Next we will discuss two important tails:

$$\begin{cases} \theta_{0,n} \geq n(\mu+\varepsilon) & \text{right tail} \\ \theta_{0.n} \leq n(\mu-\varepsilon) & \text{left tail} \end{cases}$$

$\theta = a, b, c$ or ϕ,where $a_{0,n}$,$b_{0,n}$ and $c_{0,n}$ has been defined,while $\phi_{0,n}$,which is called "Face-face first-passage time",is defined as follows:

$\phi_{0,n}$:=inf$\{T(\gamma)$:γ is a path from A to B, $\gamma \subseteq [0,n]^d\}$.

where A:=$\{0\}\times[0,n]^{d-1}$, B:=$\{$n$\}\times[0,n]^{d-1}$.

The examples of $d = 2$ and $d = 3$ can easily be imagined.

Theorem 1. minimum-cut=maximum flow.

Grimmett and Kesten (1984,PTRF) proved the following:

$$\lim_{n\longrightarrow\infty} \frac{\phi_{0,n}}{n} = \mu \qquad a.s. \quad and \quad in \quad L_1.$$

First we check the left tail.

Theorem 2.(Kesten,1986) Assume that for some $r > 0$, $Ee^{rt(e)} < \infty$ and $F(0) < p_c$, then for all $\varepsilon > 0$, there exists constants c_1 and c_2, such that

$$P(\theta_{0,n} \leq n(\mu-\varepsilon)) \leq c_1 \exp(-c_2 n).$$

$\theta = a, b, c$ or ϕ.

Remark. The assumption "$Ee^{rt(e)} < \infty$" is not in Kesten(1986). In that paper it just assumed "$Et(e) < \infty$", but given the assumption "$Ee^{rt(e)} < \infty$", the proof will be much easier.

Proof of Theorem 2 for $\theta = c$

$$\begin{aligned} P(c_{0,n} \leq n(\mu-\varepsilon)) &\leq P(c_M(a_0) + c_M(a_1) + \ldots + c_M(a_Q) \leq n(\mu-\varepsilon)) \\ &\leq P(c'_M(a_0) + c'_M(a_1) + \ldots + c'_M(a_{Q-1}) \leq n(\mu-\varepsilon)) \qquad (by \quad Lemma \quad 2) \\ &\leq P(\sum_{i=0}^{Q-1} c'_M(a_i) \leq QM(\mu-\varepsilon)) \end{aligned}$$

Since $\lim\limits_{M\longrightarrow\infty} \frac{c_{0,M}}{M} = \mu$. Take M, such that $Ec'_{0,M} \geq M(\mu - \frac{\varepsilon}{2})$.

$$\sum_{i=0}^{Q-1} [c'_M(a_i) - Ec'_M(a_i)] \leq QM(\mu-\varepsilon) - QM(\mu-\frac{\varepsilon}{2}) = -QM(\frac{\varepsilon}{2}).$$

Let $X_i = c'_M(a_i) - Ec'_M(a_i)$, then $EX_i = 0$, $\int\limits_0^1 e^{\beta x} dF_{X_1}(x) < \infty$.

For $\beta = \beta(M, F, d)$. By a standard large deviation result, $\exists c_1, c_2$, such that

$$P(c_{0,n} \leq n(\mu-\varepsilon)) \leq P(\sum_{i=1}^{Q} X_i \leq -QM(\frac{\varepsilon}{2})) \leq c_1 \exp(-c_2 n).$$

Remark. The proof of Theorem 2 for $\theta = b$ or ϕ is quite easy, while the proof of it for $\theta = a$ is relatively difficult.

Theorem 3. For all $\varepsilon > 0$, if $F(0) < p_c$, then

$$\lim_{n\longrightarrow\infty} -\frac{1}{n} log P(\theta_{0,n} \leq n(\mu-\varepsilon)) = \alpha(\varepsilon, F) > 0.$$

$\theta = a, b, c$ or ϕ.

Proof of Theorem 3 for $\theta = a$

For any $n \in Z^+$, let $X_n := -\log P(a_{0,n} \leq n(\mu-\varepsilon))$.

for all $m, n \in Z^+$, note that "$a_{0,m} \leq m(\mu-\varepsilon)$" and "$a_{m,m+n} \leq n(\mu-\varepsilon)$" are both **decreasing events**. So using FKG inequality, we get

$$P(a_{0,m} \leq m(\mu-\varepsilon))P(a_{m,m+n} \leq n(\mu-\varepsilon)) \leq P(a_{0,m} \leq m(\mu-\varepsilon), a_{m,m+n} \leq n(\mu-\varepsilon)).$$

But obviously, $P(a_{m,m+n} \leq n(\mu-\varepsilon)) = P(a_{0,n} \leq n(\mu-\varepsilon))$, $\{a_{0,m} \leq m(\mu-\varepsilon), a_{m,m+n} \leq n(\mu-\varepsilon)\} \subsetneqq \{a_{0,m+n} \leq (m+n)(\mu-\varepsilon)\}$.

So we get

$$P(a_{0,m} \leq m(\mu-\varepsilon))P(a_{0,n} \leq n(\mu-\varepsilon)) \leq P(a_{0,m+n} \leq (m+n)(\mu-\varepsilon)).$$

It equals to

$$-\log P(a_{0,m+n} \leq (m+n)(\mu-\varepsilon)) \leq [-\log P(a_{0,m} \leq m(\mu-\varepsilon))]+[-\log P(a_{0,n} \leq n(\mu-\varepsilon))].$$

Thus $X_{m+n} \leq X_m + X_n$ holds for any $m, n \in Z^+$. Use a small analytical trick we can easily get $\lim_{n\longrightarrow\infty} \frac{X_n}{n}$ exists, so

$$\lim_{n\longrightarrow\infty} -\frac{1}{n} log P(a0, n \leq n(\mu-\varepsilon)) = \alpha(\varepsilon, F) > 0.$$

Remark. When $\theta = b, c$ or ϕ, the proof of Theorem 3 is quite similar.

Next we will check the right tail.

Theorem 4.(Kesten,1986) If $F(0) < p_c$ and $Ee^{rt(e)} < \infty$ for some r, then for all $\varepsilon > 0$, there exist c_1 and c_2, such that

$$P(\theta_{0,n} \geq n(\mu+\varepsilon)) \leq c_1 \exp(-c_2 n).$$

A general case is as follows:

Theorem 5.(Chow & Zhang,2003,Annals of Applied Probability,1601-1614) If $F(0) < p_c$ and $Ee^{rt(e)} < \infty$ for some r, then for all $\varepsilon > 0$, there exist c_1 and c_2, such that

$$P(\theta_{0,n} \geq n(\mu+\varepsilon)) \leq c_1 \exp(-c_2 n^d).$$

To prove this theorem, we need to prove some lemmas first.

Lemma 3. Define

$$T_{l,k,m} := \inf\{t(\gamma) : \gamma \quad is \quad a \quad path \quad from \quad (l,0,0) \quad to \quad (k,0,0), \quad and \quad \gamma \subsetneqq [0,k]\times[-m,m]^2\}.$$

then

$$\lim_{m\longrightarrow\infty} \frac{T_{0,m,m}}{m} = \mu \qquad a.s. \quad and \quad in \quad L_1.$$

Lemma 4. If $F(0) < p_c$, $\varepsilon > 0$ and $E\exp(rt(e)) < \infty$ for some $r > 0$. Then there exists a constant $\eta > 0$, such that

$$P(T_{0,k,m} \geq k(\mu+\varepsilon)) \leq \exp(-\eta k) \qquad for \quad all \quad k \geq m.$$

We will take several steps to prove Theorem 5.

Step 1, Proof of Lemma 3.

Define

$$T_{m,n}(k) := \inf\{t(\gamma) : \gamma \quad is \quad a \quad path \quad from \quad (m,0,0) \quad to \quad (n,0,0), \quad and \quad \gamma \subseteqq [m-k, n+k] \times Z^{d-1}\}.$$

By a standard subadditive argument,

$$\lim_{n\longrightarrow\infty} \frac{t_{0,n}(k)}{n} = \mu(k) = \inf_n \frac{Et_{0,n}(k)}{n} \qquad a.s. \quad and \quad in \quad L_1.$$

$\because \forall n$, $t_{0,n}(k) \downarrow$ as $k \uparrow$,

$\therefore \mu(k) \downarrow$ as $k \uparrow$.

For fixed k and ω, let $a_{0,n}(\omega) := \lim_{k\longrightarrow\infty} t_{0,n}(k)(\omega)$, $\mu := \lim_{n\longrightarrow\infty} \frac{a_{0,n}}{n}$, then obviously, $a_{0,n} \leq t_{0,n}(k)$, $\mu \leq \mu(k)$.

Fix n, $\forall \varepsilon > 0$, take k large such that

$$\mu(k) \leq \frac{Et_{0,n}(k)}{n} \leq \frac{Ea_{0,n}}{n} + \varepsilon \leq \mu + \varepsilon.$$

So for k large, we have $\mu \leq \mu(k) \leq \mu + \varepsilon$, which is sufficient to get $\lim_{n\longrightarrow\infty} \mu(k) = \mu$.

The next thing for us to do is to compare $t_{0,k}(0)$ and $T_{0,k,m}$.

Let $\gamma \subseteqq [0,n] \times Z^{d-1}$ be a path from (0,0,0) to (n,0,0) with $T(\gamma) = t_{0,n}(0)$. Define

$$h_n(\gamma) := \max_{2\leq i\leq 3} \{|m_i| : (m_1, m_2, m_3) \in \gamma\},$$

and

$$h_n := max\{h_n(\gamma) : \gamma \quad is \quad a \quad route \quad for \quad t_{0,n}\}.$$

It is known(see **Theorem 8.15 in Smythe and Wierman(1978)**) that

$$\limsup_{n\longrightarrow\infty} \frac{h_n}{n} \leq 1 \qquad almost \quad surely. \qquad (*)$$

Let $H_n := \{\frac{h_n}{n} \leq 1\}$. Then

$$Et_{0,m}(0) \geq E(t_{0,m}(0); H_m) = E(T_{0,m,m}; H_m) = E(T_{0,m,m}) - E(T_{0,m,m}; H_m^c),$$

$$T_{0,m,m} \leq \sum_{e\in\gamma} t(e).$$

where γ is the path from (0,0,0) to (m,0,0) along the first coordinate.

By Cauchy-Schwarz inequality, we have:

$[\frac{E(T_{0,m,m};H_m^c)}{m}]^2 \leq E(\frac{\sum_{e\in\gamma} t(e)}{m})^2 \cdot P(H_m^c)$.

By (*) we have $P(H_m^c) \to 0 (m \to \infty)$. So $\lim_{m\to\infty} \frac{E(T_{0,m,m};H_m^c)}{m} = 0$.

Then $\mu = \lim\limits_{m\to\infty} \frac{Et_{0,m}(0)}{m} \ge \lim\limits_{m\to\infty} \frac{ET_{0,m,m}}{m}$.

Note that $\lim\limits_{m\to\infty} \frac{ET_{0,m,m}}{m} \ge \lim\limits_{m\to\infty} \frac{Et_{0,m}(0)}{m} \ge \mu$. So Lemma 3 follows.

Remark. For $M > 0$, let $P(0 \le t(e) \le M) = \delta > 0$.

Let A(k) be the event that all those 2k edges from (-k,0,0) to (0,0,0) and from (n,0,0) to (n+k,0,0) along the first coordinate taking values less than M, then

$P(A(k)) \ge \delta^{2k}$ on A(k), $\quad t_{-k,n+k}(0) \le t_{0,n}(k) + 2kM$.

So $\mu(0) \le \frac{Et_{-k,n+k}(0)}{n} = \frac{Et_{0,n+2k}(0)}{n} \le \frac{Et_{0,n}(k)}{n} + \frac{2kM}{n}$.

Let $n \to \infty$, we get $\mu(0) \le \mu(k) \to \mu(k \to \infty)$. $\quad \therefore \mu(0) \le \mu$.

But we have $\mu(k) \downarrow$ as $k \uparrow$ and $\mu(k) \to \mu(k \to \infty)$.

So we have $\mu(k) = \mu(\forall k)$, which is certainly a very strange thing.

Step 2, Proof of Lemma 4 from Lemma 3.

Let $k = nm$, we can obviously see from the graph that

$$P(T_{0,nm,m} \ge nm(\mu + 2\varepsilon)) \le P(\sum_{i=0}^{n-1} T_{im,(i+1)m,m} \ge nm(\mu + 2\varepsilon)).$$

where $T_{im,(i+1)m,m}$ are i.i.d., with a common distribution as $T_{0,m,m}$. We take m large such that $ET_{0,m,m} \le m(\mu + \varepsilon)$.

By Lemma 3 and a standard large deviation result, we get

$P(T_{0,nm,m} \ge nm(\mu + 2\varepsilon)) \le P(\sum\limits_{i=0}^{n-1} T_{im,(i+1)m,m} \ge nm(\mu + 2\varepsilon))$

$\le P(\sum\limits_{i=0}^{n-1} (T_{im,(i+1)m,m} - ET_{im,(i+1)m,m}) \ge nm\varepsilon) \le \exp(-Cnm\varepsilon)$,

where $C > 0$ is a constant. So Lemma 4 follows.

Step 3, Proof of Theorem 5 from Lemma 4 for $\theta = \phi$.

Take $k = mn$ and divide $[0,k]^2$ into $(\frac{k}{m})^2 = n^2$ equal subsquares of size $m \times m$, which are called $S_1, S_2, \cdots, S_{n^2}$.

Since $\{\phi([0,k] \times S_i) \ge k(\mu + \varepsilon)\}$ and $\{\phi([0,k] \times S_j) \ge k(\mu + \varepsilon)\}$ are independent for $i \ne j$, and $\{\phi_{0,k} \ge k(\mu + \varepsilon)\} \subseteqq \bigcap\limits_{i=1}^{n^2} \{\phi([0,k] \times S_i) \ge k(\mu + \varepsilon)\}$, we then have

$P(\phi_{0,k} \ge k(\mu + \varepsilon)) \le [P(\phi([0,k] \times [0,m]^2) \ge k(\mu + \varepsilon))]^{n^2}$. (1)

By Lemma 4 and translation invariance, we have

$P(\phi([0,k] \times [0,m]^2) \ge k(\mu + \varepsilon)) \le P(T_{0,k,\frac{m}{2}} \ge k(\mu + \varepsilon)) \le \exp(-\eta k)$. (2)

Combining (1) and (2), we get that for k large and $m|k$,

$$P(\phi_{0,k} \ge k(\mu + \varepsilon)) \le \exp(-\eta k n^2) = \exp(-C' k^3),$$

where $C' = \frac{\eta}{m^2}$. So Theorem 5 follows.

A further result is as follows.

Theorem 5.(Chow & Zhang,2003,Annals of Applied Probability,1601-1614) If $F(0) < p_c$ and $Ee^{rt(e)} < \infty$ for some r, then for all $\varepsilon > 0$, there exists a constant $\beta(\varepsilon, F, d)$, such that

$$\lim_{n\to\infty} \frac{-1}{n^d} \log P(\phi_{0,n} \ge n(\mu + \varepsilon)) = \beta(\varepsilon, F, d).$$