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Local Stochastic Differential Geometry, or

Feeling the shape of a manifold with Brownian motion

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Ia. Definition of Brownian Motion $(M, g) =$ d-dimensional complete Riemannian manifold.

$$\Delta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x_j}\right) \quad \text{(summation convention)}$$

$g_{ij} =$ metric tensor, $g^{ij} =$ inverse, $g = \det(g_{ij})$.

Brownian motion (X_t, P_x) is the local diffusion process generated by $\frac{1}{2}\Delta$.

WHY BROWNIAN MOTION IN DIFF GEOM?

Ib. Isotopic transport approximation

Brownian motion can be obtained as a weak limit of piecewise geodesic processes.

Ingredients: geodesics $\gamma(t) = \gamma(t, m, \xi) = \exp_m(t\xi)$ and random times $0 < t_1 < t_2 < \cdots$

Starting at $m \in M$ with initial velocity ξ_0, we move along a geodesic $\gamma(t), 0 \leq t \leq t_1$

At time t_1 choose a new direction ξ_1 at random, uniformly distributed on the tangent sphere.

Move along the geodesic $t \to \gamma(t - t_1; \gamma(t_1), \xi_1)$ for $t_1 \leq t \leq t_2$.

At time t_2 choose a new direction ξ_2 at random, uniformly distributed on the tangent sphere, independent of (t_1, t_2, ξ_2).

Move along the geodesic $t \to \gamma(t - t_2; \gamma(t_2), \xi_2)$ for $t_2 \leq t \leq t_3$.

Continue in this fashion to obtain a piecewise smooth path $(X(t), \xi(t))$.

The geodesics have constant velocity $1/\epsilon = |\gamma'(t)|$.

The random times $t_n - t_{n-1}$ are independent with the exponential distribution with mean ϵ^2.

This gives a continuous parameter Markov process on the tangent bundle, called the *isotropic transport process*.

Infinitesimal generator is

$$A^\epsilon f = \frac{Df}{\epsilon} + \frac{1}{\epsilon^2}\int_{S_m} \Big(f(m,\xi') - f(m,\xi)\Big)\,\omega_m(d\xi').$$

where D is the horizontal differentiation in the tangent bundle

ω_m is the uniform probability distribution on the unit tangent sphere S_m.

When $\epsilon \to 0$ the corresponding semigroups converge to a limit, whose generator is written

$$f \to A^0 f = \int_{S_m} D^2 f(m,\xi)\,\omega_m(d\xi) = \frac{\Delta f}{d}$$

a multiple of the Laplace-Beltrami operator of (M,g). (use results of T.G. Kurtz, JFA, 1973).

Brownian motion is obtained as the semigroup limit of a sequence of piecewise smooth Markov processes on the tangent bundle TM. The limiting process (Br. Mo) is a continuous path Markov process on M.

Ic. Horizontal flow of orthonormal frames

The isotropic transport process can be generalized to obtain a Markov process on the *frame bundle* $T^{k+1}(M)$, consisting of the set of frames of $k+1$ vectors over M. In particular we can consider orthonormal frames. In the limiting case $\epsilon \to 0$, we obtain a diffusion process on $T^k(M)$, equivalent to the process considered by Itô, Dynkin and Malliavin, called the *horizontal diffusion process* on $O(M)$ [see Ikeda and Watanabe, SDE and Diffusion Processes, 2d ed.].

Ic. Mean Value Formulas For $f \in C^\infty(M)$,

$$\begin{aligned}
\mathcal{M}_m(\epsilon, f) &:= \int_{\partial B_\epsilon} f\, dv_{d-1} \quad \text{(first meanvalue)} \\
\mathcal{L}_m(\epsilon, f) &:= \int_{S^{d-1}(1)} f(\text{exp}_m(\epsilon u))\omega(du)(\text{ 2dm.v.}) \\
\mathcal{E}_m(\epsilon, f) &= E_m[f(X_{T_\epsilon})] \quad \text{(stochastic m.v.)}
\end{aligned}$$

Here T_ϵ is the exit time of Brownian motion from a ball of radius ϵ, centered at $m \in M$; exp_m is the exponential mapping, which sends each tangent vector to the time 1 value of the geodesic which starts at m with that initial direction. ω is the normalized surface measure on the unit sphere in M_m and v_{d-1} is the normalized surface measure of the unit sphere in M.

Asymptotic expansions (P, 1981):

$$\begin{aligned}
\mathcal{M}_m(\epsilon, f) &= f(m) + \frac{\epsilon^2}{2d}\Delta f(m) + \epsilon^4 M_m^{(1)}(\epsilon, f) \\
\mathcal{L}_m(\epsilon, f) &= f(m) + \frac{\epsilon^2}{2d}\Delta f(m) + \epsilon^4 L_m^{(1)}(\epsilon, f) \\
\mathcal{E}_m(\epsilon, f) &= f(m) + \frac{\epsilon^2}{2d}\Delta f(m) + \epsilon^4 E_m^{(1)}(\epsilon, f)
\end{aligned}$$

In general $M_m^{(1)}(\epsilon, f) \neq E_m^{(1)}(\epsilon, f) \neq L_m^1(\epsilon, f)$ for small $\epsilon > 0$. Hence stochastic mean value $\neq$ geom. mean values.

Corollary. If $\forall m \in M$, either $\mathcal{E}_m(\epsilon, f) - \mathcal{M}_m(\epsilon, f) = o(\epsilon^4)$ or $\mathcal{E}_m(\epsilon, f) - \mathcal{L}_m(\epsilon, f) = o(\epsilon^4)$ then g is an Einstein metric ($\rho_{ij} = \tau g_{ij}$)

$$
\begin{aligned}
E_m^{(1)}(\epsilon, f) &= \Delta^2 f(m) + \tau(m)\Delta f(m) + o(1) \\
M_m^{(1)}(\epsilon, f) &= 3\Delta^2 f(m) - 2 < \nabla^2 f, \rho > -3 < \nabla^2 f, \rho > \\
&- \ 3 < \nabla f, \nabla \tau > +4\tau(m)\Delta f(m) + o(1) \\
L_m^{(1)}(\epsilon, f) &= \Delta^2 f(m) + \frac{1}{3} < \nabla f, \nabla \tau > \\
&+ \ < \nabla^2 f, \rho > + o(1)
\end{aligned}
$$

$$
\begin{aligned}
\rho_{ij} :&= \text{ Ricci tensor} \\
\tau :&= \sum_{i=1}^{d} \rho_{ii} = \text{ scalar curvature}
\end{aligned}
$$

IIa. Exit time and exit place

$(M, g) = d$-dimensional Riemannian manifold.

Exit time from a ball of radius r about $m \in M$

$$T_r := \inf\{t > 0 : \text{dist}(X_t, m) = r\}.$$

Mean exit time $u = E_x[T_r]$ is the solution of the equation $\frac{1}{2}\Delta u = -1$ in the ball, with $u = 0$ on the boundary.

The hitting probability measure is defined by $h = P_x[X_{T_r} \in dy]$, solution of $\Delta h = 0$ in the ball with $h = \delta_y$ (Dirac measure) on the boundary.

The joint law of (T_r, X_{T_r}) is computed from the Laplace transform $v = E_x[e^{-\alpha T_r}; X_{T_r} \in dy]$, solution of $\frac{1}{2}\Delta v = \alpha v$ in the ball, with $u = \delta_y$ on the boundary.

IIb. Br. mo.of R^d and related asymptotics

Brownian motion of Euclidean space has the well-known properties:

i) The exit time T_r has the *Brownian scaling property*; $\frac{T_r}{r^2} \equiv T_1$, (P_m).

ii) The exit place is uniformly P_m-distributed on the sphere: $r^{-1} \exp_m^{-1} X_{T_r} \equiv \mathrm{Leb}(S^{d-1})$.

iii) The exit time and exit place are independent random variables: $\forall \alpha > 0, \psi \in C(M)$

$$E_m\left(e^{-\alpha T_r}\psi(X_{T_r})\right) = E_m(e^{-\alpha T_r})E_m\left(\psi(X_{T_r})\right)$$

iv) The principal Dirichlet eigenvalue of Δ on the ball of radius r is inversely proportional to the radius^2: $\lambda_1(B_r) = z_d^2/r^2$, where z_d is the first positive zero of the Bessel function $J_{(d-2)/2}$.

For any Riemannian manifold, the above properties hold in the limit $r \downarrow 0$, in the following sense:

i')The scaled exit time T_r/r^2 converges ($r \downarrow 0$) in law to a limit $\mathbf{T}$ = exit time of Euclidean Br. Motion from the unit ball of R^d.

ii') The exit place $r^{-1}\exp_m^{-1}(X_{T_r})$ converges ($r \downarrow 0$) to normalized Lebesgue measure on the unit sphere $\mathbf{S}^{d-1} \subset \mathbf{R}^d$.

iii') The random variables (T_r, X_{T_r}) are asymptotically independent: $\forall \alpha > 0, \psi \in C(M)$

$$\lim_{r\downarrow 0} E_m\left(e^{-\alpha\frac{T_r}{r^2}}\psi(r^{-1}X_{T_r})\right) = E_m(e^{-\alpha\mathbf{T}})\int_{S^{d-1}}\psi(\theta)d\theta$$

iv') $\lambda_1(B_r) \sim z_d^2/r^2,\ (r \downarrow 0)$

To what extent do these quantities determine the local geometry of (M, g)?

III. Exit time distribution of Br. motion

Thm 3.1 (A. Gray & MP, 1983): For any (M, g), the mean exit time of Br. mo. satisfies

$$E_m(T_r) = \frac{r^2}{d} + c_1 r^4 \tau_m + c_2 r^6 \times \left(|R|_m^2 - |\rho|_m^2 + \frac{5\tau_m^2}{d} + 6\Delta\tau_m \right) + O(r^8)$$

where c_1, c_2 are constants which depend on the dimension d. R denotes the Riemann curvature tensor and ρ the Ricci tensor, defined in terms of the metric in a normal coordinate chart by

$$g_{ij}(x_1, \ldots, x_d) = \delta_{ij} - \frac{1}{3} \sum_{a,b=1}^{d} R_{iajb}(m) x_a x_b + O(|x|^3), \quad |x$$

Ricci tensor defined by

$$\rho_{ij}(m) = \sum_{a=1}^{d} R_{iaja}(m); \quad |\rho|_m^2 = \sum_{i,j=1}^{d} \rho_{ij}^2$$

Scalar curvature and norm of R defined by

$$\tau(m) = \sum_{i=1}^{d} \rho_{ii}(m); \quad |R|_m^2 = \sum_{i,a,j,b=1}^{d} R_{iajb}^2$$

Corollary 3.2 If $d = 2$ and the mean exit time satisfies $E_m(T_r) = r^2/2 + o(r^4), r \downarrow 0$, then (M, g) is locally isometric to R^2.

(The scalar curvature = Gaussian curvature is the only local geometric invariant in $d = 2$.)

In $d > 2$ dimensions we obtain converse results by writing the third term of the expansion in terms of the *Weyl tensor* C_{ijkl} by writing

$$|R|^2 - |\rho|^2 = |C|^2 + \frac{6-d}{d-2}|\rho|^2$$

Positive definite if $2 < d < 6$.

Corollary 3.3 Suppose that $2 < d < 6$ and $\forall m \in M$ we have $E_m(T_r) = r^2/d + o(r^6), r \downarrow 0$. Then (M, g) is locally isometric to R^d.

Corollary 3.4 Suppose that $d < 6$ and $\forall m \in M$ $\exists r(m) > 0$ s.t. the P_m law of T_r/r^2 doesn't depend on r for $0 < r < r(m)$. Then (M, g) is locally isometric to R^d.

END OF POSITIVE RESULTS ON EXIT TIME

Proposition 3.5(Hughes, 1992). Let $M = S^3 \times H^3$, the product of the unit 3-dimensional sphere with the three-dimensional hyperbolic space. Then for any $r < \pi$, the P_m law of T_r agrees with that of R^6. In particular $E_m(T_r) = r^2/6$, with no correction terms. But M *is not flat*.

Counter-examples my be obtained in any dimension by taking an additional flat factor, in the form $M = S^3 \times R^3 \times R^{d-6}$

IV. Exit place distribution of Br. motion

Euclidean space cannot be characterized by the exit place of Brownian motion. For example, consider a brownian particle starting at the North Pole and hitting the Arctic circle, according to the uniform distribution.

To study exit place, consider the exponential mapping $\exp_m : R^d \to M$ which sends $0 \to m \in M$ and maps straight lines to geodesics eminating from m. If $\Psi \in S^{d-1}$, define the *harmonic measure* μ_m by

$$H_r\Psi(m) := E_m\Big(\Psi(r^{-1}\exp_m(X_{T_r})\Big) = \int_{S^{d-1}} \Psi(\theta)\mu_m(r, d\theta)$$

Theorem 4.1 Ming Liao, 1988. For any (M, g), the harmonic measure operator has the asymptotic expansion

$$H_r\Psi(m)=\int_{S^{d-1}}\Psi(\theta)\left(1-\frac{r^2\rho_m^{\#}(\theta)}{12}-\frac{r^3\rho_m^{\#\#}(\theta)}{24}\right)d\theta+O$$

where $\rho^{\#}$ is the traceless Ricci tensor

$$\rho_m^{\#}(\theta)=\left(\rho_{ij}-\frac{\delta_{ij}\tau_m}{d}\right)\theta_i\theta_j \qquad \text{and}$$

$$\rho_m^{\#\#}(\theta)=\frac{\partial\rho_{ij}}{\partial x_k}\theta_i\theta_j\theta_k-\frac{1}{d+2}\theta_k\frac{\partial\tau}{\partial\theta_k}$$

(repeated indices imply a summation.) In case $d = 2$, $\rho_m^{\#} = 0$ and we have

Corollary 4.2 If $d = 2$ and the exit place law satisfies

$$E_m\Psi(X_{T_r})=\int_{S^1}\Psi(\theta)d\theta+o(r^3), r\downarrow 0,$$

then (M, g) has constant curvature.

Higher dimensional case

Corollary 4.3 Suppose that $d > 2$ and that

$$E_m \Psi(X_{T_r}) = \int_{S^{d-1}} \Psi(\theta) d\theta + o(r^3), r \downarrow 0$$

Then $\rho^{\#}(m) = 0$, esp. g is an Einstein metric (constant Ricci curvature).

Combined result on exit time and exit place for all dimensions

Corollary 4.4 Suppose that $\forall m \in M$, the exit place law satisfies

$$E_m \Psi(X_{T_r}) = \int_{S^{d-1}} \Psi(\theta) d\theta + o(r^3), r \downarrow 0$$

and the mean exit time satisfies $E_m(T_r) = r^2/d + o(r^6), r \downarrow 0$. Then (M, g) is locally isometric to the Euclidean space R^d.

V. Joint law of exit time and exit place On any space of constant sectional curvature, the exit time and exit place from a ball are independent random variables, for Brownian paths starting at the center. In order to study the joint law on more general spaces we introduce the Laplace transfrom

$$\begin{aligned} H_r(\alpha, \Psi) : &= E_m \left(e^{-\alpha \frac{T_r}{r^2}} \Psi\left(\frac{\exp_m^{-1}(X_{T_r})}{r}\right) \right) \\ &= \int_{S^{d-1}} \Psi(\theta) \mu_m^\alpha(r, d\theta) \end{aligned}$$

Keisuke Hara(1996) obtained an asymptotic expansion

$$\begin{aligned} H_r(\alpha, \Psi) &= c_0(\alpha) \int \Psi + r^2 \int u\Psi + r^3 \int v\Psi + O(r^4) \\ u(\theta) &= c_1(\alpha)\rho_{ij}\theta^i\theta^j + c_2(\alpha)\tau \\ v(\theta) &= c_3(\alpha)\partial_i\rho_{jk}\theta^i\theta^j\theta^k + c_4(\alpha)\partial_i\tau\theta^i \end{aligned}$$

where c_1, c_2, c_3, c_4 are expressed in terms of the Laplace transform of the exit time law of Euclidean Brownian motion from the unit ball.

Corollary 5.1 If $\forall m \in M$ the random variables $T_r, X_{T_r})$ are independent $\forall r < r(m)$. Then $\tau(m) =$ constant.

[also proved by Ming Liao, Kozaki-Ogura] Question: Does independence imply Einstein metric?

Example 5.2 Let $M = S^3 \times H^3$ with the product metric. Then $\forall r < \pi$, T_r and X_{T_r} are independent. But M is not an Einstein space, since Ricci curvature is not constant.

VI. Principal eigenvalue of the Laplacian

$$\lambda_1(m,r) := \inf_{(f:f\not\equiv 0, f=0 \text{ on } \partial B_r)} \frac{\int_{B_r} |df|^2}{\int_{B_r} |f|^2}.$$

Theorem 6.1(Karp and MP, 1987)

$$\begin{aligned}\lambda_1(m,r) &= \frac{z_d^2}{r^2} - \frac{\tau_m}{6} + \text{const.} r^2 \times \\ &\quad [|R|_m^2 - |\rho|_m^2 + 6\Delta\tau_m] + O(r^4), r \downarrow 0\end{aligned}$$

where constant depends only on d; z_d is the first positive zero of the Bessel function $J_{(d-2)/2}$.

Corollary 6.2 Suppose that $d < 6$ and that $\forall m \in M$ we have $\lambda_1(m,r) = z_d^2/r^2 + o(r^2), r \downarrow 0$. Then (M,g) is locally isometric to R^d.

Example 6/3 Let $M = S^3 \times H^3$. Then $\forall r < \pi$ we have $\lambda_1(m,r) = z_6^2/r^2$ (exactly as for $M = R^6$).

VII. Some methods. Local computations are made by reducing to a sequence of equations in the tangent space, based on the Laplacian of Euclidean space. Let

$$\begin{aligned}\Phi_\epsilon : C^\infty(M_m) &\mapsto C^\infty(M)\\ (\Phi_\epsilon f)(\exp_m x) : &= f(x/\epsilon).\end{aligned}$$

Then $\forall f \in C^\infty(M_m)$ we have asymp. for $\epsilon \downarrow 0$

$$\Phi_\epsilon^{-1}\Delta\Phi_\epsilon f = \epsilon^{-2}\Delta_{-2} f + \sum_{k=0}^{\infty} \epsilon^k \Delta_k f$$

where Δ_k maps a homog. polynomial of degree n to a homog. polynomial of degree $n+k$:

$$\Delta_{-2} = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}, \quad \Delta_0 = -\frac{2}{3}\sum_{i,a=1}^{d} \rho_{ia}(m) x_a \frac{\partial}{\partial x_i}$$

$$+ \ \frac{1}{3}\sum_{i,a,j,b=1}^{d} R_{iajb}(m) x_a x_b \frac{\partial^2}{\partial x_i \partial x_j}$$

In general the coefficients of Δ_k depend on the curvature tensor and its derivatives of order $\leq k$.

$\Delta_0, \Delta_1, \Delta_2$ were computed by Gray and MP (1983); Δ_3 was computed by Kozaki and Ogura (1988).

Application to mean exit time: approximate solution

$$f_\epsilon = \Phi_\epsilon(\epsilon^2 F_0 + \epsilon^4 F_2 + \epsilon^5 F_3 + \epsilon^6 F_4)$$

where $F_i = 0$ on ∂B_1 and inside satisfy

$$\begin{aligned} \Delta_{-2} F_0 + 1 &= 0 \\ \Delta_{-2} F_2 + \Delta_0 F_0 &= 0 \\ \Delta_{-2} F_3 + \Delta_1 F_0 &= 0 \\ \Delta_{-2} F_4 + \Delta_0 F_2 + \Delta_2 F_0 &= 0 \end{aligned}$$

$$\Rightarrow F_0(0) = \frac{1}{2d}, F_2(0) = \frac{\tau_m}{12d^2(d+2)}, F_3(0) = 0$$

$$\Delta f_\epsilon = -1 + O\left(\epsilon^8\right)$$

VIII. Computations on $M = S^3 \times H^3$

$$\begin{aligned}\Delta_M f &= \frac{\partial^2}{\partial r_1^2} + (2\cot r_1)\frac{\partial}{\partial r_1} + (\csc^2 r_1)\Delta^1_{S^2} \\ &+ \frac{\partial^2}{\partial r_2^2} + (2\coth r_2)\frac{\partial}{\partial r_2} + (\mathrm{csch}^2 r_2)\Delta^2_{S^2}\end{aligned}$$

Lemma. $\forall f(r_1, r_2) \in C^2(R^2)$, we have conjugacy relation

$$\Delta_M \left[\frac{f(r_1, r_2)}{q(r_1, r_2)}\right] = \frac{1}{q(r_1, r_2)} \Delta_{R^6} f(r_1, r_2)$$

where

$$q(r_1, r_2) = \frac{\sin r_1}{r_1} \frac{\sinh r_2}{r_2}$$

$$\Delta_{R^6} f = \frac{\partial^2 f}{\partial r_1^2} + \frac{2}{r_1}\frac{\partial f}{\partial r_1} + \frac{\partial^2 f}{\partial r_2^2} + \frac{2}{r_2}\frac{\partial f}{\partial r_2}$$

This allows one to go back and forth from computations on M to computations on R^6, when one restricts to *bi-radial* functions $f = f(r_1, r_2)$. Proof is by direct computation.

IX. Brownian motion from imbedded manifolds. Let M be an oriented hypersurface in R^{d+1} with unit normal vector $(n_1, \ldots, n_{d+1})$ and mean curvature $H = \text{div}\mathbf{n}$. Then

$$\Delta_M f = \sum_{i,j=1}^{d+1} (\delta_{ij} - n_i n_j) \frac{\partial^2 f}{\partial x_i \partial x_j} + dH \sum_{i=1}^{d+1} n_i \frac{\partial f}{\partial x_i}$$

$T_r^{\text{ext}} := \inf\{t > 0 : |X_t - m| = r\}$ (*extrinsic mean exit time*).

Theorem(L. Karp & MP, 1985) We have the asymptotic expansion when $r \downarrow 0$:

$$E_m(T_r^{\text{ext}}) = \frac{r^2}{2d} + \frac{r^4 H^2}{8(d+1)} + O(r^5)$$

Corollary 1. If $\forall m \in M$ $E_m(T_r^{\text{ext}}) = r^2/2d$, then M is a minimal hypersurface.

Corollary 2. If $\forall m \in M, r > 0$, the extrinsic and intrinsic mean exit times are equal: $E_m(T_r^{\text{ext}}) = E_m(T_r)$, then M is a flat hyperplane in R^{d+1}.