First Passage Percolation

Yu Zhang

June 27,2005

1 Some Results of Percolation

Hammelsley first studied the model of percolation around 1957.
Let $\mathbf{Z} = \{-\infty, \ldots, -1, 0, 1, 2, \ldots, +\infty\}$ and $\mathbf{Z^d} = \{x = (x_1, x_2, \ldots, x_d), x_i \in \mathbf{Z}, i = 1, \ldots, d\}$.
The distance of a pair of points x, y is defined by

$$l(x,y) = \sum_{i=1}^{d} \mid x_i - y_i \mid .$$

Now add a bond $e_{x,y}$ between x and y if $l(x,y) = 1$. Denote $\mathbf{E}^d = \{bond\}$ and define the bond-index sequence of i.i.d random variables $X(e) = 1$(open) with probability p, 0 (closed) with probability $1-p$.
The sample space of percolation is $\Omega = \{0,1\}^{\mathbf{E}^d}$.
An open path is an self-avoiding sequence from u to v:$u = x_0, e_1, x_1, e_2, \ldots, e_{n-1}, x_n = v$, with all the edges open. We write $u \leftrightarrow v$ if there is at least an open path from u to v. Now we define the open Cluster containing x as $C(x) = \{y \in \mathbf{Z^d} : x \leftrightarrow y\}$ and $\theta(p) = P_p(\mid C(0) \mid = \infty)$ as the critical probability of percolation model. Since the lattice is translation invariant, it follows that the above definition is well defined.
Let $p_c = p_c(d) = \sup\{p : \theta(p) = 0\}$ be the critical point of the percolation model.

Theorem 1.1 (Hammersley). $0 < p_c(d) < 1$.

Proof. Use some combinatorial estimations, it is easy to get the upper bound. The lower bound comes from the dual property of lattice $\mathbf{L}^d$.

1.1 The FKG Inequality

To state the FKG inequality we first define a partial order "$\leq$" on Ω by $\omega \leq \omega^{'}$ if $\omega(e) \leq \omega(e)^{'}$ for all $e \in \mathbf{E}^d$. An event A is said to be increasing if

$$I_A(\omega) \leq I_A(\omega') \quad for \quad all \quad \omega \leq \omega'.$$

Note that the even $\{\mid C(0) \mid = \infty\}$ is an increasing event.

Theorem 1.2 (FKG). *If both A and B are increasing events then*

$$P(AB) \geq P(A)P(B).$$

1.2 The Russo's Formula

Let A be an event, e is said to be pivotal for(A, ω) if $I_A(\omega) \neq I_A(\omega'), \omega'(e) = 1 - \omega(e)$ and $\omega'(f) = \omega(f)$ for all $f \in \mathbf{E}^d, f \neq e$.
Example: Find some pivotal edges of the event {there is at least one left-right crossing and one top-bottom crossing}

Theorem 1.3. *Let A be an increasing even depending on only finite edges of $\mathbf{E}^d$, then*

$$\frac{dP_p(A)}{dp} = \mathbf{E}_p N$$

where N is the number of pivotal edges of A.

1.3 Estimating the tail probability of the size of open cluster

Theorem 1.4. *If $p < p_c$,then there exist constants $C_1(p), C_2(p)$ so that*

$$P(\mid C(0) \mid \geq n) \leq C_1(p) \exp(-C_2(p)n)$$

Theorem 1.5. *If $p > p_c$, there exist $C_1(p)$ and $C_2(p)$, so that*

$$P(\infty >\mid C(0) \mid \geq n) \leq C_1(p) \exp(-C_2(p)n^{\frac{d-1}{d}})$$

Theorem 1.6. *If $p = p_c, d = 2, P_{p_c}(\mid C(0) \mid \geq n) \leq n^{-\delta}$*

Open Problem: If $p = p_c, d \geq 2, then \quad \theta(p_c, d) = 0$

1.4 oriented percolation

Consider the percolation on oriented lattices an in particular on the "north-east" lattice $\overrightarrow{\mathbf{L}}^d$ obtained by oriented each edge of $\mathbf{L}^d$ in the direction of increasing coordinate-value. Denote $\{u \to v\}$ as the event that there is an oriented open path. Let $\Omega_\infty^{(0,0)}$ be the event that there is at least an infinite oriented open path from $(0, 0)$ to ∞. $\theta(p) = P(\Omega_\infty^{0,0})$ is the critical probability of oriented percolation. Let $p_c = \sup\{p : \theta(p) = 0\}$ then $0 < \overrightarrow{p}_c < 1$.
In the case of $d = 2$, rotate the oriented lattice by 45°. For $p > p_c$, let $\xi_n^{(0,0)} = \{x : (0, 0) \to (x, n)\}$ and $r_n = \sup\{\xi_n^{0,0}\}$

Theorem 1.7. *If $p < p_c$, then $\lim_{n\to\infty} \frac{r_n}{n} = \alpha(p)$ a.s and in L_1.*

Proof. Subadditive method
Conjecture (Liggett):

$$\frac{d\alpha(p)}{dp} \to \infty, p \downarrow p_c.$$

2 First Passage Percolation

Consider percolation on $\mathbf{Z}^d$, for each $e \in \mathbf{E}^d$, we allocate a random time $T(e)$,which we think of as being the time required for fluid to flow along e; we assume that the family$(T(e) : e \in \mathbf{E}^d)$ of time coordinates contains independent non-negative random variables with common distribution function F and $Et(e) < \infty$. For any path π we define the passage time $T(\pi)$ by

$$T(\pi) = \Sigma_{e\in\pi} T(e).$$

The first passage time from u to v is given by

$$T(u, v) = \inf\{T(\pi), \pi \text{is a path form u to v}\},$$

or in a more general form

$$T(A, B) = \inf\{T(\pi) : \pi \text{ is a path from A to B}\}$$

for any two subsets of vertices of $\mathbf{Z}^d$.
Let

$$a_{0,n} = T((0,0,\dots,0),(n,0,\dots,0)),$$
$$b_{0,n} = T((0,0,\dots,0),H_n),$$

where $H_n = \{(n,u_2,\dots,u_d), u_i \in \mathbf{Z}\}$ is the hyperplane with the first coordinate fixed as n. Let

$$c_{0,n} = T((0,0,\dots,0),\partial B(n))$$

with $B(n) = [-n,n]^d$ and $\partial B(n) = \{x \in \mathbf{Z}^d, \mid x_i = n \quad \text{for some i} \mid\}$.

Theorem 2.1. *If $Et(e) < \infty$, then*

$$\lim_{n\to\infty} \frac{a_{0,n}}{n} = \lim_{n\to\infty} \frac{b_{0,n}}{n} = \lim_{n\to\infty} \frac{c_{0,n}}{n} = \mu a.s \text{ and } L_1$$

Sketch of proof. Kingman's subadditive argument(Liggett's version):
$\{X_{m,n}; 0 < m < n\}$ is a family of random variables such that

(1)
$$X_{0,n} \le X_{0,m} + X_{m,n} (subadditive);$$

(2)
$$X_{nk,(n+1)k}$$
is ergodic for each k;

(3)
$$X_{m+1,m+k+1}$$
has the same distribution as $X_{m,m+k}$ for all m and k.

(4)
$$EX_{0,1}^+ < \infty, EX_{0,n} \ge -cn$$
for some constant c.

Then

$$\lim_{n\to\infty} \frac{X_{0,n}}{n} = \gamma \quad \text{a.s and } L_1.$$

Let $a_{m,n} = T((m,0,\dots,0),(n,0,\dots,0))$ it is convenient to find a new open path from $(0,0)$ to $(n,0)$ by combining both segments of the open paths from $(0,0)$ to (m,n) and from (m,n) to $(n,0)$, so we get the subadditivity. Using mixing and the independent properties of the percolation and random times one can show that $X_{nk,(n+1)k}$ is ergodic. The third condition holds for the translation invariance and the independence of the lattice. $EX_{0,1} \le Et(0) < \infty$ and $EX_{0,n} \ge 0$. Then

$$\lim_{n\to\infty} \frac{a_{0,n}}{n} = \mu(F)$$

for some constant.
But we don't know any information about the limit $\mu(F)$.

Theorem 2.2 (Kesten). *If $F(0) < p_c$, then $\mu(F) > 0$. If $F(0) \ge p_c$, then $\mu(F) = 0$.*