left|G_{[n_1,n_2]}(n,n_i;E)\right| < \exp\left(-\gamma|n-n_i|\right)$.

Otherwise, n is called (γ,k)-singular.

The following central lemma describes the repulsion of singular clusters:

Lemma 27. *For every $n \in \mathbb{Z}$, $\varepsilon > 0$, $\tau < 2$, there exists $k_1 = k_1(\omega,\alpha,n,\varepsilon,\tau,E)$ such that for every*

$$k \in \mathcal{K} = \left\{ k \in \mathbb{Z}_+ : \exists\, \tilde\omega \in \mathbb{T} \text{ with } |P_k(\tilde\omega,E)| \ge \frac{1}{\sqrt{2}}\, e^{k\gamma(E)} \right\}$$

with $k > k_1$, we have that

$$m,n \text{ are both } (\gamma(E)-\varepsilon,k)\text{-singular and } |m-n| > \frac{k+1}{2} \;\Rightarrow\; |m-n| > k^\tau.$$

Proof of Theorem 10. Let $E(\omega)$ be a generalized eigenvalue of $H_\omega^{\lambda,\alpha}$ and denote the corresponding generalized eigenfunction by u_E. Notice that every point $n \in \mathbb{Z}$ with $u_E(n) \neq 0$ is (γ,k)-singular for $k > k_2 = k_2(E,\gamma,\omega,n)$.

Assume without loss of generality $u_E(0) \neq 0$ (otherwise replace zero by one). Thus, by Lemma 27, if

$$|n| > \max\left\{k_1(\omega,\alpha,0,\varepsilon,1.5,E), k_2(E,\gamma(E)-\varepsilon,\omega,0)\right\} + 1,$$

the point n is $(\gamma(E)-\varepsilon,k)$-regular for some $k \in \{|n|-1,|n|,|n|+1\} \cap \mathcal{K} \neq \emptyset$, since 0 is $(\gamma(E)-\varepsilon,k)$-singular. Thus, there exists an interval $[n_1,n_2]$ of length k containing n such that

$$\frac{1}{5}(|n|-1) \le |n-n_i| \le \frac{4}{5}(|n|+1)$$

and

$$\left|G_{[n_1,n_2]}(n,n_i;E)\right| < e^{-(\gamma(E)-\varepsilon)|n-n_i|}.$$

From this and Lemma 24, we therefore see that

$$|u_E(n)| \le 2C(u_E)(2|n|+1)e^{-\left(\frac{\gamma(E)-\varepsilon}{5}\right)(|n|-1)}.$$

By the uniform lower bound $\gamma(E) \geq \log\lambda$, this implies exponential decay if ε is chosen small enough. $\square$

Sketch of the proof of Lemma 27. Assume that m_1 and m_2 are both $(\gamma(E)-\varepsilon, k)$-singular with

$$d = m_2 - m_1 > \frac{k+1}{2}.$$

We set $n_i = m_i - \left\lfloor \frac{3}{4}k \right\rfloor$, $i = 1, 2$.

It may be shown that there is a polynomial Q_k of degree k such that

$$P_k(\tilde\omega) = Q_k\left(\cos\left(2\pi(\tilde\omega + \frac{k-1}{2}\alpha\right)\right). \tag{12}$$

Let

$$\omega_j = \begin{cases} \omega + \left(n_1 + \frac{k-1}{2} + j\right)\alpha, & j = 0, 1, \ldots, \left\lfloor \frac{k+1}{2} \right\rfloor - 1, \\ \omega + \left(n_2 + \frac{k-1}{2} + j - \left\lfloor \frac{k+1}{2} \right\rfloor\right)\alpha, & j = \left\lfloor \frac{k+1}{2} \right\rfloor, \left\lfloor \frac{k+1}{2} \right\rfloor + 1, \ldots, k. \end{cases}$$

Lagrange interpolation then shows

$$|Q_k(z)| = \left| \sum_{j=0}^{k} Q_k(\cos(2\pi\omega_j) \frac{\prod_{l\neq j}(z - \cos(2\pi\omega_l)}{\prod_{l\neq j}(\cos(2\pi\omega_j) - \cos(2\pi\omega_l))} \right|. \tag{13}$$

By $(\gamma(E)-\varepsilon, k)$-singularity and Lemma 25, we have for k sufficiently large,

$$|Q_k(\cos(2\pi\omega_j)| < \exp\left(\frac{k}{8}(\gamma(E) - \varepsilon)\right), \quad j = 0, 1, \ldots, k. \tag{14}$$

Using that the Diophantine and non-resonance assumptions, one may show that if $d < k^\tau$ for some $\tau < 2$, we have for large k,

$$\frac{|\prod_{l\neq j}(z - \cos(2\pi\omega_l)|}{|\prod_{l\neq j}(\cos(2\pi\omega_j) - \cos(2\pi\omega_l))|} \leq \exp\left(\frac{k\varepsilon}{16}\right) \quad \text{for } z \in [-1,1],\ 0 \leq j \leq k. \tag{15}$$

Given $\tau < 2$, consider $k \in \mathcal{K}$ large enough and $\tilde\omega$ with

$$|P_k(\tilde\omega)| \geq \frac{1}{\sqrt{2}}\, e^{k\gamma(E)}.$$

But assuming $d < k^\tau$, we also have the following upper bound,

$$|P_k(\tilde\omega)| \leq (k+1)\exp\left(\frac{k}{8}(\gamma(E) - \varepsilon)\right)\exp\left(\frac{k\varepsilon}{16}\right),$$

which follows from (12)–(15) for $z = \cos\left(2\pi(\tilde\omega + \frac{k-1}{2}\alpha\right)$. This contradiction shows that $d < k^\tau$ is impossible. $\square$

5. Cantor Spectrum via Aubry Duality

In this section we present Puig's proof of the striking fact that localization for the operator family $\{H_\omega^{\lambda,\alpha}\}_{\omega\in\mathbb{T}}$, as established in the previous lecture, implies via Aubry duality and reducibility Cantor spectrum for the dual family $\{H_\omega^{\lambda^{-1},\alpha}\}_{\omega\in\mathbb{T}}$. Once one has shown Cantor spectrum for $0 < \lambda < 1$, Aubry duality applied again yields Cantor spectrum for $\lambda > 1$. That is, we will show how Theorem 11 follows from Jitomirskaya's localization result.

Consider the equations

$$u(n+1) + u(n-1) + 2\lambda\cos(2\pi n\alpha)u(n) = Eu(n), \tag{16}$$

$$u(n+1)+u(n-1)+2\lambda^{-1}\cos(2\pi(\omega+n\alpha))u(n)=(\lambda^{-1}E)u(n) \tag{17}$$

and recall from Section 2 how exponentially decaying eigenfunctions and highly regular quasi-periodic solutions are related to each other via Aubry duality:

Lemma 28. (a) *Suppose u is an exponentially decaying solution of* (16). *Consider its Fourier series*

$$\hat{u}(\omega)=\sum_{m\in\mathbb{Z}}u(m)e^{2\pi im\omega}.$$

Then, $\hat{u}$ is real-analytic on $\mathbb{T}$, it extends analytically to a strip, and the sequence $\tilde{u}(n)=\hat{u}(\omega+n\alpha)$ is a solution of (17).

(b) *Conversely, suppose u is a solution of* (17) *with $\omega=0$ of the form $u(n)=g(n\alpha)$ for some real-analytic function g on $\mathbb{T}$. Consider the Fourier series*

$$g(\omega)=\sum_{n\in\mathbb{Z}}\hat{g}(n)e^{2\pi in\omega}.$$

Then, the sequence $\{\hat{g}(n)\}$ is an exponentially decaying solution of (16).

Proof. (a) Since u is exponentially decaying, $\hat{u}$ extends to a function that is analytic in a neighborhood of the unit circle $\mathbb{T}$. The other statement was shown in Section 2.

(b) Since u is a solution of (17) with $\omega=0$ and we have $u(m)=g(m\alpha)$, we have

$$g((m+1)\alpha)+g((m-1)\alpha)+2\lambda^{-1}\cos(2\pi m\alpha)g(m\alpha)=(\lambda^{-1}E)g(m\alpha).$$

Rewriting this in terms of the Fourier expansion, we find

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\hat{g}(n)e^{2\pi in(m+1)\alpha}+\sum_{n\in\mathbb{Z}}\hat{g}(n)e^{2\pi in(m-1)\alpha}+\lambda^{-1}\left(e^{-2\pi im\alpha}+e^{2\pi im\alpha}\right)\sum_{n\in\mathbb{Z}}\hat{g}(n)e^{2\pi inm\alpha}\\ =(\lambda^{-1}E)\sum_{n\in\mathbb{Z}}\hat{g}(n)e^{2\pi inm\alpha}.\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}E\hat{g}(n)e^{2\pi inm\alpha}\\ &=\sum_{n\in\mathbb{Z}}\left(\lambda\hat{g}(n)e^{2\pi in(m+1)\alpha}+\lambda\hat{g}(n)e^{2\pi in(m-1)\alpha}+\left(e^{-2\pi im\alpha}+e^{2\pi im\alpha}\right)\hat{g}(n)e^{2\pi inm\alpha}\right)\\ &=\sum_{n\in\mathbb{Z}}\left(2\lambda\cos(2\pi n\alpha)\hat{g}(n)+\hat{g}(n-1)+\hat{g}(n+1)\right)e^{2\pi inm\alpha}\end{aligned}$$

Since g is real-analytic on $\mathbb{T}$ with analytic extension to a strip, it follows that the Fourier coefficients of g decay exponentially and satisfy the difference equation (16). □

Next we use the information provided by the previous lemma to reduce the situation at hand to constant coefficients. We prove a general statement to this effect:

Lemma 29. *Let $\alpha\in\mathbb{T}$ be Diophantine and suppose $A:\mathbb{T}\to\mathrm{SL}(2,\mathbb{R})$ is a real-analytic map, with analytic extension to the strip $|\Im\omega|<\delta$ for some $\delta>0$. Assume that there is a non-vanishing real-analytic map $v:\mathbb{T}\to\mathbb{R}^2$ with analytic extension to the same strip $|\Im\omega|<\delta$ such that*

$$v(\omega+\alpha)=A(\omega)v(\omega)\quad \textit{for every } \omega\in\mathbb{T}.$$

Then, there are a real number c and a real-analytic map $B : \mathbb{T} \to \mathrm{SL}(2, \mathbb{R})$ with analytic extension to the strip $|\Im\omega| < \delta$ such that with

$$C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \tag{18}$$

we have

$$B(\omega + \alpha)^{-1} A(\omega) B(\omega) = C \quad \textit{for every } \omega \in \mathbb{T}. \tag{19}$$

Proof. Since v does not vanish, $d(\omega) = v_1(\omega)^2 + v_2(\omega)^2$ is strictly positive and hence we can define

$$B_1(\omega) = \begin{pmatrix} v_1(\omega) & -\frac{v_2(\omega)}{d(\omega)} \\ v_2(\omega) & \frac{v_1(\omega)}{d(\omega)} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

for $\omega \in \mathbb{T}$. We have

$$A(\omega) B_1(\omega) = \begin{pmatrix} v_1(\omega + \alpha) & * \\ v_2(\omega + \alpha) & * \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \tag{20}$$

and hence

$$A(\omega) B_1(\omega) = B_1(\omega + \alpha) \tilde{C}(\omega)$$

with

$$\tilde{C}(\omega) = \begin{pmatrix} 1 & \tilde{c}(\omega) \\ 0 & 1 \end{pmatrix},$$

where $\tilde{c} : \mathbb{T} \to \mathbb{R}$ is analytic. Indeed, by (20) the first column of $\tilde{C}(\omega)$ is determined and then its $(2, 2)$ entry must be one since $\tilde{C}(\omega) = B_1(\omega + \alpha)^{-1} A(\omega) B_1(\omega) \in \mathrm{SL}(2, \mathbb{R})$. Now let

$$c = \int_{\mathbb{T}} \tilde{c}(\omega) \, d\omega.$$

and define the matrix C as in (18).

We claim that we can find $b : \mathbb{T} \to \mathbb{R}$ analytic (with analytic extension to a strip) such that

$$b(\omega + \alpha) - b(\omega) = \tilde{c}(\omega) - c \quad \text{for every } \omega \in \mathbb{T}. \tag{21}$$

Indeed, expand both sides of the hypothetical identity (21) in Fourier series:

$$\sum_{k \in \mathbb{Z}} b_k e^{2\pi i(\omega + \alpha)k} - \sum_{k \in \mathbb{Z}} b_k e^{2\pi i \omega k} = \sum_{k \in \mathbb{Z}} \tilde{c}_k e^{2\pi i \omega k} - c.$$

Since we have $\tilde{c}_0 = c$, the $k = 0$ terms disappear on both sides and hence all we need to do is to require

$$b_k (e^{2\pi i \alpha k} - 1) = \tilde{c}_k \quad \text{for every } k \in \mathbb{Z} \setminus \{0\}.$$

In other words, if we set $b_0 = 0$ and

$$b_k = \frac{\tilde{c}_k}{e^{2\pi i \alpha k} - 1} \quad \text{for every } k \in \mathbb{Z} \setminus \{0\},$$

then

$$b(\omega) = \sum_{k \in \mathbb{Z}} b_k e^{2\pi i \omega k}$$

satisfies (21). Since $\tilde{c}(\cdot)$ has an analytic extension to a strip, the coefficients $\tilde{c}_k$ decay exponentially. On the other hand, the Diophantine condition which α satisfies ensures that the coefficients b_k decay exponentially as well and hence $b(\cdot)$ is real-analytic with an extension to the same open strip.

Setting

$$B_2(\omega) = \begin{pmatrix} 1 & b(\omega) \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

and using (21), we find

$$\begin{aligned} B_2(\omega+\alpha)^{-1}\tilde{C}(\omega)B_2(\omega) &= \begin{pmatrix} 1 & -b(\omega+\alpha) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{c}(\omega) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b(\omega) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -b(\omega+\alpha) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b(\omega)+\tilde{c}(\omega) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & b(\omega)+\tilde{c}(\omega)-b(\omega+\alpha) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \\ &= C \end{aligned}$$

for every $\omega \in \mathbb{T}$. Thus, setting $B(\omega) = B_1(\omega)B_2(\omega)$, we obtain (19). □

Proof of Theorem 11. Consider first a coupling constant $\lambda > 1$. We have seen above that Aubry duality maps the energy E to the dual energy $\lambda^{-1}E$. We will establish below that if E is an eigenvalue of $H_0^{\lambda,\alpha}$, then the dual energy $\lambda^{-1}E$ is an endpoint of a gap of the spectrum of $H_0^{\lambda^{-1},\alpha}$. Since we already know that for Diophantine α, $H_0^{\lambda,\alpha}$ has pure point spectrum, we can consider energies belonging to the countable dense set of eigenvalues. It then follows that the dual energies are all endpoints of gaps and hence the gaps are dense because the spectra are just related by uniform scaling.

Let us implement this strategy. Consider an eigenvalue E of $H_0^{\lambda,\alpha}$ and a corresponding exponentially decaying eigenfunction. Then, Lemma 28 yields the real-analytic function $\hat{u}$, which has an analytic extension to a strip, and a quasi-periodic solution of the dual difference equation at the dual energy. Using this as input to Lemma 29, we then obtain that

$$A(\omega) = \begin{pmatrix} \lambda^{-1}E - 2\lambda^{-1}\cos(2\pi\omega) & -1 \\ 1 & 0 \end{pmatrix}$$

may be analytically conjugated via $B(\cdot)$ to the constant

$$C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

Let us show that $c \neq 0$. Assume to the contrary $c = 0$. Then, $A(\omega) = B(\omega + \alpha)B(\omega)^{-1}$ for every $\omega \in \mathbb{T}$ and therefore, all solutions of (17) are analytically quasi-periodic! Indeed,

$$\begin{aligned} \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} &= A(\omega+(n-1)\alpha) \begin{pmatrix} u(n-1) \\ u(n-2) \end{pmatrix} \\ &= \cdots \\ &= A(\omega+(n-1)\alpha) \times \cdots \times A(\omega) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} \\ &= B(\omega+n\alpha)B(\omega)^{-1} \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}, \end{aligned}$$

that is,

$$u(n) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, B(\omega + n\alpha)B(\omega)^{-1} \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} \right\rangle,$$

and hence $u(n) = g(n\alpha)$ with a real-analytic function g on $\mathbb{T}$. Now consider two linearly independent solutions of (17) and associate with them via Lemma 28 the corresponding exponentially decaying solutions of the dual equation (16). They must be linearly independent too, which yields the desired contradiction since by constancy of the Wronskian there cannot be two linearly independent exponentially decaying solutions. This contradiction shows $c \neq 0$.

Let us now perturb the energy and consider

$$\tilde{A}(\omega) = \begin{pmatrix} (\lambda^{-1}E + \lambda^{-1}\delta) - 2\lambda^{-1}\cos(2\pi\omega) & -1 \\ 1 & 0 \end{pmatrix} = A(\omega) + \begin{pmatrix} \lambda^{-1}\delta & 0 \\ 0 & 0 \end{pmatrix}.$$

One can show that there is $\delta_0 > 0$ such that

$$0 < |\delta| < \delta_0 \text{ and } \delta c < 0 \quad \Rightarrow \quad \lambda^{-1}E + \lambda^{-1}\delta \notin \sigma(H_0^{\lambda^{-1},\alpha}). \tag{22}$$

Since the E's in question are dense in $\sigma(H_0^{\lambda,\alpha})$, Aubry duality shows that the $\lambda^{-1}E$'s in question are dense in $\sigma(H_0^{\lambda^{-1},\alpha})$. By (22) all these energies are endpoints of gaps of $\sigma(H_0^{\lambda^{-1},\alpha})$. Thus, $\sigma(H_0^{\lambda^{-1},\alpha})$ does not contain an interval. Recall that by general principles, $\sigma(H_0^{\lambda^{-1},\alpha})$ is closed and does not contain isolated points. Consequently, $\Sigma^{\lambda^{-1},\alpha} = \sigma(H_0^{\lambda^{-1},\alpha})$ is a Cantor set. Then, by Aubry duality again, $\Sigma^{\lambda,\alpha} = \lambda\Sigma^{\lambda^{-1},\alpha}$ is a Cantor set, too. Putting everything together, it follows that for every Diophantine α and every $\lambda \in (0,\infty) \setminus \{1\}$, $\Sigma^{\lambda,\alpha}$ is a Cantor set. $\square$

6. Liouville Frequencies

So far we have shown the main theorems, that is, an identification of the almost sure spectral type and the Cantor structure of the spectrum, for Diophantine frequencies α and non-critical coupling constants λ. In this final section we discuss the case of Liouville frequencies in more depth. They have already been considered when we showed that there are never any eigenvalues and hence the expected localization at super-critical coupling fails in these cases.

As a consequence of this non-result, we cannot deduce using Aubry duality that the spectrum at sub-critical coupling is almost surely purely absolutely continuous nor that it is a Cantor set. Thus, a different approach is required to prove these statements, which turn out to be indeed true. Since Liouville numbers are very well approximated by rational numbers, it is natural to try and prove the expected statements by periodic approximation. This is also the philosophy that is implemented with the help of Gordon's lemma to prove the absence of eigenvalues. Periodic operators always have purely absolutely continuous spectrum, so one needs a method to push this through to the quasi-periodic limit. Moreover, the spectra of periodic operators have many gaps and a suitable continuity statement for spectra could feasibly identify a dense set of gaps in the spectrum of the quasi-periodic operator if the rate of approximation is sufficiently good.

We have the following pair of theorems. For simplicity, we work with the notion of a Liouville number as introduced earlier. Stronger results are known but the proofs are (much) more difficult.

Theorem 30 (Choi-Elliott-Yui 1990). *Suppose α is Liouville. Then, for every $\lambda > 0$, $\Sigma^{\lambda,\alpha}$ is a Cantor set.*

Theorem 31 (Avila-Damanik 2008). *Suppose α is Liouville. Then, for every $0 < \lambda < 1$ and almost every $\omega \in \mathbb{T}$, $H_\omega^{\lambda,\alpha}$ has purely absolutely continuous spectrum.*

We will only make these results plausible by describing the main ideas and tools used in the proofs. As pointed out above, it will be essential to obtain a good understanding of the periodic approximants to $H_\omega^{\lambda,\alpha}$, which arise when α is replaced by a rational number close to it. So, from now on, we will drop the requirement that α is irrational and instead consider $H_\omega^{\lambda,\alpha}$ for $\lambda > 0$ and $\alpha, \omega \in \mathbb{T}$. Given such λ and α, we define

$$\Sigma^{\lambda,\alpha} = \bigcup_{\omega \in \mathbb{T}} \sigma(H_\omega^{\lambda,\alpha}).$$

For irrational α, this definition coincides with the one we had previously, since the spectrum of $H_\omega^{\lambda,\alpha}$ is ω-independent in this case.

We first discuss Cantor spectrum. By Aubry duality, it suffices to consider $0 < \lambda \le 1$. This will sometimes be relevant below. The first question we address is the continuity of $\Sigma^{\lambda,\alpha}$ as a function of α.

Lemma 32 (Avron-van Mouche-Simon 1990). *For every $\lambda > 0$, there exists $\delta > 0$ such that if $|\alpha - \alpha'| < \delta$, then*

$$\mathrm{dist}_H \left(\Sigma^{\lambda,\alpha}, \Sigma^{\lambda,\alpha'} \right) \le 6 \left(2\lambda |\alpha - \alpha'| \right)^{1/2}.$$

Here, $\mathrm{dist}_H(\Sigma^{\lambda,\alpha}, \Sigma^{\lambda,\alpha'})$ denotes the Hausdorff distance between $\Sigma^{\lambda,\alpha}$ and $\Sigma^{\lambda,\alpha'}$. The proof uses test functions with a linear cut-off. This continuity result shows that any gap of $\Sigma^{\lambda,p/q}$ corresponds to a gap of $\Sigma^{\lambda,\alpha}$ if its length is larger than $6\left(2\lambda|\alpha - \alpha'|\right)^{1/2}$. Figure 1 shows the sets $\Sigma^{1,p/q}$ for $0 \le p/q \le 1$ with $q \le 50$. This plot, which has a beautiful self-similar structure, is known as the *Hofstadter butterfly*.

The following result gives quite detailed information about the number and size of the gaps of $\Sigma^{\lambda,p/q}$. The result should be compared with the plot of the Hofstadter butterfly.

Lemma 33 (Choi-Elliott-Yui 1990). *Suppose $0 < \lambda \le 1$ and p, q are coprime positive integers. Write $q = 2m + 1$ or $q = 2m + 2$. Then, $\Sigma^{\lambda,p/q}$ has exactly $2m$ gaps. Each of these gaps is of length at least $\lambda^m 8^{-q}$.*

This produces many gaps in $\Sigma^{\lambda,\alpha}$ when α is Liouville. Indeed, choose a sequence $p_k/q_k \to \alpha$ with p_k, q_k coprime and $|\alpha - p_k/q_k| < k^{-q_k}$. Then, for k large enough, there are at least $q_k - 2$ many gaps in $\Sigma^{\lambda,\alpha}$. It remains to show that these gaps are dense. To this end, the following observation does the job.

Lemma 34 (Choi-Elliott-Yui 1990). *Suppose $0 < \lambda \le 1$ and p, q are coprime positive integers. For every gap of $\Sigma^{\lambda,p/q}$, there is another gap of $\Sigma^{\lambda,p/q}$ within distance $8\pi/q$.*

Let us now discuss how to prove purely absolutely continuous spectrum at subcritical coupling for Liouville frequencies. The spectral type is much less robust under perturbations than, for example, the spectrum. Simple approximation of $H_\omega^{\lambda,\alpha}$ by $H_\omega^{\lambda,p/q}$ does not look promising at first. It turns out that robustness improves if one averages over ω!

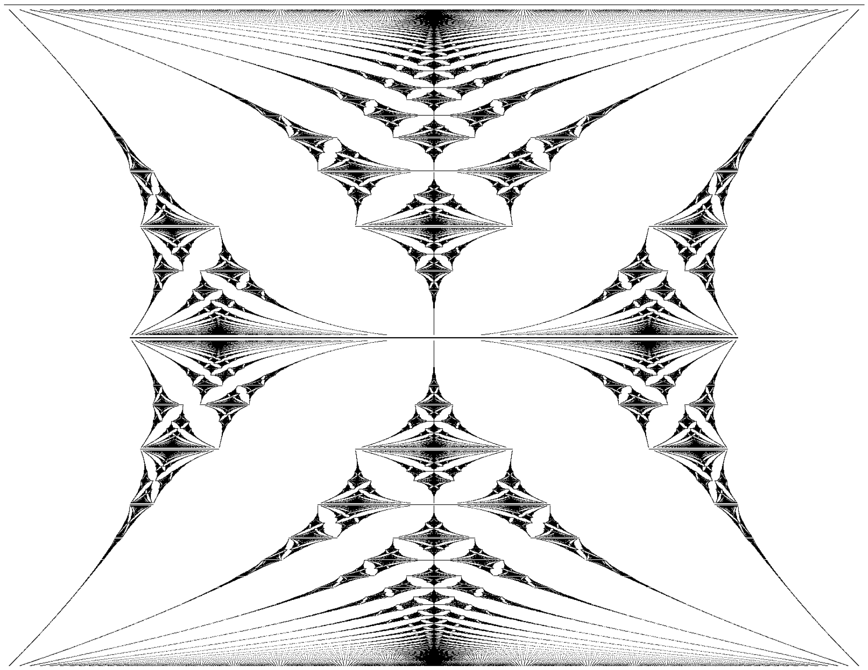

FIGURE 1. The Hofstadter butterfly: The x-axis corresponds to the energy E and the y-axis corresponds to the frequency α.

Recall that the spectral measure $d\mu_\omega$ associated with $H_\omega^{\lambda,\alpha}$ and the state $\delta_0 \in \ell^2(\mathbb{Z})$ is the Borel probability measure with the property that

$$\langle \delta_0, g(H_\omega^{\lambda,\alpha})\delta_0 \rangle = \int_{\mathbb{R}} g(E)\, d\mu_\omega(E)$$

for all bounded, Borel measurable functions g. The density of states measure dN is formally given by

$$dN(E) = \int_{\mathbb{T}} d\mu_\omega(E)\, d\omega.$$

More precisely, it is the Borel probability measure that satisfies

$$\int_{\mathbb{T}} \langle \delta_0, g(H_\omega^{\lambda,\alpha})\delta_0 \rangle \, d\omega = \int_{\mathbb{R}} g(E)\, dN(E)$$

for all bounded, Borel measurable functions g. Consider the Lebesgue decomposition of these measures. Denote the Radon-Nikodym derivative of the absolutely continuous component of $d\mu_\omega$ (resp., dN) by $\mu_{\omega,\mathrm{ac}}(E)$ (resp., $N_{\mathrm{ac}}(E)$).

Lemma 35 (Kotani 1997)**.** *For almost every $E \in \{E : \gamma(E) = 0\}$, we have*

$$N_{\mathrm{ac}}(E) = \int_{\mathbb{T}} \mu_{\omega,\mathrm{ac}}(E)\, d\omega.$$

Theorem 36 (Kotani 1997)**.** *Suppose that the Lyapunov exponent vanishes on $\Sigma^{\lambda,\alpha}$ and*

$$\int_{\mathbb{R}} N_{\mathrm{ac}}(E)\, dE = 1.$$

Then, $H_\omega^{\lambda,\alpha}$ *has purely absolutely continuous spectrum for almost every* $\omega \in \mathbb{T}$*.*

Since the periodic approximants have purely absolutely continuous spectrum, the associated density of states measures are absolutely continuous so we want to take a limit of these quantities as $p_k/q_k \to \alpha$. What about the Lyapunov exponents? It is well known that the Lyapunov exponent vanishes on the spectrum in the periodic case. For the almost Mathieu operator, we have the following result:

Theorem 37 (Bourgain-Jitomirskaya 2002)**.** *If* $\lambda > 0$ *and* α *is irrational, then* $\gamma(E) = \max\{\log \lambda, 0\}$ *for every* $E \in \Sigma^{\lambda,\alpha}$*.*

This shows that for $0 < \lambda \le 1$, the Lyapunov exponent vanishes on the spectrum and we can attempt to apply Kotani's theorem in this coupling regime. We mention in passing that for $\lambda = 1$, the spectrum has zero Lebesgue measure and hence dN is purely singular. For $0 < \lambda < 1$, on the other hand, and a Liouville α it is indeed possible to show that the irrational case is so well approximated by the rational case that $\int_{\mathbb{R}} N_{\mathrm{ac}}(E)\, dE = 1$. Kotani's theorem then yields the result.

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