Lecture 5 of First Passage Percolation (2005/7/6)

Theorem 1 *(Kesten 1993) If $F(0) < p_c$, there exist $c_1, c_2 > 0$,*

$$P(|\theta_{0n} - E\theta_{0n}| \geq t\sqrt{n}) \leq c_1 \exp(-c_2 t^2) \quad \theta = a, b, c, \phi.$$

This implies :

$$\begin{aligned} Var(\theta_{0n}) &\leq cn \\ P(\theta_{0n} \geq n(\mu + \varepsilon)) &\leq c_1 \exp(-c_2 n) \\ P(\theta_{0n} \leq n(\mu + \varepsilon)) &\leq c_1 \exp(-c_2 n) \\ c, c_1, c_2 &> 0. \end{aligned}$$

Theorem 2 *(Talagrand 1995)(Isoperimetric inequality)*

$$S = \{path\ \gamma\ from\ \vec{0}\ to\ (n, 0, \cdots, 0) : T(\gamma) = a_{0n}\} \qquad r = \sup_{s \in S} |s|,$$

Denote by M the medium of a_{0n}. Then there exist $c_1, c_2 > 0$, such that

$$P(|a_{0n} - M| \geq \mu) \leq c_1 \exp(-c_2 \min(\frac{u^2}{r}, u))$$

and there exist $c_3, c_4, c_5 > 0$, such that

$$P(\exists \gamma, |\gamma|_e = n, t(\gamma) \leq c_3 n) \leq c_4 \exp(-c_5 n)$$

Ising first passage percolation.

Notation: $\Omega = \{1, -1\}^{Z^2}$ Spin configuration.
$\omega(x)$: the spin value at x. $V \subseteq Z^2$. $l(x, y) = \sum_{i=1}^{2} |x_i - y_i|,\ x, y \in Z^2$

$$\text{Hamiltonian} : H^{\omega}_{V,h}(\sigma) = -\frac{1}{2} \sum_{\substack{l(x,y)=1 \\ x,y \in V}} \sigma(x)\sigma(y) - \sum_{x \in V} [h + \sum_{\substack{y \in V^c \\ l(x,y)=1}} \omega(y)]\sigma(x)$$

where $\sigma \in \{-1, 1\}^V$, h$\in R$, is called the external field.
Gibbs measure:

$$Q^{\omega}_{V,\beta,h}(\sigma) = [\sum_{\sigma' \in \Omega_V} \exp(-\beta H^{\omega}_{V,h}(\sigma'))]^{-1} \exp(-\beta H^{\omega}_{V,h}(\sigma))$$

β is called inverse temperature. Then there exists a probability measure on Ω.

$$\mu_{\beta,h}(\ \cdot\ |\mathcal{F}_{V^c})(\omega) = Q^{\omega}_{V,\beta,h}(\cdot)$$

let β_c be the critical value of β, such that if $\beta < \beta_c$, the Gibbs measure is unique.
$C^+(0)$: positive cluster at $\vec{0}$.

$C^-(0)$: negative cluster.

percolation probability: $\theta(\beta,h)=\mu_{\beta,h}(|C^+(0)|=\infty)=\mu_{\beta,h}(|C^-(0)|=\infty))$.

$h_c(\beta)=\sup\{h\in R;\ \theta(\beta,h)=0\}$.

Higuchi (1993, PTRF) if $\beta>\beta_c$, then $h_c(\beta)=0$; if $\beta<\beta_c$, and $h<h_c$, then there exist $c_1,c_2\in(0,\infty)$,such that

$\mu_{\beta,h}(|C^+(0)|\geq n)\leq c_1\exp(-c_2 n)$ and $\mu_{\beta,h}(|C^-(0)|\geq n)\leq c_1\exp(-c_2 n)$.

$$t(e)=\begin{cases}1, & \sigma(u)\neq\sigma(v), l(u,v)=1;\\ 0, & \text{otherwise};\end{cases}\tag{1}$$

For any path $r\in Z^2$, $t(r)=\sum_{e\in r}t(e)$.

$$T(A,B)\stackrel{\triangle}{=}\inf\{t(r):r\ \text{ is a path from } A \text{ to } B\}$$

Then $a_{0n}=T((0,0),(n,0))$. T is called the first passage time.

By Kingman's subadditive

$$\lim_{n\to\infty}\frac{a_{0n}}{n}=\inf_n\frac{E_{\beta,h}a_{0n}}{n}=\nu \qquad \text{a.s. and in } L_1$$

Question: Whether there exist $c_1,c_2>0$, such that

$$\mu_{\beta,h}(|\theta_{0n}-E\theta_{0n}|\geq t\sqrt{n})\leq c_1\exp(-c_2t^2)$$

Theorem 3 *(Higuchi+ Zhang 2000 Ann of Prob 353-) If $\beta<\beta_c, |h|<h_c(\beta)$, then there exist $c_1,c_2>0$*

$$\mu_{\beta,h}(|\theta_{0n}-E_{\beta,h}\theta_{0n}|\geq t\sqrt{n}\log^2 n)\leq c_1\exp(-c_2t^2) \qquad \theta=a,b,c,\phi$$

A sketch of proof for $\theta=c$.

1: Recall mixing property.

If $\beta<\beta_c$ or $h\neq 0, V_1,V_2\subset Z^2, A\subset\mathcal{F}_{V_1}, B\subset\mathcal{F}_{V_2}$, then

$$|\mu_{\beta,h}(A\cap B)-\mu_{\beta,h}(A)\mu_{\beta,h}(B)|\leq c_1(\beta,h)(|V_1|\wedge|V_2|)\exp(-c_2(\beta,h)dist(V_1,V_2))$$

2: We redefine e is open, if

(1) : $t(e)=0$; and

(2) : e does not belong to a open cluster with size larger than $\log^2 n$.

otherwise, We call e is closed. Letting

$$X(e)=\begin{cases}0, & \text{e is open};\\ 1, & \text{e is closed}.\end{cases}\tag{2}$$

$\tilde{c}(0)$: the open cluster containing the origin. Then $\tilde{c}(0)$ is finite.
By Higuchi's exponential decay result, If $\beta < \beta_c, |h| < H_c(\beta)$, then there exist infinitely many closed dual circuits $\Lambda_1^*, \cdots \Lambda_n^* \cdots \quad \Lambda_i^* \bigcap \Lambda_j^* = \emptyset, x \in I(\Lambda_i^*)$, $y \in O(\Lambda_i^*)$, then $T(0,x) = i-1, T(0,y) = i$. By the definition of $X(e)$, $\{\Lambda_i = \Gamma^*\}$ only depends on $\omega(x)$, where $\text{dist}(x, A(\Gamma^*)) \leq \log^2 n, x \in \partial A(\Lambda_i^*), y \in \partial A(\Lambda_{i+1}^*)$, and $||x-y|| \leq \log^2 n, A(\Lambda_i^*)$ is the Area covered by Λ_i^*. Define $\mathcal{F}_k = \sigma$-field generated by $X(e)$, for $e \in A(\Gamma_1^*)$, where $\mathcal{F}_k$ consists of unions of sets of the form $\{(X(e_1), X(e_2) \cdots X(e_m)) \in B : \Lambda_k^* = \Gamma^*, |A(\Gamma^*)|_e = m\}.\Gamma^*$ is a dual circuit. $\{e_1, e_2, \cdots e_k\} \subset A(\Gamma_1^*)$. B is a m-dimensional Borel set. We set $\mathcal{F}_0 = \{\emptyset, \Omega\}$, then $\mathcal{F}_0 \subset \mathcal{F}_1 \cdots \subset \mathcal{F}_k \subset \cdots$. and

$$\tilde{c}_{0n} - E\tilde{c}_{0n} = \sum_{i=0}^{Mn} [E(\tilde{c}_{0n}|\mathcal{F}_{i+1}) - E(\tilde{c}_{0n}|\mathcal{F}_i)].$$

since $|E(\tilde{c}_{0n}|\mathcal{F}_{i+1}) - E(\tilde{c}_{0n}|\mathcal{F}_i)| \leq \log^2 n$, this is the mixing condition. Apply Azuma inequality, we can get that, there exist $c_1, c_2 > 0$, such that

$$P(|\tilde{c}_{0n} - E\tilde{c}_{0n}| \geq t\sqrt{n}\log^2 n) \leq \exp(-\frac{t^2 n \log^4 n}{\sum_{i=1}^{Mn} c \log^4 n}) \leq c_1 e^{-c_2 t^2}$$

Meanwhile we can get that, there exist $c_3, c_4 > 0$ such that

$$P(|\tilde{c}_{0n} - c_{0n}| \geq t\sqrt{n}\log^2 n) \leq P(A) \leq c_3 \exp(-c_4\sqrt{n}\log^2 n)$$

where

$$A = \{\exists v_1, v_2, \cdots, v_m,\ v_i \neq v_j, i \neq j, m \geq t\sqrt{n}\log^2 n, |c(v_i)|_e \geq \log^2 n\ , i = 1, 2, \cdots, m\}$$