Lecture 4 of First passage percolation.(2005.7.4)

1 A Sketch Proof of the right tail limit theorem for $\phi_{0,n}$

Theorem 1.1 *If $F(0) < p_c$ and $Ee^{rt(e)} < \infty$ for some r, then for all $\epsilon > 0$, there exists a constant $\beta(\epsilon, F, d)$, such that*

$$\lim_{n\to\infty} -\frac{1}{n^d} \log P(\phi_{0,n} \geq n(\mu+\epsilon)) = \beta(\epsilon, F, d).$$

Sketch of proof for d=2, and t(e)=0 and 1.

By the max-flow=min-cut theorem, $\phi_{0,n}$=the number of path(1) in the dual. So, we can get that(Multisubadditive)

$$P(\phi([0,n]^2) \geq n(\mu+\epsilon)) \leq n\binom{n}{\frac{n}{2}} P(\phi([0,n]^2) \geq n(\mu+\epsilon); x_1, \cdots, x_m \; in\, the\, boundary\, fixed),$$

$$P(\phi([0,n]\times[n,2n]) \geq n(\mu+\epsilon)) \leq n\binom{n}{\frac{n}{2}} P(\phi([0,n]\times[n,2n]) \geq n(\mu+\epsilon); x_1, \cdots, x_m \; in\, the\, boundary\, fixed),$$

$$P(\phi([n,2n]\times[0,n]) \geq n(\mu+\epsilon)) \leq n\binom{n}{\frac{n}{2}} P(\phi([n,2n]\times[0,n]) \geq n(\mu+\epsilon); x_1, \cdots, x_m \; in\, the\, boundary fixed),$$

and

$$P(\phi([n,2n]\times[n,2n]) \geq n(\mu+\epsilon)) \leq n\binom{n}{\frac{n}{2}} P(\phi([n,2n]\times[n,2n]) \geq n(\mu+\epsilon); x_1, \cdots, x_m \; in\, the\, boundary\, fixed)$$

Then

$$P(\phi_{0,n} \geq n(\mu+\epsilon)) \leq n^4 \binom{n}{\frac{n}{2}}^4 P(\phi_{0,2n} \geq 2n(\mu+\epsilon)) \leq n^4 2^{4n} P(\phi_{0,2n} \geq 2n(\mu+\epsilon)).$$

Since $\frac{n^4 2^{4n}}{e^{n^2}} \to 0$, thus the limit exists.

2 Long Common Subsequences (match pairs)

Chvatal and Sankoff(1975)

Probability Theory and Combinatorial optimization (1997) Steele Siam.

Consider two sequence of

$$X_1, X_2, \cdots, X_n, \cdots,$$

$$Y_1, Y_2, \cdots, Y_n, \cdots.$$

of independent identically distributed random variables with values in a finite alphabet.

$L_n = \max\{k : X_{i_1} = Y_{j_1}, X_{i_2} = Y_{j_2}, \cdots, X_{i_k} = Y_{j_k}\}$, the maximum is taken over all pairs of subsequences $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n, 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n$.

Example:$HHTHHTHHT, THHTHTTHT, L_n = 7$.

By Kingman's subadditive argument, $\lim_{n\to\infty} \frac{L_n}{n} = \gamma(F), a.s.\ L_1$.

Arratin and Waterman(1994)

$$S(i,j)=\begin{cases}1 & X_i=Y_j\\ 0 & X_i\neq Y_j\end{cases}.$$

1.one unit to right from (i,j) to (i+1,j) or (i,j+1) which cost time 0.

2.diagonally from (i,j) to(i+1,j+1) which cost time zero if s(i+1,j+1)=0 or time -1 if s(i+1,j+1)=1.

Then, $-L_n=\inf_\gamma\{t(\gamma): \gamma\ is\ a\ path\ from\ (0,0)\ to\ (n,n)\}$.

Theorem 2.1 *(***Hoeffding-Azumn inequality 1963,1967***)*
Let $\{M_i\}$ *be a martingale sequence ,* $d_i=M_{i+1}-M_i$*(martingale difference), and* $|d_i|\le c_i$*, then*

$$P(\sum_{i=1}^n d_i\ge\lambda)\le 2\exp(-\frac{\lambda^2}{2\sum_{i=1}^n c_i^2}).$$

Let $Z_i=(X_i,Y_i)$, $\mathcal{F}_0=\{\emptyset,\Omega\}$, $\mathcal{F}_k=\sigma(Z_1,Z_2,\cdots,Z_k)$, $M_i=E(L_n|\mathcal{F}_i)$ is a martingale sequence.

$d_i=E(L_n|\mathcal{F}_{i+1})-E(L_n|\mathcal{F}_i)$, $L_n-E(L_n)=\sum_{i=0}^{n-1}d_i$, and

$$E(L_n|\mathcal{F}_i)=\sum_{z_{i+1},\cdots,z_n}L(Z_1,Z_2,\cdots,Z_i,z_{i+1},\cdots,z_n)P(Z_{i+1}=z_{i+1},\cdots,Z_n=z_n),$$

$$E(L_n|\mathcal{F}_{i+1})=\sum_{z_{i+2},\cdots,z_n}L(Z_1,Z_2,\cdots,Z_{i+1},z_{i+2},\cdots,z_n)P(Z_{i+2}=z_{i+2},\cdots,Z_n=z_n)$$
$$=\sum_{z_{i+1},\cdots,z_n}L(Z_1,Z_2,\cdots,Z_{i+1},z_{i+2},\cdots,z_n)P(Z_{i+1}=z_{i+1},Z_{i+2}=z_{i+2},\cdots,Z_n=z_n).$$

Since $|L(Z_1,Z_2,\cdots,Z_i,z_{i+1},\cdots,z_n)-L(Z_1,Z_2,\cdots,Z_{i+1},z_{i+2},\cdots,z_n)|\le 2$, then $|d_i|\le 2$.

So we get this theorem:

Theorem 2.2

$$P(|L_n-EL_n|\ge t)\le 2\exp(-\frac{t^2}{8n}).$$

Application:

1.$Var(L_n)=\int_0^\infty 2tP(|L_n-EL_n|\ge t)dt\le n.$

2.$\forall\epsilon\ge 0$, take n, s.t. $\gamma n-\frac{\epsilon}{2}n\le EL_n\le n+\frac{\epsilon}{2}n$, so

$$P(L_n\ge n(\gamma+\epsilon))\le P(L_n-EL_n\ge\frac{\epsilon}{2}n)\le P(|L_n-EL_n|\ge\frac{\epsilon}{2}n)\le 2\exp(-\frac{\epsilon^2n^2}{32}),$$

$$P(L_n\le n(\gamma-\epsilon))\le P(L_n-EL_n\le\frac{-\epsilon}{2}n)\le P(|L_n-EL_n|\ge\frac{\epsilon}{2}n)\le 2\exp(-\frac{\epsilon^2n^2}{32}).$$

but maybe the two inequality is not sharp.

Theorem 2.3 *(First passage percolation,1993,Kesten,Ann. of Prob. 296-338.)*
If $F(0) \leq 0$, $Ee^{-rt(e)} < \infty$ *for some* r*, there exist* $c_1 = c_1(F, d)$*, such that*

$$P(|\theta_n - E\theta_n| \geq tn^{\frac{1}{2}}) \leq c_1 \exp(-c_2 t^2).$$

where $\theta = a, b, c, \phi$*.*

Sketch of Kesten's proof:
Order the bonds of Z^d, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_k = \sigma(t(e_1), t(e_2), \cdots, t(e_k))$, $M_i = E(\theta_n|\mathcal{F}_i)$ is a martingale sequence.

$d_i = E(\theta_n|\mathcal{F}_{i+1}) - E(\theta_n|\mathcal{F}_i)$, $\theta_n - E(\theta_n) = \sum_{i=0}^{n-1} d_i$.

Then estimate that $\sum_{k=1}^{n} d_k^2 \leq Cn$, finally use the Hoeffding-Azumn inequality.