Stein's Method and Normal Approximation

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1. Introduction

Let $\{X_i\}$ be a sequence of independent random variables with $EX_i = 0$ and $EX_i^2 < \infty$. Put

$$S_n = \sum_{i=1}^n X_i, \quad B_n^2 = \sum_{i=1}^n EX_i^2.$$

- Lindeberg-Feller:

  In order that

$$B_n^{-2} \max_{1 \le i \le n} EX_i^2 \to 0$$

  and

$$S_n/B_n \xrightarrow{d.} N(0,1),$$

it is necessary and sufficient that Lindeberg's condition be satisfied

$$\forall\ \varepsilon > 0,\ \ B_n^{-2} \sum_{i=1}^{n} EX_i^2 I_{\{|X_i| > \varepsilon B_n\}} \longrightarrow 0$$

- The Berry-Esseen inequality (Berry (1941), Esseen (1942)):

  If $\{X_i\}$ is sequence of i.i.d. random variables with $EX_i = 0$ and $E|X_i|^3 < \infty$, then

$$\sup_z |F_n(z) - \Phi(z)| \leq Cn^{-1/2} E|X_1|^3 / \sigma^3$$

  where $\sigma^2 = EX_1^2$ and $F_n(z) = P(\frac{S_n}{\sqrt{\mathsf{Var}(S_n)}} \leq z)$.

- Esseen's inequality (1945):

$$\sup_z |F_n(z) - \Phi(z)| \leq CB_n^{-3} \sum_{i=1}^{n} E|X_i|^3$$

- Feller (1968):

$$\sup_z |F_n(z) - \Phi(z)| \leq 6\{B_n^{-2} \sum_{i=1}^n E X_i^2 I_{\{|X_i| > B_n\}} + B_n^{-3} \sum_{i=1}^n E|X_i|^3 I_{\{|X_i| \leq B_n\}}\}.$$

- Non-uniform estimates
  - Nagaev (1965):
    If $\{X_i\}$ i.i.d with $EX_i = 0$ and $E|X_i|^3 < \infty$, then
    $$|F_n(z) - \Phi(z)| \le \frac{CE|X_1|^3}{(1+|z|^3)\sqrt{n}\sigma^3}$$
  - Bikelis (1966):
    $$|F_n(z) - \Phi(z)| \le \frac{C\sum_{i=1}^n E|X_i|^3}{(1+|z|^3)B_n^3}$$

2. Stein's Method

A totally new method of normal approximation, introduced by Stein in 1972. It works well not only for independent random variables but also for dependent variables. Stein's ideas can be also applied to many other probability approximations, notably to Poisson, Poisson process, compound Poisson and binomial approximations.

2.1 The Stein equation

Let $Z \sim N(0,1)$, and let $\mathcal{C}_{bd}$ be the set of continuous and piecewise continuously differential functions $f : R \rightarrow R$ with $E|f'(Z)| < \infty$. Stein's method rests on the following observation.

Lemma 0.2.1 *Let W be a real valued random variable. Then W has a standard normal distribution it is necessary and sufficient that for all $f \in \mathcal{C}_{bd}$*

$$Ef'(W) = EWf(W). \tag{0.2.1}$$

Stein's equation:

$$f'(w) - wf(w) = I_{\{w \le z\}} - \Phi(z). \tag{0.2.2}$$

where $z \in R^1$ is fixed.

Solution to the equation:

$$\begin{aligned} f_z(w) &= e^{w^2/2} \int_{-\infty}^{w} [I_{\{x \le z\}} - \Phi(z)] e^{-x^2/2} dx \\ &= -e^{w^2/2} \int_{w}^{\infty} [I_{\{x \le z\}} - \Phi(z)] e^{-x^2/2} dx \\ &= \begin{cases} \sqrt{2\pi} e^{w^2/2} \Phi(w)[1 - \Phi(z)] & \text{if } w \le z, \\ \\ \sqrt{2\pi} e^{w^2/2} \Phi(z)[1 - \Phi(w)] & \text{if } w \ge z. \end{cases} \end{aligned} \tag{0.2.3}$$

The general Stein equation:

Let h be a real valued measurable function with $E|h(Z)| < \infty$.

$$f'(w) - wf(w) = h(w) - Eh(Z). \tag{0.2.4}$$

The solution $f = f_h$ is given by

$$\begin{aligned} f_h(w) &= e^{w^2/2} \int_{-\infty}^{w} [h(x) - Eh(Z)] e^{-x^2/2} dx \\ &= -e^{w^2/2} \int_{w}^{\infty} [h(x) - Eh(Z)] e^{-x^2/2} dx. \end{aligned} \tag{0.2.5}$$

2.2 Properties of solutions to the Stein equations

Let f_z be the solution to $(0.2.3)$.

Lemma 0.2.2 *We have*

$$wf_z(w) \text{ is an increasing function of } w, \tag{0.2.6}$$

$$|wf_z(w)| \le 1, \quad |wf_z(w) - uf_z(u)| \le 1 \tag{0.2.7}$$

$$|f_z'(w)| \le 1, \ |f_z'(w) - f_z'(v)| \le 1 \tag{0.2.8}$$

$$0 < f_z(w) \le \min(\sqrt{2\pi}/4, 1/|z|) \tag{0.2.9}$$

and

$$|(w+u)f_z(w+u) - (w+v)f_z(w+v)| \le (|w| + \sqrt{2\pi}/4)(|u| + |v|) \tag{0.2.10}$$

for all real $w,\ u$, *and* v.

- For $w > 0$

$$\frac{we^{-w^2/2}}{(1+w^2)\sqrt{2\pi}} \leq 1 - \Phi(w) \leq \frac{e^{-w^2/2}}{w\sqrt{2\pi}}$$

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}e^{-w^2/2}\left(\frac{1}{w} - \frac{1}{w^3}\right) &\leq 1 - \Phi(w) \\ &\leq \frac{1}{\sqrt{2\pi}}e^{-w^2/2}\left(\frac{1}{w} - \frac{1}{w^3} + \frac{3}{w^5}\right) \end{aligned}$$

For general h,

Lemma 0.2.3 *For absolutely continuous function h: $R \to R$*

$$\sup_w |f_h(w)| \leq \min(\sqrt{\pi/2} \sup_w |h(w) - Eh(Z)|, 2 \sup_w |h'(w)|), \quad (0.2.11)$$

$$\sup_w |f'_h(w)| \leq \min(2 \sup_w |h(w) - Eh(Z)|, 4 \sup_w |h'(w)|) \quad (0.2.12)$$

and

$$\sup_w |f''_h(w)| \leq 2 \sup_w |h'(w)|. \quad (0.2.13)$$

2.3 The main idea of the Stein approach

The Stein equation $(0.2.4)$ is the starting point for normal approximations. To illustrate the main idea of this approach, let $\xi_1, \xi_2, \cdots, \xi_n$ be independent random variables satisfying $E\xi_i = 0$ for each $1 \le i \le n$ and $\sum_{i=1}^n E\xi_i^2 = 1$. Put

$$W = \sum_{i=1}^n \xi_i, \; W^{(i)} = W - \xi_i \qquad (0.2.14)$$

Let h be a measurable function with $E|h(Z)| < \infty$, and $f = f_h$ be the solution of the Stein equation $(0.2.4)$.

Aim: Estimate

$$Eh(W) - Eh(Z) = Ef'(W) - EWf(W).$$

Since ξ_i and $W^{(i)}$ are independent and $E\xi_i = 0$ for each $1 \le i \le n$, we have

$$\begin{aligned} EWf(W) &= \sum_{i=1}^{n} E\xi_i f(W) \\ &= \sum_{i=1}^{n} E\xi_i [f(W) - f(W^{(i)})] \\ &= \sum_{i=1}^{n} E\xi_i \int_0^{\xi_i} f'(W^{(i)} + t)dt \\ &= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} f'(W^{(i)} + t)\xi_i (I_{\{0 \le t \le \xi_i\}} - I_{\{\xi_i \le t < 0\}})dt \\ &= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} f'(W^{(i)} + t)K_i(t)dt, \end{aligned} \tag{0.2.15}$$

where

$$K_i(t) = E\xi_i(I_{\{0\le t\le \xi_i\}} - I_{\{\xi_i \le t < 0\}}). \tag{0.2.16}$$

- $K_i(t) \ge 0$
- $\int_{-\infty}^{\infty} K_i(t)dt = E\xi_i^2$
- $\int_{-\infty}^{\infty} |t|K_i(t)dt = E|\xi_i|^3/2.$

From

$$\sum_{i=1}^n \int_{-\infty}^{\infty} K_i(t)dt = \sum_{i=1}^n E\xi_i^2 = 1,$$

it follows that

$$Ef'(W) = \sum_{i=1}^n E\int_{-\infty}^{\infty} f'(W)K_i(t)dt. \tag{0.2.17}$$

Thus, by $(0.2.15)$ and $(0.2.17)$

$$Ef'(W) - EWf(W) = \sum_{i=1}^{n} E \int_{-\infty}^{\infty} [f'(W) - f'(W^{(i)} + t)] K_i(t) dt. \tag{0.2.18}$$

Equations $(0.2.15)$ and $(0.2.18)$ play a key role in proving a Berry-Esseen type inequality.

2.4 Expectation of smooth functions

Equation $(0.2.18)$ is ready to drive a Berry-Esseen type bound for smooth function h.

Theorem 0.2.1 *For any absolutely continuous function h satisfying $\sup_x |h'(x)| \leq c_1$*

$$|Eh(W) - Eh(Z)| \leq 3c_1 \sum_{i=1}^{n} E|\xi_i|^3. \tag{0.2.19}$$

In particular, we have

$$|E|W| - \sqrt{\frac{2}{\pi}}| \leq 3 \sum_{i=1}^{n} E|\xi_i|^3.$$

We note that it is not necessary to assume the existence of finite third moments in Theorem 0.2.1.

Theorem 0.2.2 *Let* h *be absolutely continuous with* $|h'| \le c_1$*. Then*

$$|Eh(W) - Eh(Z)| \le 4c_1(4\beta_2 + 3\beta_3), \tag{0.2.20}$$

where

$$\beta_2 = \sum_{i=1}^{n} E\xi_i^2 I_{\{|\xi_i|>1\}} \text{ and } \beta_3 = \sum_{i=1}^{n} E|\xi_i|^3 I_{\{|\xi_i|\le 1\}}. \tag{0.2.21}$$

2.5 The Lindeberg central limit theorem

Although Theorem $(0.2.2)$ does not give a sharp Berry-Esseen bound directly, one can use a bounded absolutely continuous function to approximate the indicator function and then apply Theorem 0.2.2 to obtain a weak version of the Berry-Esseen bound which is good enough to recover the Lindebderg central limit theorem.

Theorem 0.2.3 *We have*

$$\sup_z |P(W \leq z) - \Phi(z)| \leq 2.2(4\beta_2 + 3\beta_3)^{1/2}, \tag{0.2.22}$$

where β_2 *and* β_3 *are defined in* $(0.2.21)$*.*

Although Theorem 0.2.3 does not give a sharp Berry-Esseen bound, it does provide a self-contained proof for the central limit theorem under Lindeberg's condition.

Let $X_1, X_2, \cdots, X_n$ be independent random variables with $EX_i = 0$ and $EX_i^2 < \infty$ for each $1 \le i \le n$. Put

$$S_n = \sum_{i=1}^n X_i \text{ and } B_n^2 = \sum_{i=1}^n EX_i^2.$$

To apply Theorem 0.2.3, let

$$\xi_i = X_i/B_n \text{ and } W = S_n/B_n. \tag{0.2.23}$$

Observe that for any $0 < \varepsilon < 1$

$$\beta_2 + \beta_3 = \frac{1}{B_n^2}\sum_{i=1}^n EX_i^2 I_{\{|X_i|>B_n\}} \tag{0.2.24}$$

$$
\begin{aligned}
&+\frac{1}{B_n^3}\sum_{i=1}^{n} E|X_i|^3 I_{\{|X_i|\le B_n\}} \\
\le\ & \frac{1}{B_n^2}\sum_{i=1}^{n} EX_i^2 I_{\{|X_i|>B_n\}} && (0.2.25) \\
&+\frac{1}{B_n^3}\sum_{i=1}^{n} B_n EX_i^2 I_{\{\varepsilon B_n\le|X_i|\le B_n\}} \\
&+\frac{1}{B_n^3}\sum_{i=1}^{n} \varepsilon B_n EX_i^2 I_{\{|X_i|<\varepsilon B_n\}} \\
\le\ & \varepsilon+\frac{1}{B_n^2}\sum_{i=1}^{n} EX_i^2 I_{\{|X_i|>\varepsilon B_n\}}. && (0.2.26)
\end{aligned}
$$

If Lindeberg's condition is satisfied, then $(0.2.26)$ implies $\beta_2+\beta_3\to 0$ as

$n \longrightarrow \infty$ since ε is arbitrary. This shows

$$\sup_z |P(S_n/B_n \leq z) - \Phi(z)| \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

by Theorem 0.2.3.

3. Uniform Berry-Esseen Bounds

Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent random variables with zero means, finite second moment.

$$\sum_{i=1}^{n} E\xi_i^2 = 1$$

Use the notation in the previous section,

$$W = \sum_{i=1}^{n} \xi_i, \; W^{(i)} = W - \xi_i$$

$$K_i(t) = E\xi_i(I_{\{0 \le t \le \xi_i\}} - I_{\{\xi_i \le t < 0\}}).$$

Let f_z be the solution of the Stein equation $(0.2.1)$. Our goal is to use

Stein's method to prove the uniform Berry-Esseen inequality

$$\sup_z |P(W \le z) - \Phi(z)| \le C \sum_{i=1}^{n} E|\xi_i|^3.$$

3.1 Bounded random variables

For bounded ξ_i, we are ready to apply $(0.2.15)$ to obtain the following Berry-Esseen type bound.

Theorem 0.3.1 *If* $|\xi_i| \leq \delta_0$ *for* $1 \leq i \leq n$, *then*

$$\sup_z |P(W \leq z) - \Phi(z)| \leq 3.3\delta_0 \tag{0.3.1}$$

Proof. Write $f = f_z$. It follows from $(0.2.15)$ that

$$
\begin{aligned}
EWf(W) &= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} f'(W^{(i)} + t) K_i(t) dt \\
&= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \{(W^{(i)} + t) f(W^{(i)} + t) \\
&\quad + I_{\{W^{(i)} + t \le z\}} - \Phi(z)\} K_i(t) dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{n} \int_{-\infty}^{\infty} P(W^{(i)} + t \le z) K_i(t) dt - \Phi(z) \\
&= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \{Wf(W) - (W^{(i)} + t) f(W^{(i)} + t)\} K_i(t) dt.
\end{aligned}
\tag{0.3.2}
$$

By $(0.2.10)$,

$$\sum_{i=1}^{n} E\int_{-\infty}^{\infty} |Wf(W) - (W^{(i)}+t)f(W^{(i)}+t)|K_i(t)dt$$

$$\leq \sum_{i=1}^{n}\int_{-\infty}^{\infty} E(|W^{(i)}| + \sqrt{2\pi}/4)(|\xi_i| + |t|)K_i(t)dt$$

$$\leq (1+\sqrt{2\pi}/4)\sum_{i=1}^{n}\int_{-\infty}^{\infty}(E|\xi_i| + |t|)K_i(t)$$

$$= (1+\sqrt{2\pi}/4)\sum_{i=1}^{n}\{E|\xi_i|E\xi_i^2 + 0.5E|\xi_i|^3\}$$

$$\leq 1.5(1+\sqrt{2\pi}/4)\sum_{i=1}^{n} E|\xi_i|^3. \qquad (0.3.3)$$

Noting that the assumption $|\xi_i| \leq \delta_0$ implies $K_i(t) = 0$ for $|t| > \delta_0$, we have

$$\sum_{i=1}^{n}\int_{-\infty}^{\infty} P(W^{(i)}+t\le z)K_i(t)dt$$
$$= \sum_{i=1}^{n}\int_{|t|\le\delta_0} P(W-\xi_i+t\le z)K_i(t)dt$$
$$\ge \sum_{i=1}^{n}\int_{|t|\le\delta_0} P(W\le z-2\delta_0)K_i(t)dt$$
$$= P(W\le z-2\delta_0).$$

Combining with $(0.3.2)$ and $(0.3.3)$ gives

$$P(W\le z-2\delta_0)-\Phi(z-2\delta_0)$$
$$\le \Phi(z)-\Phi(z-2\delta_0)+1.5(1+\sqrt{2\pi}/4)\sum_{i=1}^{n}E|\xi_i|^3$$

$$\leq \frac{2\delta_0}{\sqrt{2\pi}} + 1.5(1 + \sqrt{2\pi}/4)\delta_0 \leq 3.3\delta_0 \tag{0.3.4}$$

Similarly, we have

$$\sum_{i=1}^{n} \int_{-\infty}^{\infty} P(W^{(i)} + t \leq z) K_i(t) dt \leq P(W \leq z + 2\delta_0).$$

and

$$P(W \leq z + 2\delta_0) - \Phi(z + 2\delta_0) \geq -3.3\delta_0 \tag{0.3.5}$$

This proves $(0.3.1)$ by $(0.3.4)$ and $(0.3.5)$.

3.2 The concentration inequality approach

Let $\gamma = \sum_{i=1}^{n} E|\xi_i|^3$.

Proposition 0.3.1 *We have*

$$P(a \leq W^{(i)} \leq b) \leq \sqrt{2}(b-a) + (1+\sqrt{2})\gamma \qquad (0.3.6)$$

for $a < b$ *and* $1 \leq i \leq n$.

Theorem 0.3.2 *We have*

$$\sup_z |P(W \leq z) - \Phi(z)| \leq 7\gamma. \tag{0.3.7}$$

Remark. One can prove

$$\sup_z |P(W \le z) - \Phi(z)| \le 4.1 \sum_{i=1}^{n} (E\xi_i^2 I_{\{|\xi_i|>1\}} + E|\xi_i|^3 I_{\{|\xi_i|\le 1\}}).$$

3.3 A randomized concentration inequality

Let Δ_1 and Δ_2 be real-valued Borel measurable functions of $(\xi_i, 1 \le i \le n)$.

Theorem 0.3.3 *We have*

$$\begin{aligned} & P(\Delta_1 \le W \le \Delta_2) \\ & \le E|W(\Delta_2 - \Delta_1)| + 2\gamma \\ & \quad + \sum_{i=1}^{n} \{E|\xi_i(\Delta_1 - \Delta_{1,i})| + E|\xi_i(\Delta_2 - \Delta_{2,i})|\} \end{aligned} \tag{0.3.8}$$

where $\Delta_{1,i}$ *and* $\Delta_{2,i}$ *are Borel measurable functions of* $(\xi_j, 1 \le j \le n,\ j \ne i)$, *and*

$$\gamma = \sum E|\xi_i|^3$$

Theorem 0.3.4 *Let* $\Delta = \Delta(\xi_1, \cdots, \xi_n) : R^n \longrightarrow R^1$ *be a Borel measurable function. Then we have Then we have*

$$\sup_z |P(W + \Delta \le z) - \Phi(z)| \le 9\gamma + E|W\Delta| + \sum_{i=1}^{n} E|\xi_i(\Delta - \Delta_i)|, \qquad (0.3.9)$$

where Δ_i *is a measurable function of* $(\xi_j, 1 \le j \le n, j \ne i)$.

An application to U-statistic

Let $X_1, X_2, \cdots, X_n$ be a sequence of independent identically distributed random variables, and let $h(x, y)$ be a real-valued Borel measurable symmetric function, i.e., $h(x, y) = h(y, x)$. Define the U-statistic with the kernel h by

$$U_n = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} h(X_i, X_j).$$

Theorem 0.3.5 *Assume that* $Eh(X_1, X_2) = 0$ *and* $\sigma^2 = Eh^2(X_1, X_2) < \infty$. *Let* $g(x) = Eh(x, X_2)$ *and* $\sigma_1^2 = Eg^2(X_1)$. *If* $\sigma_1 > 0$, *then*

$$\sup_z |P(\frac{\sqrt{n}U_n}{2\sigma_1} \leq z) - \Phi(z)| \leq \frac{2\sigma}{(n-1)^{1/2}\sigma_1} + \frac{9E|g(X_1)|^3}{n^{1/2}\sigma_1^3}.$$

To illustrate the main idea of the proof of Theorem 3.3, consider $a = \Delta_1 < b = \Delta_2$, where a and b are real numbers, and show that

$$P(a \le W \le b) \le (b-a) + 2\gamma,$$

where $\gamma = \sum_{i=1}^{n} E|\xi_i|^3$.

Proof. Let $\delta = \gamma/2$,

$$f(w) = \begin{cases} -\frac{1}{2}(b-a) - \delta & \text{for } w < a - \delta, \\ w - \frac{1}{2}(b+a) & \text{for } a - \delta \le w \le b + \delta, \\ \frac{1}{2}(b-a) + \delta & \text{for } w > b + \delta \end{cases}$$

and

$$\begin{aligned} M_j(t) &= \xi_j\{I(-\xi_j \le t \le 0) - I(0 < t \le -\xi_j)\}, \\ M(t) &= \sum_{j=1}^n M_j(t). \end{aligned}$$

Then

$$\begin{aligned} |f| &\le \frac{b-a}{2} + \delta, \;\; f' \ge 0, \\ f'(w) &= 1 \;\; \text{for } a - \delta < w \le b + \delta. \end{aligned}$$

Observe that

$$\begin{aligned} & EWf(W) \\ &= \sum_{j=1}^n E\xi_j[f(W) - f(W - \xi_j)] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} E\xi_j \int_{-\xi_j}^{0} f'(W+t)dt \\
&= \sum_{j=1}^{n} E \int_{-\infty}^{\infty} f'(W+t)M_j(t)dt \\
&= E \int_{-\infty}^{\infty} f'(W+t)M(t)dt \\
&\geq E \int_{|t|\leq\delta} f'(W+t)M(t)dt \\
&\geq EI(a \leq W \leq b) \int_{|t|\leq\delta} M(t)dt \\
&= EI(a \leq W \leq b) \sum_{j=1}^{n} |\xi_j| \min(\delta, |\xi_j|) \\
&\geq P(a \leq W \leq b)H_1 - H_2,
\end{aligned}
$$

where $\eta_j = |\xi_j| \min(\delta, |\xi_j|)$,

$$H_1 = \sum_{j=1}^{n} E\eta_j, \ H_2 = E|\sum_{j=1}^{n} \eta_j - E\eta_j|.$$

It is easy to see that

$$\min(x, y) \geq x - x^2/(4y)$$

for $x > 0$ and $y > 0$. Thus,

$$\begin{aligned} \sum_{j=1}^{n} E\eta_j &= \sum_{j=1}^{n} E|\xi_j| \min(\delta, |\xi_j|) \\ &\geq \sum_{j=1}^{n} E|\xi_j|^2 - E|\xi_j|^3/(4\delta) \\ &= \frac{1}{2}. \end{aligned}$$

By the Hölder inequality,

$$H_2 \le \Big(\sum_{j=1}^{n} E\xi_j^2 \min(\delta^2, \xi_j^2)\Big)^{1/2} \le \delta.$$

Combining the above inequalities yields

$$\begin{aligned} P(a \le W \le b) &\le 2(H_2 + EWf(W)) \\ &\le 2(\delta + E|W|\{\frac{(b-a)}{2} + \delta\}) \\ &\le (b-a) + 2\gamma. \end{aligned}$$

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