Theorem 1 (Kesten,1986) If $F(0) < p_c$, then $\mu(F) > 0$;

If $F(0) \geqslant p_c$, then $\mu(F) = 0$.

To prove this theorem, we need the following lemma.

Lemma 1 If $F(0) < p_c$, there exist constants $c_1, c_1, c_3 > 0$ such that

$$P(\text{There exists a self-avoiding path } \gamma \text{ from 0 that contains at least } n \text{ bonds but } T(\gamma) \leqslant c_1 n) \leqslant c_2 \mathrm{e}^{-c_3 n}.$$

We prove the theorem with the lemma first, and then we give the proof of the lemma.

Proof : For $F(0) > p_c$, we consider the following percolation:

$$e = \begin{cases} \text{open}, & t(e) = 0, \\ \text{closed}, & t(e) > 0. \end{cases}$$

Let $C(0)$ be the open cluster containing the origin. Since $F(0) > p_c$, we have $P(C(0) = \infty) > 0$. That is to say, with a positive probability, we have $C_{0,n} = 0$ for all $n \in \mathbb{N}$. Therefore,

$$P\big(\lim_{n\to\infty} \frac{C_{0,n}}{n} = 0\big) > 0.$$

But we have

$$\lim_{n\to\infty} \frac{C_{0,n}}{n} = \mu \quad a.s. \qquad ,$$

we can conclude that $\mu = 0$.

For $F(0) < p_c$, consider

$$E\theta_{0,n} = \sum_i P(\theta_{0,n} > i), \qquad \theta = a, b, c.$$

With lemma 1 we have

$$\begin{aligned} & P(\theta_{0,n} \leqslant c_1 n) \\ \leqslant & P(\text{There is a self-avoiding path } \gamma \text{ that contains at least n bonds satisfying } T(\gamma) \leqslant c_1 n) \\ \leqslant & c_2 \exp(-c_3 n) < \frac{1}{2} \ , \end{aligned}$$

for large n.

Consequently, $P(\theta_{0,n} > c_1 n) \geqslant 1/2$. Therefore, $E\theta_{0,n} \geqslant c_1 n/2$,

$$\mu = \lim_{n\to\infty} \frac{E\theta_{0,n}}{n} \geqslant \frac{1}{2} c_1 > 0.$$

The case $F(0) = p_c$ results from the continuity.

Now we prove lemma 1.

Proof: We use the method of renormalization.

(i) Assume $F(x) = p\mathbf{1}_{(x\geqslant 0)} + (1-p)\mathbf{1}_{(x\geqslant 1)}$. Here we only consider the case $d = 3$.

For integer $k \geqslant 1$, $u \in \mathbb{Z}^3$, consider the cube

$$B_k(u) = \prod_{i=1}^{3} [ku_i, ku_i + k)$$

with lower left hand corner at ku.

For each $B_k(u)$, it has at most 26 neighbor cubes, where we count its diagonal neighbors.

Given a path γ, we say a cube $B_k(u)$ is good if $\gamma \cap B_k(u) \neq \emptyset$.

For each good cube $B_k(u)$, we say it is excellent if $\exists\ e \in \gamma \cap B_k(u) + [-k, k]^3$ such that $t(e) = 1$. Otherwise we say it is bad.

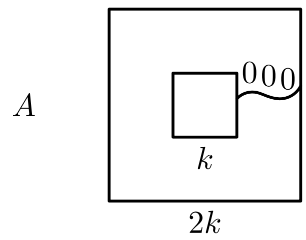
Let Q be the number of good cubes. Since γ contains n edges, we get $n(k+1)^{-d} \leqslant Q \leqslant n$.

Note that these good cubes are connected. If $T(\gamma) \leqslant c_1 n$ for $c_1 < (k+1)^{-d}/54$, there are at most $Q/2$ excellent cubes among the Q good cubes and there are at least $Q/2$ bad cubes. Of these bad cubes, we consider the bigger cubes $B_k(u) + [-k, k]^3$. At least $Q/54$ of them are disjoint.

Now we estimate the probability of a bad cube. See the figure in the next page.

We have

$$P(\text{bad cube}) = P(A) \leqslant k^d c_4 \exp(-c_5 k) \leqslant c_6 \exp(-c_7 k).$$



where c_i is independent of k. Therefore,

$$
\begin{aligned}
&P(\exists\, \gamma \text{ with } |\gamma| = n,\ T(\gamma) \leqslant c_1 n) \\
\leqslant &\sum_{Q \geqslant n(k+1)^{-d}} (26)^Q \sum_{m \geqslant Q/54} \binom{Q}{m} \big[P(\text{bad cube})\big]^m \\
\leqslant &\sum_{Q \geqslant n(k+1)^{-d}} (26)^Q \cdot 2^Q \cdot [c_6 \exp(-c_7 k)]^{Q/54} \\
\leqslant &\sum_{Q \geqslant n(k+1)^{-d}} \exp(Q \log 52 + \frac{Q}{2} \log c_6 + \log Q - c_7 k \cdot \frac{Q}{54}) \\
\leqslant &\, c_8 \exp(-c_9 n).
\end{aligned}
$$

when we take k large enough.

(ii) For any distribution F with $F(0) < p_c$, there exists η such that

$$
\begin{aligned}
&P(t(e) < \eta) < p_c, \\
&P(t(e) \geqslant \eta) > 1 - p_c.
\end{aligned}
$$

The same argument as (i) shows that

$$P(\exists\, \gamma \text{ with } T(\gamma) \leqslant c_1 n) \leqslant c_8 \exp(-c_9 n).$$

This is the proof of lemma 1.

Now we give an alternative proof of lemma 1. We will use the following lemma.

Lemma 2 $P($There exists a self-avoiding path γ from v to w

that passes $v = v_1, \cdots v_l = w$ with $T(\gamma) \leqslant x)$

$$\leqslant P\Big(\sum_{i=1}^{l-1} T'(v_i, v_{i+1}) < x\Big).$$

where $T'(v_i, v_{i+1})$ are independent and $T'(v_i, v_{i+1}) \stackrel{d}{=} T(v_i, v_{i+1})$.

This lemma can be proved with BK Inequality.

Proof of lemma 1 from lemma 2:

We can find a sequence of points $a_0, a_1, \cdots, a_Q$ along the path where there exists $k = k(i)$ such that $a_i, a_{i+1} \in B_k(u)$. Then

$$
\begin{aligned}
& P(\exists\ \gamma \text{ with } T(\gamma) \leqslant cn,\ |\gamma| = n) \\
\leqslant & \sum_{Q \geqslant n(k+1)^{-d}} \Big[\sum_{a_1, \cdots, a_Q} P\Big(\sum_{i=0}^{Q-1} T(a_i, a_{i+1}) \leqslant c_1 n \Big) \Big] \\
\leqslant & \sum_{Q \geqslant n(k+1)^{-d}} \Big[\sum_{a_1, \cdots, a_Q} P\Big(\sum_{i=0}^{Q-1} T'(a_i, a_{i+1}) \leqslant c_1 n \Big) \Big] \qquad \text{(Lemma 2)} \\
\leqslant & \, \mathrm{e}^{\xi c_1 n} \sum_{Q \geqslant n(k+1)^{-d}} \Big[\sum_{a_1, \cdots, a_Q} \prod_{i=0}^{Q-1} E \mathrm{e}^{-\xi T'(a_i, a_i+1)} \Big] \\
\leqslant & \, \mathrm{e}^{\xi c_1 n} \sum_{Q \geqslant n(k+1)^{-d}} \Big[\sum_{a \in B(k)} E \mathrm{e}^{-\xi T'(0,a)} \Big]^Q \\
\leqslant & \, \mathrm{e}^{\xi c_1 n} \sum_{Q \geqslant n(k+1)^{-d}} \Big(\frac{1}{2} \Big)^Q \qquad (*) \\
\leqslant & \, \mathrm{e}^{\xi c_1 n} \Big(\frac{1}{2} \Big)^{n(k+1)^{-d} - 1} \leqslant c_2 \exp(-c_3 n).
\end{aligned}
$$

The last $\leqslant$ we take ξ small enough that $\mathrm{e}^{\xi c_1} (1/2)^{(k+1)^{-d}} < 1$.