Lecture 6 of First passage percolation(2005.7.8)

1 A Lower Bound of the Variations

For $x, y \in \mathbb{R}^d$, $T(x,y) = T(x', y')$, $x', y' \in \mathbb{Z}^d$ are the nearest neighbors of x, y. Given a unit vector $\vec{x} \in \mathbb{R}^d$, by the Kingman's subadditive argument,

$$\lim_{n\to\infty} \frac{1}{n} T(0, n\vec{x}) = \mu_F(\vec{x}), \ \ a.s. \ and \ L_1.$$

Theorem 1.1 (Kesten) *$\mu_F(\vec{x}) = 0$ iff $F(0) \geq p_c$.*

Clearly $\mu_F(\vec{x} + \vec{y}) \leq \mu_F(\vec{x}) + \mu_F(\vec{y})$. Let $B_d = \{\vec{x} \in \mathbb{R}^d : \mu_F(\vec{x}) \leq 1\}$.

Theorem 1.2 (Cox+Durrett(1983) Shape Theorem) *Let $B(t) = \{v \in \mathbb{R}^d : T(0,v) \leq t\}$. If $Et(e) < \infty$, for $\forall \epsilon > 0$, $\exists t_0$, $t > t_0$ such that*

$$tB_d(1-\epsilon) \subset B(t) \subset tB_d(1+\epsilon);$$

and let $G(t) = \{v \in \mathbb{R}^d : Et(0,v) \leq t\}$, then

$$G(t)(1-\epsilon) \subseteq B(t) \subseteq G(t)(1+\epsilon).$$

If $F(0) < p_c$, B_d is a compact convex set and ∂B_d is a continuous convex closed curve.

Iden's Growth Model

Time 1:one cell(a unit square), and each integer time a new cell is chosen from the unit squares adjacent to the existed cells with a probability proportional to the $\sharp$ of edges and it has in common with these cells.

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-x} & \text{if } x \geq 0. \end{cases}$$

Let $t_n = \inf\{t : B(t) \text{ contains } n \text{ vertices}\}$, A_n has the same distribution as $B(t_n)$, then $\lim_{n\to\infty} \frac{A_n}{\sqrt{n}}$ exists.
QUESTION: what is the shape?
Let $\lambda = \inf\{x : P(t(e) \leq x) > 0\}$,

Theorem 1.3 (Durrett+Liggett(1981)) *If $\lambda > 0$, $P(t(e) = \lambda) > \vec{p}_c(d)$, B_d contains a flat edge.*

Theorem 1.4 (Kesten) *If $d > 5000$, $F(0) < p_c$, then $\mu_F(x_1) < \mu_F(x_0)$, where $x_0 = (1, 0, ..., 0)$ and $x_1 = (\frac{1}{\sqrt{d}}, ..., \frac{1}{\sqrt{d}})$.*

Theorem 1.5 (Newman+Piza(1995,Ann of Prob. 977-1005)) *If $Et^2(e) < \infty$, $Var(t(e)) > 0$, (1) $\lambda = 0$ and $P(t(e) = 0) = F(0) < p_c(d)$ or (2) $\lambda > 0$ and $P(t(e) = \lambda) < \vec{p}_c(d)$, then*

$$Var(\theta_{0n}) \geq c \log n, \ \theta = a, b, c.$$

Remark. QUESTIONS:
(1) $Var(\theta_{0n}) \geq n^{\alpha}$? $\theta = a, b, c$.
(2) For $d = 2$, if $\lambda > 0$ and $P(t(e) = \lambda) > \vec{p}_c$, $Var(a_{0n}) \geq c \log n$?
(3) For long common subsequence model and match pair model, $Var(L_n) \geq c \log n$?

Lemma 1.1 *For any positive a_k, $m \geq 1$*

$$\sum_{k=1}^{m} a_k^2 \geq \frac{1}{12}(\sum_{k=1}^{m} k^{-1})^{-1}(\sum_{k=1}^{m-1} k^{-1}[k^{-\frac{1}{2}} \sum_{j=1}^{k} a_j])^2$$

Proof of Theorem 1.5. Order bonds in a spiral order $e_1, e_2, ..., e_k, ...$ $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_k = \sigma(e_1, e_2, ..., e_k)$.

$$\begin{aligned} Var(c_{0n}) &= Var(c_{0n} - Ec_{0n}) \\ &= Var(\sum_{i=0}^{c_{n^2}}(E(c_{0n}|\mathcal{F}_{i+1}) - E(c_{0n}|\mathcal{F}_i))) \\ &= \sum_{i=0}^{c_{n^2}} E(E(c_{0n}|\mathcal{F}_{i+1}) - E(c_{0n}|\mathcal{F}_i))^2, \end{aligned}$$

Suppose γ is the route of c_{0n}, that is γ is a path from $(0, ...0)$ to $\partial B(n)$ satisfying $t(\gamma) = c_{0n}$. Consider the case when $c_{0n} = i$, $e_i \in \gamma$ and $t(e_i) = 1$; if $t(e_i)$ is turned to 0, then we get $c_{0n} = i - 1$. Let $F_k = \{t(e_k) = 1, e_k \in \text{a route}\}$, then by Lemma1.1,

$$\begin{aligned} Var(c_{0n}) &\geq \sum_{k=1}^{c_{n^2}} P^2(F_k) \\ &\geq C(\sum_{k=1}^{c_{n^2}} k^{-1})^{-1}(\sum_{k=1}^{c_{n^2}} k^{-1})^2 \\ &\geq C_1(\ln n)^{-1}(\ln n)^2 = C_1 \ln n. \end{aligned}$$

Theorem 1.6 *If $P(\frac{H_n}{n} \leq n^{1-\delta}) > 0$, then $Var(a_{0n}) \geq n^{C(\delta)}$.*

2 Convergence Speed

Given a subadditive ergodic process:
(i) $X_{l,n} \leq X_{l,m} + X_{m,n}$, $0 \leq l < m < n$;
(ii) $\{X_{nk,(n+1)k}\}$ is ergodic for each k;
(iii) $X_{m+1,m+k+1}$ has the same distribution as $X_{m,m+k}$ for all m and k;
(iv) $E\exp(rX_{0,1}) < \infty$, for some r and $EX_{0,n} \geq -cn$, $c > 0$;
then $\lim_{n\to\infty} \frac{X_{0,n}}{n} = \gamma$ a.s. and in L_1.
Let $S_n = M \log^k n$,
(1) $X_{0,n}$ has at least a convergence speed n^{α},

$$P(|X_{0,n} - rn| \geq S_n n^{\alpha}) \leq \exp(-cS_n);$$

(2)$X_{0,n}$ has at least a concentration speed n^α,

$$P(|X_{0,n} - EX_{0,n}| \geq S_n n^\alpha) \leq \exp(-cS_n);$$

(3) For each n, $\exists A_n$ such that

$$P(A_n) \geq \exp(-n^{\frac{\alpha}{2}}), \text{ and on } A_n \;\; X_{0,n} + X_{n,2n} \leq X_{0,2n}.$$

Theorem 2.1 *(1)⇔(2)+(3).*

Application of Theorem 2.1.
There exist $v_0 \in \{0\} \times [0, n^2]$ and $v_{\frac{n}{2}} \in \{\frac{n}{2}\} \times [0, n^2]$, such that $P(A_1) \geq \frac{c}{n^d}$, where $A_1 = \{\omega : T(v_0, v_{\frac{n}{2}})(\omega) = \phi_{0,\frac{n}{2}}\}$. And also there exists $v_n \in \{n\} \times [0, n^2]$ and let $A_2 = \{\omega : T(v_0, v_n)(\omega) = \phi_{0,n}\}$, then $P(A_2) \geq \frac{c}{n^{2d}}$. So we have $P(A_1 \cap A_2) \geq \frac{c}{n^{4d}}$. Similarly $P(A_1 \cap A_2 \cap A_3 \cap A_4) \geq \frac{c}{n^{16d}}$, where $A_3 = \{\omega : T(v_0, v_{\frac{3}{2}n})(\omega) = \phi_{0,\frac{3}{2}n}\}$ and $A_4 = \{\omega : T(v_0, v_{2n})(\omega) = \phi_{0,2n}\}$and on $A_1 \cap A_2 \cap A_3 \cap A_4$, $a_{0,n} + a_{n,2n} \leq a_{0,2n}$, then by Theorem 2.1 we get the convergence speed,

$$P(|a_{0,n} - \mu n| \geq t\sqrt{n}) \leq \exp(-t).$$