Potential Theory of Geometric Stable Processes

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Outline

- Definitions
- Potential Theory of geo stable subordinators
- Potential Theory of sym geo stable subordinators
- iterated subordinators and iterated subordinate BM
- Applications: capacity, Harnack inequality, etc.

Geo stable distributions and geo id distributions were first introduced by Klebanov, Maniya and Melamed in 1984. Geometric stable distributions play an important role in heavy-tail modeling of economic data.

Let F be a distribution in $\mathbb{R}^d$. F is called geo id if for any $p \in (0,1)$, there exist a sequence of iid random variables $Z_1, Z_2, \cdots$ and a geo random variable $T(p)$ (with parameter $p \in (0,1)$) independent of $Z_1, Z_2, \ldots$ such that the distribution of $\sum_{j=1}^{T(p)} Z_j$ is F.

Geo id distributions are id and they are best described in terms of ch functions. In fact, there is a one-to-one correspondence between the ch functions of geo id distributions and those of the id distributions. Namely, a distribution F is geo id if and only if its ch function has the form $\exp(-\Psi(\xi))$ with

$$\Psi(\xi) = \log(1 + \Phi(\xi)), \quad \xi \in \mathbb{R}^d,$$

where $\exp(-\Phi)$ is the ch function of some id distribution.

A rv is called geo id if its distribution function is geo id.

Construction of geo id rv:

Let $Y = (Y_t, t \geq 0)$ be a Lévy process with ch exponent Φ and let τ be an exponential rv with parameter 1 which is independent of Y. Then $X = Y(\tau)$ is a geo id rv with ch function $\exp(-\Psi)$, where Ψ is given by

$$\Psi(\xi) = \log(1 + \Phi(\xi)), \quad \xi \in \mathbb{R}^d,$$

Therefore the distribution of X is equal to the 1-potential of the process Y.

It is also known (see Bertoin's book) that the Lévy measure of X is given by

$$\mu(A) = \int_0^\infty t^{-1} e^{-t} \mathbf{P}(Y_t \in A) dt, \quad A \subset \mathbb{R}^d \setminus \{0\}.$$

Let F be a distribution in $\mathbb{R}^d$, $Z_1, Z_2, \ldots$ be independent rv's with common distribution F. Let $T(p)$ represent a geo rv with parameter $p \in (0,1)$, independent of $Z_1, Z_2, \ldots$. The distribution F is said to be geo strictly stable if for any $p \in (0,1)$, there is a constant $a = a(p) > 0$ such that the distribution of

$$a \sum_{j=1}^{T(p)} Z_j$$

is also F.

A geo strictly stable distribution is also id. There is also a one-to-one correspondence between the ch functions of geo strictly stable distributions and those of the strictly stable distributions. Namely, a distribution F is geo strictly stable if and only if its ch function Ψ has the form $\exp(-\Psi(\xi))$ with

$$\Psi(\xi) = \log(1 + \Phi(\xi)), \quad \xi \in \mathbb{R}^d,$$

where $\exp(-\Phi)$ is the ch function of some strictly stable distribution.

A rv is called a geo strictly stable rv if its distribution is geo strictly stable.

A rv Z is geo strictly α-stable if its ch function is $\exp(-\Psi)$, where Ψ is given by

$$\Psi(\xi) = \log(1 + \Phi(\xi)), \quad \xi \in \mathbb{R}^d,$$

with $\exp(-\Phi)$ being the ch function of a strictly α-stable rv ($\alpha \in (0, 2]$).

We will always assume that $\alpha \in (0, 2]$.

A Lévy process $X = (X_t, \mathbf{P}_x)$ is called a geo α-stable process if its ch. exponent $\Psi(\xi) = -\log(\mathbf{E}_x(e^{i\xi\cdot(X_1 - X_0)}))$ is given by

$$\Psi(\xi) = \log(1 + \Phi(\xi)), \quad \xi \in \mathbb{R}^d,$$

with $\exp(-\Phi)$ being the ch function of some strictly α-stable distribution.

We will be mainly interested in the rotationally invariant geo strictly α-stable process, that is in the case when

$$\Psi(\xi) = \log(1 + |\xi|^\alpha), \quad \xi \in \mathbb{R}^d.$$

We will simply call these processes sym geo. α-stable processes. The sym geo 2-stable process also goes by the name of sym variance gamma process and it is also used by some researchers (D. Madan, M. Yor) to study heavy-tailed financial models.

Despite the applications of geo stable processes in mathematical finance and other fields, there has not been much study about the potential theory of these processes.

For any $\alpha \in (0, 2]$, the function

$$\phi(\lambda) = \log(1 + \lambda^{\alpha/2}), \quad \lambda > 0$$

is a complete Bernstein function. An increasing Lévy process $S = (S_t : t \geq 0)$ with the Laplace exponent ϕ given above is called a geo $\alpha/2$-stable subordinator.

Let $S = (S_t : t \geq 0)$ be a geo $\alpha/2$-stable subordinator and let $Y = (Y_t : t \geq 0)$ be a BM in $\mathbb{R}^d$ with generator Δ. If Y and S are independent, then the subordinate process $Y = (Y_t : t \geq 0)$ defined by

$$Y_t = X(S_t), \quad t \geq 0,$$

is a sym geo α-stable process. This construction of sym geo α-stable processes will play an important role in our argument.

Assume that $S = (S_t : t \geq 0)$ is a geo $\alpha/2$-stable subordinator. Its Laplace exponent is given by

$$\phi(\lambda) = \log(1 + \lambda^{\alpha/2}), \quad \lambda > 0.$$

The function ϕ above can be written in the form

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\mu(dt), \quad \lambda > 0$$

for some σ-finite measure μ on $(0, \infty)$ (the Lévy measure of S). Since ϕ is complete Bernstein, the Lévy measure μ has a complete monotone density $\mu(t)$.

The potential distribution of the subordinator S is defined by

$$U(x) = \mathbf{E} \int_0^\infty 1_{(S_t \in [0,x])} \, dt \,,$$

and its Laplace transform is given by

$$\mathcal{L}U(\lambda) = \frac{1}{\phi(\lambda)} = \frac{1}{\log(1 + \lambda^{\alpha/2})} \,.$$

Since $\lim_{\lambda \to \infty} \phi(\lambda) = \infty$, we must have $\mu((0, \infty)) = \infty$. Using this one can show that the potential measure U has a density u which is completely monotone on $(0, \infty)$.

A function $l : (0, \infty) \to (0, \infty)$ is said to be slowly varying at ∞ (resp. at $0+$) if for every $\lambda > 0$

$$\lim \frac{l(\lambda x)}{l(x)} = 1, \quad x \to \infty(\text{ resp. } x \to 0+ .$$

A function $f : (0, \infty) \to (0, \infty)$ is said to be slowly varying at ∞ (resp. at $0+$) if for every $\lambda > 0$, the ratio $\frac{f(\lambda x)}{f(x)}$ converges to a positive number as $x \to \infty$ (resp $x \to 0+$).

If a function f is slowly varying at infinity (resp at $0+$), then there is a real number ρ such that

$$\lim \frac{f(\lambda x)}{f(x)} = \lambda^{\rho}, \quad x \to \infty(\text{ resp. } x \to 0+)$$

for every $\lambda > 0$.

Karamata's Tauberian theorem

Let l be slowly varying at ∞ (resp. at $0+$) and $\rho \geq 0$. The following are equivalent:

(i) As $x \to \infty$ (resp. $x \to 0+$)

$$U(x) \sim x^{\rho} l(x)/\Gamma(1+\rho);$$

(ii) As $\lambda \to \infty$ (resp. $\lambda \to 0+$)

$$\mathcal{L}U(\lambda x) \sim \lambda^{-\rho} l(1/\lambda).$$

Karamata's Monotone Density Theorem. *Suppose that $dU(x) = u(x)dx$, where u is monotone. If there is a real number ρ and a function l that is slowly varying at ∞ (resp at $0+$) such that*

$$U(x) \sim x^{\rho} l(x), \quad x \to \infty (\textit{resp. } x \to 0+),$$

then

$$u(x) \sim \rho x^{\rho-1} l(x), \quad x \to \infty (\textit{ resp. } x \to 0+)$$

de Haan's Tauberian Theorem. *If l is slowly varying at ∞ (resp. at $0+$), $c \geq 0$, the following are equivalent:*

(i) As $x \to \infty$ (resp. $x \to 0+$)

$$\frac{U(\lambda x) - U(x)}{l(x)} \to c \log \lambda, \quad \forall \lambda > 0.$$

(ii) As $x \to \infty$ (resp. $x \to 0+$)

$$\frac{\mathcal{L}U(\frac{1}{\lambda x}) - \mathcal{L}U(\frac{1}{x})}{l(x)} \to c \log \lambda, \quad \forall \lambda > 0.$$

de Haan's Monotone Density Theorem. *Let $dU(x) = u(x)dx$ with u monotone, and l slowly varying at ∞ (resp. at $0+$). Assume that $c > 0$. Then the following are equivalent:*

(i) As $x \to \infty$ (resp. $x \to 0+$)

$$\frac{U(\lambda x) - U(x)}{l(x)} = c \log \lambda, \quad \forall \lambda > 0.$$

(ii) As $x \to \infty$ (resp. $x \to 0+$)

$$u(x) \sim c x^{-1} l(x).$$

Theorem. *For any* $\alpha \in (0, 2]$, *we have*

$$u(x) \sim \frac{2}{\alpha x (\log x)^2}, \quad x \to 0+ .$$

Proof. Since

$$\frac{\mathcal{L}U(\frac{1}{t\lambda}) - \mathcal{L}U(\frac{1}{\lambda})}{(\log \lambda)^{-2}} \to \frac{2}{\alpha} \log t, \quad \forall t > 0$$

as $\lambda \to 0$, we have by de Haan's Tauberian theorem that

$$\frac{U(tx) - U(x)}{(\log x)^{-2}} \to \frac{2}{\alpha} \log t, \quad t > 0$$

as $x \to 0$. Now we can apply the monotone density theorem to get that

$$u(x) \sim \frac{2}{\alpha x (\log x)^2}$$

as $x \to 0$.

Theorem. *For any* $\alpha \in (0, 2]$, *we have*

$$u(x) \sim \frac{1}{\Gamma(\alpha/2)} x^{\alpha/2 - 1}, \quad x \to \infty$$

The distribution $F_{\alpha/2}$ of S_1 is absolutely continuous and the density $f_{\alpha/2}$ is decreasing on $(0,\infty)$. When $\alpha=2$ we have

$$f_1(x)=e^{-x}, \quad x>0.$$

Theorem. *For any* $\alpha\in(0,2)$, *we have*

$$f_{\alpha/2}(x)\sim\frac{1}{\Gamma(\alpha/2)}x^{\alpha/2-1}, \quad x\to 0+$$

and

$$f_{\alpha/2}(x)\sim 2\pi\Gamma(1+\frac{\alpha}{2})\sin(\frac{\alpha\pi}{4})x^{-1-\frac{\alpha}{2}}, \quad x\to\infty.$$

When $\alpha = 2$, the Lévy density of S is given by

$$\mu(t) = t^{-1}e^{-t}, \quad t > 0.$$

For $\alpha \in (0, 2)$, we have

$$\mu(x) = \frac{\alpha}{2x}(1 - F_{\alpha/2}(x)), \quad x > 0.$$

Hence we have

Theorem. *For any* $\alpha \in (0, 2]$, *we have*

$$\mu(x) \sim \frac{\alpha}{2x}, \quad x \to 0+ .$$

By considering the "dual" subordinator of S and using the Tauberian theorem and the monotone density theorem we can get the following

Theorem. *For any* $\alpha \in (0, 2)$, *we have*

$$\mu(x) \sim \frac{\alpha}{2\Gamma(1 - \alpha/2)}x^{-\alpha/2-1}, \quad x \to \infty.$$

$d \geq 3$, $\alpha \in (0, 2]$.

$Y = (Y_t, t \geq 0)$: BM in $\mathbb{R}^d$ with generator Δ.

$S = (S_t, t \geq 0)$: geometric $\alpha/2$-stable subordinator with Laplace exponent $\log(1 + \lambda^{\alpha/2})$.

Y and S: independent.

$X_t = Y(S_t)$: sym. geometric α-stable process.

u: the potential density of S.

The potential density of X is given by

$$G(x) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) u(t)\, dt,$$

Theorem. *For any* $\alpha \in (0, 2]$, *we have*

$$G(x) \sim \frac{\Gamma(d/2)}{2\alpha\pi^{d/2}|x|^d \log^2 \frac{1}{|x|}}, \quad |x| \to 0.$$

and

$$G(x) \sim \frac{1}{\pi^{d/2} 2^\alpha} \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} |x|^{\alpha-d} \quad |x| \to \infty.$$

Recall $\mu(t)$ is the Lèvy density of S. The jumping function of X is given by $J(x,y) = J(y-x)$ with

$$J(x) = \frac{1}{2}\int_0^\infty (4\pi t)^{-d/2} \exp(-\frac{|x|^2}{4t})\mu(t)dt.$$

We also have

$$J(x) = \frac{1}{2}\int_0^\infty t^{-1}e^{-t}p_\alpha(t,0,x)dt$$

where p_α is the density of a sym. α-stable process in $\mathbb{R}^d$.

Theorem. *For any $\alpha \in (0,2]$ we have*

$$J(x) \sim \frac{\Gamma(d/2)}{\alpha|x|^d}, \quad |x| \to 0.$$

Theorem. *For any $\alpha \in (0,2)$ we have*

$$J(x) \sim \frac{\alpha}{2^{\alpha+1}\pi^{d/2}} \frac{\Gamma(\frac{d+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}|x|^{-d-\alpha}, \quad |x| \to \infty.$$

and when $\alpha = 2$ we have

$$J(x) \sim 2^{-\frac{d}{2}}\pi^{-\frac{d-1}{2}} \frac{e^{-|x|}}{|x|^{\frac{d+1}{2}}}, \quad |x| \to \infty.$$

Let $q_\alpha(t,x,y) = q_\alpha(t, y-x)$ be the transition density of X. Then we have

Theorem. *For $\alpha \in (0,2)$ we have*

$$q_\alpha(1,x) \sim \frac{\alpha 2^{\alpha-1} \sin\frac{\alpha\pi}{2}\Gamma(\frac{d+\alpha}{2})\Gamma(\frac{\alpha}{2})}{\pi^{\frac{d}{2}+1}}|x|^{-d-\alpha}, \quad |x| \to \infty.$$

For $\alpha = 2$ we have

$$q_2(1,x) \sim 2^{-\frac{d}{2}}\pi^{-\frac{d-1}{2}}\frac{e^{-|x|}}{|x|^{\frac{d-1}{2}}}, \quad |x| \to \infty.$$

Theorem. *Suppose $\alpha \in (0,2)$. There exist positive constants C_1 and C_2 such that for all $x, y \in \mathbb{R}^d$ and $t < 1 \wedge \frac{d}{2\alpha}$,*

$$C_1 t\left(\frac{1}{|x-y|^{d+\alpha}} \wedge \frac{1}{|x-y|^{d-t\alpha}}\right) \le p_\alpha(t,x,y)$$

$$p_\alpha(t,x,y) \le C_2\, t\left(\frac{1}{|x-y|^{d+\alpha}} \wedge \frac{1}{|x-y|^{d-t\alpha}}\right).$$

$\alpha \in (0, 2]$, $\phi^{(1)}(y) = \phi(y) = \log(1 + y^{\alpha/2})$, $\phi^{(n)}(y) = \phi(\phi^{(n-1)}(y))$ for $n > 1$. $\phi^{(n)}$ is a complete Bernstein function.

Let $S^{(n)} = (S_t^{(n)} : t \geq 0)$ be a subordinator with Laplace exponent $\phi^{(n)}$. $\mu^{(n)}$: The Lévy measure of $S^{(n)}$. $u^{(n)}$: the potential density of $S^{(n)}$.

For convenience, we introduce the following:

$$l_n(y) = \log\log\cdots\log y, \quad n \text{ times}$$

and

$$L_n(y) = l_1(y)l_2(y)\cdots l_n(y).$$

Theorem *For* $\alpha \in (0, 2]$,

$$u^{(n)}(x) \sim \frac{2}{\alpha x L_{n-1}(\frac{1}{x}) l_n^2(\frac{1}{x})}, \quad x \to 0+.$$

and

$$u^{(n)}(x) \sim \frac{1}{\Gamma((\alpha/2)^n)} x^{(\alpha/2)^n - 1}, \quad x \to \infty.$$

For $\alpha \in (0, 2)$

$$\mu^{(n)}(x) \sim \frac{(\alpha/2)^n}{x^{(\alpha/2)^n+1}\Gamma(1 - (\alpha/2)^n)}, \quad x \to \infty.$$

$Y = (Y_t : t \geq 0)$: BM in R^d. $S^{(n)}$: the subordinator with Laplace exponent $\phi^{(n)}$ as before. Y and $S^{(n)}$ independent. Define $X^{(n)} = (X_t^{(n)} : t \geq 0)$ by $X_t^{(n)} = Y(S_t^{(n)})$.

$G^{(n)}(x, y) = G^{(n)}(y - x)$: the Green function of $X^{(n)}$.

$J^{(n)}(x, y) = J^{(n)}(y - x)$: the jumping function of $X^{(n)}$.

Theorem *For $\alpha \in (0, 2]$ and $n \geq 1$, we have*

$$G^{(n)}(x) \sim \frac{\Gamma(d/2)}{2\alpha\pi^{d/2} L_{n-1}(1/|x|^2) l_n^2(1/|x|^2)}, \quad |x| \to 0$$

and as $|x| \to \infty$,

$$G^{(n)}(x) \sim C|x|^{\alpha(\alpha/2)^{n-1}-d}, \quad |x| \to \infty,$$

where

$$C = \frac{1}{\pi^{d/2} 2^{\alpha(\alpha/2)^{n-1}}} \frac{\Gamma((d - \alpha(\alpha/2)^{n-1})/2)}{\Gamma((\alpha/2)^2)}.$$

Now suppose that X a transient sym Lévy process on $\mathbb{R}^d$ with a potential density $G(x, y) = G(y - x)$ such that G is a positive and decreasing function on $[0, \infty)$ and $G(0) = \infty$. Let Cap denote the (0-order) capacity with respect to X. Then we can show that

Theorem. *There exist positive constants $c_1 < c_2$ such that*

$$\frac{c_1 r^d}{\int_{\bar{B}(0,r)} G(|y|)dy} \leq \mathrm{Cap}(\bar{B}(0, r)) \leq \frac{c_2 r^d}{\int_{\bar{B}(0,r)} G(|y|)\, dy}.$$

Specialize to sym iterated geo stable processes we get

Corollary. *For any $\alpha \in (0, 2]$ and $n \geq 1$ we have*

$$\mathrm{Cap}(\bar{B}(0, r)) \asymp r^d l_n(\frac{1}{r}), \quad r \to 0.$$

In particular, for $n = 1$, we have

$$\mathrm{Cap}(\bar{B}(0, r)) \asymp r^d \log \frac{1}{r}, \quad r \to 0.$$

and for $n = 2$,

$$\mathrm{Cap}(\bar{B}(0, r)) \asymp r^d \log \log \frac{1}{r}, \quad r \to 0.$$

Now suppose that X a transient sym Lévy process on $\mathbb{R}^d$ with a potential density $G(x, y) = G(y - x)$ such that G is a positive and decreasing function on $[0, \infty)$ and $G(0) = \infty$. We further assume that $G : (0, \infty) \to (0, \infty)$ is regularly varying at $0+$. Then we can show that

Prop. *There exists positive constant c such that for any $r \in (0, 1/2)$ we have*

$$c \int_{B(0,r/6)} G(|y|)dy \le \inf_{z \in B(0,r/6)} \mathbf{E}_z \tau_{B(0,r)}$$

$$\le \sup_{z \in B(0,r)} \mathbf{E}_z \tau_{B(0,r)} \le \int_{B(0,r)} G(|y|)dy.$$

Specialize to the case of sym iterated geo stable processes we get

Corollary. *For any $\alpha \in (0, 2]$ and $n \ge 1$ we have*

$$c_1 / l_n(\frac{1}{r}) \le \inf_{z \in B(0,r/6)} \mathbf{E}_z \tau_{B(0,r)}$$

$$\le \sup_{z \in B(0,r)} \mathbf{E}_z \tau_{B(0,r)} \le c_2 / l_n(\frac{1}{r}).$$

Now we assume $\alpha \in (0, 2]$ when $n = 1$, and $\alpha \in (0, 2)$ for $n \geq 2$.

Theorem. *For any $r \in (0, 1/2)$, there exists a constant $C > 0$ such that for any $z_0 \in \mathbb{R}^d$ and any nonnegative bounded function u in $\mathbb{R}^d$ which is harmonic with respect to $X^{(n)}$ in $B(z_0, r)$ we have*

$$u(x) \leq Cu(y), \quad x, y \in B(z_0, r/2).$$

Note that the constant C above depends on r. We would like to get rid of this dependence on r.

Theorem. *Let D be a domain in $\mathbb{R}^d$ and K be a compact subset of D. Then there exists $C = C(D, K) > 0$ such that for any nonnegative function u in $\mathbb{R}^d$ which is harmonic with respect to $X^{(n)}$ in D we have*

$$u(x) \leq Cu(y), \quad x, y \in K.$$

Future work:

Fine potential theoretic properties of GS processes.

sample paths properties of GS processes.

heat kernel estimates and parabolic Harnack inequalities

Potential theory of killed GS processes