Contact Processes on Scale-free Networks*

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Abstract We investigate the contact process on random graphs generated from the configuration model for scale-free complex networks with the power law exponent $\beta \in (2, 3]$. Using the neighborhood expansion method, we show that, with positive probability, any disease with an infection rate $\lambda > 0$ can survive for exponential time in the number of the vertices of the graph. This strongly supports the view that stochastic scale-free networks are remarkably different from traditional regular graphs, such as Z^d and classical Erdős-Rényi random graphs.

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It is observed that many networks, such as the Internet, World Wide Web, the scientific collaboration network, social networks etc, have scale-free structures in the sense that the degree distribution of these networks follows a power law [1], i.e. the probability of a randomly chosen node having k neighbors is asymptotically equal to $k^{-\tau}$ for some constant $\tau > 0$ independent of k. Since the appearance of Barabási-Albert's paper [1] on the scaling law of degree sequences of complex networks, many research works were carried out in this cross-disciplinary field. A great deal of studies, both rigorous and non-rigorous, have been made to capture and describe the detailed structural properties of such networks with a power law degree distribution. Many random graph models have been proposed besides the Barabási-Albert's original preferential attachment model. However, thus far, there are much less works on the effect of the topology of these scale-free structures on stochastic processes taking place on these networks, such as the spread of viruses on the Internet.

The contact process, or the susceptible-infected-susceptible(SIS) model in physics literature, is usually used in the study of the spread of some virus on network structures

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[2, 3, 4]. In this process, a vertex of the graph is either infected or healthy (but susceptible) at any time. A healthy vertex is infected with rate proportional to the number of its infected neighbors, and an infected vertex recovers with rate one independently of the status of its neighbors.

The contact process has been studied intensely in the mathematical community [5], but it is usually considered on homogeneous graphs or nonhomogeneous graphs with bounded-degree. The fundamental results in those studies are concerned with phase transitions, i.e. there are thresholds $0 < \lambda_1 \leq \lambda_2$ of the infection rate λ on an infinite graph. If $\lambda > \lambda_2$, then with positive probability the disease can spread and infect every vertex of the graph infinitely often. If $\lambda_1 < \lambda < \lambda_2$, then the disease can survive with positive probability but infects every vertex of the graph for finite times only. If $\lambda < \lambda_1$, then the disease dies out almost surely. It turns out that $\lambda_1 = \lambda_2$ for Z^d and $\lambda_1 < \lambda_2$ for regular tree T_d with $d \geq 3$. See [5] for details.

In physics literature, as far as we know, Pastor-Satorras and Vesignani [2, 3] were the first group to study contact processes on scale-free networks. They argued that the thresholds of the contact processes on scale-free networks are zero by applying simulation and (non-rigorous) mean-field methods.

In this paper, the contact process on scale-free structures is defined based on the Chung and Lu's random graph model [6]. It will be rigorously proven by neighborhood expansion tools that, with positive probability, the disease with any positive infection rate λ will survive on power law graph with exponent $\beta \in (2, 3]$. So the threshold of the contact process is exactly zero. This shows the contact process on conventional concrete structures are dramatically different from that on scale-free structures which are heterogeneous and have vertices with unbounded degrees.

This paper is organized as follows. In Section 1, we present the main results after the definitions of the power law random graphs and the contact process. In Sections 2 and 3 we prove the main results introduced in Section 1. For convenience, c is a positive constant which may be changed from line to line.

1 Models and Main Results

1.1 Power Law Random Graphs

We consider graphs with n vertices and take $V = \{1, 2, \cdots, n\}$ to be the vertex set. Let $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ be a sequence of nonnegative integers for each integer $n > 0$ and $d = (\sum_{i=1}^n w_i)/n$ be the arithmetical average of these integers. Assume $\max_{1 \leq i \leq n} w_i^2 < nd$. Let $\mathcal{G}(\mathbf{w})$ be the set of all graphs with vertex set V in which the edge joining vertex i and j appears independently and with probability

$$p_{ij} = \frac{w_i w_j}{\sum_{i=1}^n w_i}.$$

In particular, we focus on the power law case throughout this paper that

$$w_i = c i^{-\frac{1}{\beta - 1}}, \quad i_0 \leq i \leq i_0 + n, \tag{1}$$

where

$$c = \frac{\beta - 2}{\beta - 1} dn^{\frac{1}{\beta-1}}, \quad \text{and} \quad i_0 = n(\frac{d(\beta - 2)}{m(\beta - 1)})^{\beta-1}.$$

m is the minimum degree and $\beta > 1$ is called the exponent of the power law. We shall say that *almost surely* a random graph G in $\mathcal{G}(\mathbf{w})$ has property Q if $\mathbf{P}(Q) \to 1$ as $n \to \infty$, in which $\mathbf{P}(\cdot)$ is the probability measure corresponding to the random graph model $\mathcal{G}(\mathbf{w})$. A fundamental phenomenon in the model $\mathcal{G}(\mathbf{w})$ is that *almost surely* there is a unique giant component whose volume is much larger than the volume of any other component (see Lemma 2.4). For more details we refer to [6, 7, 8].

1.2 The Contact Process

The contact process is usually studied as a model of spreading some disease. Intuitively, a vertex is either infected or healthy. Waiting for exponential time with mean one, an infected vertex becomes healthy independent of other vertices. During its infection time, a vertex infects its healthy neighbors at rate $\lambda > 0$. So a healthy vertex would be infected at rate proportional to the number of its infected neighbors.

Formally, the contact process with infection parameter λ on graph $G(\mathrm{V}, \mathrm{E})$ is a continuous-time Markov process $\{\eta_t; \quad t \geq 0\}$ which can be identified at time t by subset $\mathrm{A}_t = \{v \in \mathrm{V}; \eta_t(v) = 1\}$ of V. The vertices in A_t are regarded *infected* and the rest are considered as being *healthy.* The transition rates of η_t are defined by

$$\mathrm{A} \to \mathrm{A} \setminus \{v\} \quad for\ v \in \mathrm{A} \quad at\ rate\ 1$$

and

$$\mathrm{A} \to \mathrm{A} \cup \{v\} \quad for\ v \notin \mathrm{A} \quad at\ rate\ \lambda \mid \{u \in \mathrm{A}; \{u, v\} \in \mathrm{E}\} \mid$$

where $\{u, v\}$ denotes the edge joining vertices u and v.

For more details, please refer to [5]. Usually, one vertex is infected at time $t = 0$. It is known that the contact process becomes healthy eventually on any finite graph. Consequently, one would like to study the typical behavior of the extinction time of the contact process on finite graphs.

Let $\sigma_n = \inf\{t \geq 0, \mathrm{A}_t = \emptyset\}$ be the extinction time of the contact process on a graph with n vertices. As in [5], the extinction time of the contact process in $\{1, \cdots, n\}^d$ grows exponentially in the number of the vertices, i.e. $\sigma_{n^d} \sim \exp(cn^d)$ for some $c > 0$ independent of n in the supercritical region of the contact process in Z^d, but $\sigma_{n^d} \sim c_1 \ln n$ for some $c_1 > 0$ when λ is less than the threshold of the process in Z^d.

We call the contact process becomes an *epidemic* if σ_n increases exponentially as the number n of the vertices of the graph increases.

1.3 Main Results

As we will show in Lemma 2.4, *almost surely* there is a unique giant component in random graph G of $\mathcal{G}(\mathbf{w})$. So we just run the contact process on that giant component and ignore other cases.

Theorem 1. For every $1 \gg \lambda > 0$, there exists N such that for a typical sample of the power law random graph with exponent $2 < \beta < 3$ of size $n > N$, if vertex v_n is chosen uniformly in the giant component, then, an infection starting at v_n will become an epidemic with probability bounded below by

$$\lambda^{c \ln(\lambda^{-1})}$$

for some constant $c > 0$.

Theorem 2. For every $1 \gg \lambda > 0$, there exists N such that for a typical sample of the power-law graph with exponent $\beta = 3$ of size $n > N$, if vertex v_n is chosen uniformly in the giant component, then, an infection starting at v_n will become an epidemic with probability bounded below by

$$\lambda^{c \frac{\ln(\lambda^{-1})}{\ln \ln(\lambda^{-1})}}$$

for some constant $c > 0$.

Remark. Noam Berger *et al* [9] proved that the same lower bound on the survival probability in the case of the Barabási-Albert model in which the exponent of the power law degree sequence is 3. Another work has been done by A. Ganesh *et al* [10]. Considering the contact process on several kinds of graphs including the power law random graphs, they proved similar results but paid little attention to the survival probability. For power law random graphs with exponent $\beta > 3$, the following conjecture first appeared in the physical literature and was also proposed by Rick Durrett [8].

Conjecture: If $\beta > 3$, then $\lambda_c > 0$, where λ_c is the threshold of infection rate such that the contact process on the power law random graph with rate $\lambda > \lambda_c$ will become an epidemic.

2 Proof of Theorem 1

The proof of Theorem 1 is divided into three steps. First, we apply the neighborhood expansion tools to prove Proposition 2.5 which asserts that one can find a vertex of large degree in a small neighborhood of any uniformly chosen vertex in the giant component. Second, we can find a short path from this vertex to another vertex whose degree is at least n^γ for some constant $\gamma > 0$. Finally, we consider the contact process on this path and show the disease persists around the end vertex of the path for long time.

Let S be a subset of vertices, *i.e.* $\mathrm{S} \subset \mathrm{V}$. For $k \geq 1$, define the k-th moment of the expected volume by $\mathrm{Vol}_k(\mathrm{S}) = \sum_{v_i \in \mathrm{S}} w_i^k$. For $k = 1$, write $\mathrm{Vol}_k(\mathrm{S}) = \mathrm{Vol}(\mathrm{S})$ for convenience. Recall that $\mathrm{Vol}(\mathrm{G}) = \sum_{i=1}^n w_i = nd$ where $d = n^{-1} \sum_{i=1}^n w_i > 1$. Define $\Gamma(\mathrm{S}) = \{v \in \mathrm{V} : v \sim u \quad \text{and} \quad v \notin \mathrm{S}\}$.

Lemma2.1. (Lemma 2 of [6]) In a random graph $\mathrm{G} \in \mathcal{G}(\mathbf{w})$, for any two subsets S and T of vertices,

$$\mathbf{P}\Big(\mathrm{Vol}(\Gamma(\mathrm{S}) \cap \mathrm{T}) \geq (1 - 2\epsilon)\mathrm{Vol}(\mathrm{S})\frac{\mathrm{Vol}_2(\mathrm{T})}{\mathrm{Vol}(\mathrm{G})}\Big) \geq 1 - e^{-c},$$

provided Vol(S) satisfies

$$\frac{2c\mathrm{Vol}_3(\mathrm{T})\mathrm{Vol}(\mathrm{T})}{\epsilon^2 \mathrm{Vol}_2^2(\mathrm{T})} \leq \mathrm{Vol}(\mathrm{S}) \leq \frac{\epsilon \mathrm{Vol}_2(\mathrm{T})\mathrm{Vol}(\mathrm{G})}{\mathrm{Vol}_3(\mathrm{T})}. \tag{2}$$

Let $d(\mathrm{S},\mathrm{T}) = \min\{d(u,v), u \in \mathrm{S}, v \in \mathrm{T}\}$ be the distance between S and T.

Lemma 2.2. (Lemma 3 of [6]) For any two disjoint subsets S and T of vertices with $\mathrm{Vol(S)Vol(T)} > c\mathrm{Vol(G)}$,

$$\mathrm{P}(d(\mathrm{S},\mathrm{T}) > 1) < e^{-c}.$$

Lemma 2.3. (Lemma 4 of [6]) Suppose that $\mathrm{G} \in \mathcal{G}(\mathbf{w})$ of n vertices such that for a fixed value $c > 0$, G has $o(n)$ vertices of degree less than c. For any fixed vertex v in the giant component, let $\mathrm{S} = \{v\}$, $\Gamma_1(\mathrm{S}) = \Gamma(\mathrm{S})$ and $\Gamma_i(\mathrm{S}) = \Gamma(\Gamma_{i-1}(\mathrm{S}))$ for $i > 1$. If $\tau = o(\sqrt{n})$, then there is an index $i_0 \leq c_0\tau$ such that

$$\mathbf{P}(\mathrm{Vol}(\Gamma_{i_0}(\mathrm{S})) \geq \tau) \geq 1 - c_1\tau^{3/2}e^{-c_2\tau},$$

where c_0, c_1, c_2 are constants depending only on c and $d = n^{-1}\sum_{i=1}^n w_i > 1$.

Lemma 2.4.(Theorem 1.3 of [11]) Suppose that $\mathrm{G} \in \mathcal{G}(\mathbf{w})$. If $d = n^{-1}\sum_{i=1}^n w_i > 1$, then the following statements hold.

1. Almost surely G has a unique giant component. Furthermore, the volume of the giant component is at least $(1 - 2/\sqrt{de} + o(1))\mathrm{Vol(G)}$ if $d \geq 4/e$, and is at least $(1 - (1 + \ln d)/d + o(1))\mathrm{Vol(G)}$ if $d < 2$;

2. All other components almost surely have volume at most $O(\log_d n)$.

The first three lemmas, usually called the neighborhood expansions, are useful tools to prove facts about short diameters of connected graphs. Now we prove a proposition by applying these lemmas. The *k-neighborhood* of some vertex v is the set $\{u \in V; d(u,v) \leq k\}$ of vertices.

Proposition 2.5. For sufficiently small λ and for $\mathrm{G}_n \in \mathcal{G}(\mathbf{w})$ with exponent $2 < \beta < 3$, with probability at least $1 - O(\exp(c\lambda^{-2(3-\beta)}))$ for some constant $c > 0$, the $O(\ln \lambda^{-1})$-neighborhood of a randomly chosen vertex v_n in G_n contains a vertex with degree larger than λ^{-2}.

Here is an observation leading to the above proposition and its proof. Note that the diameter determines how far the distance between a typical pair of vertices of the graph. We know by [12, 6] that the average distances of the power law random graphs with n vertices have the form of $\ln\ln n$ for exponent $\beta \in (2,3)$. Intuitively, when λ is small enough, the $O(\ln \lambda^{-1})$-neighborhood will be similar to the whole random graph $G \in \mathcal{G}(\mathbf{w})$ which has vertices of large degrees. Therefore we can apply the neighborhood expansion lemmas to find the desired vertex of large degree in some j-neighborhood with small j. First, we show that some i-neighborhood of v_n will grow "large" enough by Lemma 2.3. Second, applying Lemma 2.1 to prove the neighborhood of the i-neighborhood of v_n will grow exponentially fast. Finally, after at most $O(\ln \lambda^{-1})$ steps, the volume of reachable vertices is large enough to reach the desired vertex of large degree with another step.

Proof of Proposition 2.5. Recall (1) and note that the minimum expected degree of the degree sequence is

$$w_{\mathtt{min}} = (1+o(1))\frac{d(\beta-2)}{\beta-1}.$$

Define $\mathrm{T} = \{\nu_i \in \mathrm{V}; w_i \in [w_{\mathtt{min}}, aw_{\mathtt{min}}]\}$ where $a = \ln\ln(\lambda^{-1})$. A little calculation yields that

$$\begin{aligned}
\mathrm{Vol}(\mathrm{T}) &\approx nd(1-a^{2-\beta}),\\
\mathrm{Vol}_2(\mathrm{T}) &\approx nd^2(1-\frac{1}{\beta-1})^2\frac{\beta-1}{3-\beta}a^{3-\beta}, \quad \text{and}\\
\mathrm{Vol}_3(\mathrm{T}) &\approx nd^3(1-\frac{1}{\beta-1})^3\frac{\beta-1}{4-\beta}a^{4-\beta}.
\end{aligned}$$

Let S be i-th neighborhood of v_n, consisting of all vertices within distance i from v_n. Choose

$$c = \ln\ln(\lambda^{-1}), \quad \epsilon = 1/4, \quad \tau = a^{\beta}.$$

By Lemma 2.3, there are constants c_0, c_1, c_2 and an index $i_0 \le c_0\tau$ such that

$$\mathbf{P}(\mathrm{Vol}(\Gamma_{i_0}(v_n)) \ge \tau) \ge 1 - \frac{c_1\tau^{3/2}}{e^{c_2\tau}}.$$

Then (2) is satisfied. By Lemma 2.1, with probability at least

$$1-e^{-c} = 1-e^{-a} = 1+(\ln\lambda)^{-1},$$

the volume of $\Gamma_i(v_n)$ for $i > i_0$ will grow at rate greater than

$$(1-2\epsilon)\frac{\mathrm{Vol}_2(\mathrm{T})}{\mathrm{Vol}(\mathrm{G})} = (1-2\epsilon)\frac{d(\beta-2)^2}{(3-\beta)(\beta-1)}a^{3-\beta}.$$

After at most $\frac{4\ln(\lambda^{-1})}{(3-\beta)\ln a} = o(\ln\lambda^{-1})$ steps, the volume of the set of reachable vertices is at least

$$[\frac{d(\beta-2)^2}{2(\beta-1)(3-\beta)}a^{3-\beta}]^{\frac{4\ln\lambda^{-1}}{(3-\beta)\ln a}} \ge \lambda^{-4}.$$

To apply Lemma 2.2, let $\widetilde{\mathrm{T}} = \{v_i; w_i \ge \lambda^{-2}\}$. If $w_i \ge \lambda^{-2}$, then $i \le i_0$ where $i_0 = n[\lambda^2 d(\beta-2)/(\beta-1)]^{\beta-1}$. It follows that

$$\mathrm{Vol}(\widetilde{\mathrm{T}}) = \sum_{i=0}^{\lfloor i_0 \rfloor} w_i \approx \int_1^{i_0} \frac{\beta-2}{\beta-1}(\frac{n}{x})^{\frac{1}{\beta-1}}dx \approx n\lambda^{2(\beta-1)}(\frac{\beta-1}{d(\beta-2)})^{2-\beta}.$$

We denote $\rho = 2(3-\beta) > 0$ for $\beta \in (2,3)$. Then

$$\mathrm{Vol}(\widetilde{\mathrm{T}})\lambda^{-4} \approx dn\lambda^{-\rho}(\frac{\beta-1}{\beta-2})^{2-\beta}d^{\beta-3} \ge c\mathrm{Vol}(\mathrm{G})\lambda^{-\rho}.$$

Lemma 2.2 implies that a vertex with degree larger than λ^{-2} can be reached in one additional step with probability at least $1-\exp(-c\lambda^{-\rho})$. The total number of steps is at most

$$c_0\tau + O(\ln(\lambda^{-1})) + 1 = O(\ln(\lambda^{-1})).$$

The total probability of failure is at most

$$\frac{c_1 a^{3\beta/2}}{e^{c_2 a^\beta}} + e^{-a} + e^{-c\lambda^{-\rho}} = O(e^{-c\lambda^{-\rho}}) < \varepsilon_0,$$

for fixed ε_0 and small λ. This completes the proof. □

To complete the proof of Theorem 1, we need the following lemmas.

Lemma 2.6. With probability at least $1-\exp(-c\lambda^{-\rho})$, the vertex $u^{(1)}$ chosen in Proposition 2.5 has a neighbor $u^{(2)}$ with degree larger than $(1+\widetilde{\epsilon})\lambda^{-2}$ for fixed $\widetilde{\epsilon}\in(0,1)$, and constant $c>0$ depending on $\widetilde{\epsilon}, d$ and β only.

Proof. Let $\{u^{(1)}\sim u_i\}$ be the event that $u^{(1)}$ is joined with u_i by an edge. According to the definition of the power law random graph, all the events of this kind are mutually independent.

$$\begin{aligned}
&\mathrm{P}(u^{(1)} \text{ is connected with a vertex of degree larger than} (1+\widetilde{\epsilon})\lambda^{-2})\\
=&1-\mathrm{P}(\cap_{i=1}^{i_0}\{u^{(1)}\sim u_i\}^c) = 1-\prod_{i=1}^{i_0}\mathrm{P}(\{u^{(1)}\sim u_i\}^c)\\
=&1-\prod_{i=1}^{i_0}(1-\frac{w_{u^{(1)}}w_{u_i}}{nd}) \;= 1-\exp(-w_{u^{(1)}}\sum_{i=1}^{i_0}\frac{w_{u_i}}{nd}) + o(1)\\
\geq&1-\exp(-\frac{1}{\lambda^2}\sum_{i=1}^{i_0}\frac{(1+\widetilde{\epsilon})\lambda^{-2}}{nd}) \;= 1-\exp(-c\lambda^{-\rho}).
\end{aligned}$$

for some constant $c>0$, where

$$i_0 = \lfloor n[\lambda^2\frac{d(\beta-2)}{(\beta-1)(1+\widetilde{\epsilon})}]^{\beta-1}\rfloor$$

is the smallest integer i such that $w_i \geq \lambda^{-2}(1+\widetilde{\epsilon})$ for fixed $\widetilde{\epsilon}$. □

By the same argument as above and by induction we can get the following lemma.

Lemma 2.7. Fix $\widetilde{\epsilon}\in(0,1)$. Suppose we have found the vertices $u^{(1)}, u^{(2)}, \ldots, u^{(j)}$ such that the degree of $u^{(i)} \geq (1+\widetilde{\varepsilon})^{i-1}\lambda^{-2}$ and $u^{(i)}\sim u^{(i-1)}$ for all $j\geq i\geq 2$. Then with probability at least $1-\exp(-c\lambda^{-\rho})$, a neighbor $u^{(j+1)}$ of $u^{(j)}$ can be found such that the degree of $u^{(j+1)} \geq (1+\widetilde{\varepsilon})^j\lambda^{-2}$.

The following lemma is about the contact process on a star-shaped graph. It shows that with high probability the contact process on a star-shaped graph will survive for a exponential time in the number of the leaves of the star.

Lemma 2.8. (Lemma 5.3 of [9]) Let G be a star-shaped graph, with center x and n leaves. Let λ be the parameter of the contact process on the star-shaped graph. $\lambda \to 0$ and $\lambda^2 n \to \infty$ as $n \to \infty$. Let A_t be the set of vertices infected at time t. Suppose that $\mathrm{A}_0 = \{x\}$. There exists a constant $c > 0$ such that

$$\mathrm{P}(\sigma_{n+1} \geq \exp(c\lambda^2 n)) = 1 - o(n).$$

Proof of Theorem 1. By Lemmas 2.6 and 2.7, with probability at least $[1-\exp(-c\lambda^{-\rho})]^{c\ln n}$, there exists a path from v_n to some vertex u with degree larger than $(1+\widetilde{\epsilon})^{c\ln n}\lambda^{-2}$.

By Proposition 2.5 there exists a path $v^{(1)} = v_n, v^{(2)}, \ldots, v^{(k_0)} = u^{(1)}$ for $k_0 = O(\ln \lambda^{-1})$, starting at v_n and ending at $u^{(1)}$. Compare the contact process with a jump process restricted to this path starting at v_n. At each jump, the infection can reach the next vertex of the path with probability $\lambda/(1+\lambda)$. Then, with probability

$$(\frac{\lambda}{1+\lambda})^{k_0} \geq \lambda^{-c\ln \lambda}$$

for some constant $c > 0$, the infection will reach $u^{(1)}$.

Conditioned on the event that the infection reaches $u^{(1)}$, by iterative applications of Lemma 2.8, with probability bounded away from zero, the infection will reach $u^{(c\ln n)}$, and by another application of Lemma 2.8, the disease will survive up to time at least

$$\exp(c\lambda^{-2}(1+\widetilde{\epsilon})^{c\ln n}) = \exp(n^{\gamma}),$$

for some constant $\gamma > 0$. Thus, the infection becomes an epidemic. □

3 Proof of Theorem 2

The strategy of proving Theorem 2 is very similar to that of Theorem 1. But the difference between the exponents of power law distributions in two cases leads to the following subtle modifications. We have to prove a proposition which is the analogy of Proposition 2.5.

Proposition 3.1. For sufficiently small λ and for the power law random graph G_n with exponent $\beta = 3$, with probability at least $1 - O(\exp(-c\lambda^{-1}))$ for some constant $c > 0$, the $O(\ln \lambda^{-1}/\ln\ln \lambda^{-1})$-neighborhood of a randomly chosen vertex ν_n in G_n contains a vertex with degree larger than λ^{-2}.

To prove Proposition 3.1, we first prove some lemmas.

Lemma 3.2. Randomly pick a vertex v_n in the giant component, with probability at least $1 - (\ln \lambda^{-1})^{-3}$, the volume of $\Gamma_{i_0}(v_n)$ is at least $(\ln\ln \lambda^{-1})^6$ for some $i_0 = O((\ln\ln \lambda^{-1})^6)$.

Proof. Let $\tau = (\ln\ln \lambda^{-1})^6$. The claim follows directly from Lemma 2.3. □

Lemma 3.3. (Lemma 7 of [6]) Suppose that S is a subset of vertices, $\beta = 3$, $\epsilon < 1/2$ and $c > 0$. If

$$4c\frac{t}{d}(\epsilon \ln \frac{2t}{d})^{-2} < \mathrm{Vol(S)} \leq n^{2/3},$$

then

$$\mathbf{P}\left(\frac{\mathrm{Vol}(\Gamma(\mathrm{S}))}{\mathrm{Vol}(\mathrm{S})} > (1-2\epsilon)\frac{d}{2}\ln\frac{2t}{d}\right) \geq 1 - e^{-c}.$$

Lemma 3.4. Suppose that S is a subset of vertices and $\mathrm{Vol}(\mathrm{S}) \geq (\ln\ln\lambda^{-1})^6$. Let $\Gamma_0(\mathrm{S}) = \mathrm{S}$, and define inductively that $\Gamma_{k+1}(\mathrm{S}) = \Gamma(\Gamma_k(\mathrm{S}))$. Then

$$P(\mathrm{Vol}(\Gamma_i(\mathrm{S})) > \lambda^{-5}) \geq 1 - o((\ln\lambda^{-1})^{-2})$$

if $i \geq c(\ln\lambda^{-1})/(\ln\ln\lambda^{-1})$.

Proof. We adopt the same approach in the proof of Claim 6 in [6]. Define a sequence $\{a_i;\ i \geq 1\}$ recursively as follows. Let a_0 be some number larger than $(\ln\ln\lambda^{-1})^6$. Define $a_{i+1} = (d/10)a_i \ln a_i$ for $i \geq 1$. Note that $a_{i+1} > a_i$, and $a_i \geq (\ln\ln\lambda^{-1})^6$. Furthermore

$$a_i \geq (i+s)^{i+s}, \qquad \text{for } s = \exp(10e/d) \text{ and for any } i \geq 0. \tag{3}$$

We shall prove this claim later. Choose

$$\epsilon = (\ln\ln\lambda^{-1})^{-1}, \quad c = (\ln\ln\lambda^{-1})^2, \quad \text{and} \quad t = (\ln\frac{\epsilon^2 a_i}{2c})^2 d\epsilon^2 a_i/(4c).$$

Inductively, suppose $\mathrm{Vol}(\Gamma_i(\mathrm{S})) \geq a_i$ for some $i \geq 0$. Note that

$$\frac{2t/d}{\ln^2(2t/d)} = \frac{\epsilon^2 a_i}{2c}\frac{\ln^2\frac{\epsilon^2 a_i}{2c}}{(\ln\frac{\epsilon^2 a_i}{2c} + 2\ln\ln\frac{\epsilon^2 a_i}{2c})^2} \leq \frac{\epsilon^2 a_i}{2c} \leq \frac{\epsilon^2}{2c}\mathrm{Vol}(\Gamma_i(\mathrm{S})).$$

Hence by Lemma 3.3 and by inequality (3),

$$\mathbf{P}\left(\mathrm{Vol}(\Gamma_{i+1}(\mathrm{S})) \geq (1-2\epsilon)a_i\frac{d}{2}\ln\frac{2t}{d} \geq \frac{d}{10}a_i\ln a_i = a_{i+1}\right) \geq 1 - \exp(-c).$$

Let $i = (1+o(1))(\ln\lambda^{-5})(\ln\ln\lambda^{-5})^{-1}$. Then

$$\mathrm{Vol}(\Gamma_i(\mathrm{S}) \geq a_i \geq (i+s)^{i+s} \geq e^{\ln\lambda^{-5} - o(\ln\lambda^{-5})} = \lambda^{-5} + o(1).$$

This is the desired conclusion of Lemma 3.4. □

Proof of Inequality (3). First, $a_0 = (\ln\ln\lambda^{-1})^6 \geq s^s$ by the fact that s is bounded and λ can be small enough. We now proceed by induction and assume that the claim holds for a_i. Then

$$\begin{aligned} a_{i+1} &= \frac{d}{10}a_i\ln a_i \geq \frac{d}{10}(i+s)^{i+s}\ln(i+s)^{i+s} \\ &= \frac{d}{10}(1 - \frac{1}{i+s+1})^{i+s+1}(i+s+1)^{i+s+1}\ln(i+s) \\ &\geq (i+s+1)^{i+s+1}. \end{aligned}$$

Therefore $a_i \geq (i+s)^{i+s}$ for $i \geq 0$. □

Proof of Proposition 3.1. Combining Lemma 2.2 and Lemma 3.4, the same arguments used in the proof of Proposition 2.1 yield that, a vertex of degree larger than λ^{-2} can be reached by one additional step with probability at least $1-e^{-c/\lambda}$. The total number of steps is at most

$$O((\ln\ln\lambda^{-1})^6)+O(\frac{\ln\lambda^{-1}}{\ln\ln\lambda^{-1}})+1=O(\frac{\ln\lambda^{-1}}{\ln\ln\lambda^{-1}}).$$

The total probability of failure is at most

$$\frac{c_1\tau^{3/2}}{e^{c_2\tau}}+O(\frac{1}{\ln^2\lambda^{-1}})+e^{-c/\lambda}=O(e^{-c/\lambda}).$$

This completes the proof. □

Proof of Theorem 2. Applying Proposition 3.1, the same arguments used in the proof of Theorem 1 lead to Theorem 2, with $k_0=O(\frac{\ln\lambda^{-1}}{\ln\ln\lambda^{-1}})$ instead. □

Discussions and a Question

In the proof of Theorem 1, we showed the probability that there is a path from $u^{(1)}$ to some vertex with degree larger than $(1+\widetilde{\epsilon})^{c\ln n}\lambda^{-2}$ bounded below by

$$[1-\exp(-c\lambda^{-2(3-\beta)})]^{c\ln n},$$

which tends to zero as $n\to\infty$. In [10], A. Ganesh *et al* used the fact that the probability that an infection starting from a randomly infected vertex spreads to the maximum degree node is at least $O(n^{-c\ln n})$ for some constant $c>0$ independent of n. Our estimation is stronger than it. Berger *et al* [9] proved that with probability at least 1/4 such a path can be found in Barabasi-Albert's model, which is independent of the number of the vertices of the graph. So a consequent question is as follows. *Is there a path with its length less than $c\ln n$ from $u^{(1)}$ to some vertex with degree larger than $O(\lambda^{-2}(1+\widetilde{\epsilon})^{c\ln n})$ as $n\to\infty$?*

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