3. High moment asymptotics.

One of the goals in this section is to establish

$$\lim_{m\to\infty}\frac{1}{m}\log\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\left[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\right]^p = p\log\rho \tag{3.1}$$

where

$$\rho=\sup_f\iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x-y)f(x)f(y)dxdy$$

where the supremum is taken for all f on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d}|f(x)|^{\frac{2p}{2p-1}}dx=1$$

and where

$$G(x)=\int_0^\infty e^{-t}\frac{1}{(2\pi t)^{d/2}}\exp\Big\{-\frac{|x|^2}{2t}\Big\}dt \qquad x\in\mathbb{R}^d$$

A more general problem is to investigate the limit behaviors of the integral

$$\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\left[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\right]^p$$

where $x_{\sigma(0)}=x_0$ is a fixed point in Ω. The idea was first introduced by Mörters and König (2002). The argument we present here simplifies their original proof. The study in this direction might hold key to the solutions of several hard problems in the large deviations. Here we give a lower bound result.

Theorem 3.1. *Let $p\ge 1$ be a constant, let $(\Omega,\mathcal{A},\pi)$ be a meassure space and let K: $\Omega\times\Omega\longrightarrow\mathbb{R}^+$ be a measurable, non-negative function satisfying the following assumptions:*

(1). Symmetry: $K(x,y)=K(y,x)$ for any $x,y\in\Omega$.

(2). Irreducibility: For any $x\in\Omega$,

$$\pi\Big(\{y\in\Omega;\quad G(x,y)=0\}\Big)=0$$

(3). Double integrablity: For any $f\in\mathcal{L}^{\frac{2p}{2p-1}}(\Omega,\mathcal{A},\pi)$,

$$\iint_{\Omega\times\Omega}K(x,y)f(x)f(y)\pi(dx)\pi(dy)<\infty$$

Then

$$\liminf_{m\to\infty}\frac{1}{m}\log\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p \\ \geq p\log\sup_f\iint_{\Omega\times\Omega}K(x,y)f(x)f(y)\pi(dx)\pi(dy) \tag{3.2}$$

where the supremum is taken over all functions f on Ω satisfying

$$\int_\Omega |f(x)|^{\frac{2p}{2p-1}}\pi(dx)=1 \tag{3.3}$$

Proof. Given $\epsilon>0$, write $\Omega_\epsilon=\{y\in\Omega;\quad K(x_0,y)\geq\epsilon\}$, $\mathcal{A}_\epsilon=\mathcal{A}\cap\Omega_\epsilon$ and $\pi_\epsilon(\cdot)=\pi(\cdot\cap\Omega_\epsilon)$. For any function h on Ω_ϵ with $\inf_{x\in\Omega}h(x)>0$ and

$$\int_{\Omega_\epsilon}h^q(x)\pi(dx)=1$$

we have

$$\Big\{\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\Big\}^{1/P}$$
$$\geq\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\Big(\prod_{k=1}^m h(x_k)\Big)\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})$$
$$=\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\Big(\prod_{k=1}^m h(x_k)\Big)\prod_{k=1}^m K(x_{k-1},x_k)$$
$$\geq\epsilon\inf_{x\in\Omega}h(x)\int_{\Omega_\epsilon^m}\pi(dx_1)\cdots\pi(dx_m)\prod_{k=2}^m\sqrt{h(x_{k-1})}K(x_{\sigma(k-1)},x_{\sigma(k)})\sqrt{h(x_k)}$$

Define the linear operator T: $\mathcal{L}^2(\Omega_\epsilon,\mathcal{A}_\epsilon,\pi_\epsilon)\longrightarrow\mathcal{L}^2(\Omega_\epsilon,\mathcal{A}_\epsilon,\pi_\epsilon)$ as

$$Tg(x)=\int_{\Omega_\epsilon}\sqrt{h(x)}K(x,y)\sqrt{h(y)}g(y)\pi(dy)\qquad g\in\mathcal{L}^2(\Omega_\epsilon,\mathcal{A}_\epsilon,\pi_\epsilon)$$

One can see that T is self-adjoint: $\langle g_1,Tg_2\rangle=\langle Tg_1,g_2\rangle$ for any $g_1,g_2\in\mathcal{L}^2(\Omega_\epsilon,\mathcal{A}_\epsilon,\pi_\epsilon)$. Let $g\in\mathcal{L}^2(\Omega_\epsilon,\mathcal{A}_\epsilon,\pi_\epsilon)$ such that $\sup_{x\in\Omega_\epsilon}g(x)<\infty$ and

$$\int_{\Omega_\epsilon}g^2(x)\pi(dx)=1$$

Then we have

$$\Big\{\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\Big\}^{1/P}$$
$$\geq \epsilon\Big(\inf_{x\in\Omega}h(x)\Big)\Big(\sup_{x\in\Omega_\epsilon}g(x)\Big)^{-1}\int_{\Omega_\epsilon^m}\pi(dx_1)\cdots\pi(dx_m)g(x_1)$$
$$\times\Big(\prod_{k=2}^m\sqrt{h(x_{k-1})}K(x_{k-1},x_k)\sqrt{h(x_k)}\Big)g(x_m)$$
$$=\epsilon\Big(\inf_{x\in\Omega}h(x)\Big)\Big(\sup_{x\in\Omega_\epsilon}g(x)\Big)^{-1}\langle g,T^{m-1}g\rangle$$

Consider the spectrul representation

$$\langle g,Tg\rangle=\int_{-\infty}^{\infty}\lambda\mu_g(d\lambda)$$

where μ_g is a probability measure on $\mathbb{R}$. By Jensen inequality,

$$\langle g,T^{m-1}g\rangle=\int_{-\infty}^{\infty}\lambda^{m-1}\mu_g(d\lambda)\geq\Big(\int_{-\infty}^{\infty}\lambda\mu_g(d\lambda)\Big)^{m-1}=\Big(\langle g,T^{m-1}g\rangle\Big)^{m-1}$$

Summarizing what we have,

$$\liminf_{m\to\infty}\frac{1}{m}\log\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p$$
$$\geq p\log\langle g,Tg\rangle=p\log\int_{\Omega_\epsilon\times\Omega_\epsilon}K(x,y)\sqrt{h(x)}g(x)\sqrt{h(y)}g(y)\pi(dx)\pi(dy)$$

Given a function f on Ω_ϵ with

$$0<\inf_{x\in\Omega_\epsilon}f(x)\leq\sup_{x\in\Omega_\epsilon}f(x)<\infty \tag{3.4}$$

and

$$\int_{\Omega_\epsilon}|f(x)|^{\frac{2p}{2p-1}}\pi(dx)=1 \tag{3.5}$$

if we take $h=f^{\frac{2(p-1)}{2p-1}}$ and $g=f^{\frac{p}{2p-1}}$ then $f(x)=\sqrt{h(x)}g(x)$. Consequently,

$$\liminf_{m\to\infty}\frac{1}{m}\log\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p$$
$$\geq p\log\int_{\Omega_\epsilon\times\Omega_\epsilon}K(x,y)f(x)f(y)\pi(dx)\pi(dy)$$

Extend f on Ω by $f(x)=0$ on $\Omega\setminus\Omega_\epsilon$. Then the above establishment can be rewritten as

$$\liminf_{m\to\infty}\frac{1}{m}\log\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\left[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\right]^p$$
$$\geq p\log\int_{\Omega\times\Omega}K(x,y)f(x)f(y)\pi(dx)\pi(dy)$$

Notice that the set consisting of the functions f supported by Ω_ϵ for some $\epsilon>0$ (depending on f) and satisfying (3.4) and (3.5) is dense in the unit sphere of $\mathcal{L}^{\frac{2p}{2p-1}}(\Omega,\mathcal{A},\pi)$. Taking supremum over such f on the right hand side gives (3.2). $\square$

The upper bound is much more difficult. On the other hand, the following result indicate that we are in the right track.

Theorem 3.2. *Let $p\geq 1$ be a constant, let Ω be a finite set and let K: $\Omega\times\Omega\longrightarrow\mathbb{R}^+$ be non-negative function such that $K(x,y)=K(y,x)$ for any $x,y\in\Omega$. Let π be a non-negative function on Ω. Then*

$$\limsup_{m\to\infty}\frac{1}{m}\log\sum_{x_1,\cdots,x_m\in\Omega}\pi(x_1)\cdots\pi(x_m)\left[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\right]^p$$
$$\leq p\log\sup_f\sum_{x,y\in\Omega}K(x,y)f(x)f(y)\pi(x)\pi(y) \tag{3.6}$$

where the supremum is taken over all functions f on Ω satisfying

$$\sum_{x\in\Omega}|f(x)|^{\frac{2p}{2p-1}}\pi(x)=1 \tag{3.7}$$

Proof. We may assume that $\pi(\cdot)$ is a probability density on Ω and $\pi(x)>0$, for otherwise we can replace $\pi(\cdot)$ by $\pi(\cdot)/\pi(\Omega)$ and remove all zero points of π from A. Let

$$\mu=L_m^{\mathbf{x}}=\frac{1}{m}\sum_{k=1}^m\delta_{x_k}$$

be the empirical measure generated by $\mathbf{x}=(x_1,\cdots,x_m)$. Notice that for each $\sigma|in\Sigma_m$

$$\sum_{y_1,\cdots,y_m\in\Omega}1_{\{L_m^{\mathbf{y}}=\mu\}}1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}}=1$$

We have

$$\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})$$
$$=\sum_{y_1,\cdots,y_m\in\Omega}1_{\{L_m^{\mathbf{y}}=\mu\}}\sum_{\sigma\in\Sigma_m}1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})$$
$$=\sum_{y_1,\cdots,y_m\in\Omega}1_{\{L_m^{\mathbf{y}}=\mu\}}\prod_{k=1}^m K(y_{k-1},y_k)\sum_{\sigma\in\Sigma_m}1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}}$$

By a simple combinatorial argument,

$$\sum_{\sigma\in\Sigma_m} 1_{\{\mathbf{x}\circ\sigma=\mathbf{y}\}} = \prod_{x\in\Omega}(m\mu(x))!$$

Consequently,

$$\begin{aligned}&\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\\ &=\prod_{x\in\Omega}(m\mu(x))!\sum_{y_1,\cdots,y_m\in\Omega}1_{\{L_m^{\mathbf{y}}=\mu\}}\prod_{k=1}^m K(y_{k-1},y_k)\end{aligned}$$

Let $q>1$ be the conjugate number of p and define $\phi_\mu(x)=\mu(x)^{1/q}\pi(x)^{1/p}$. Then

$$\begin{aligned}&\sum_{y_1,\cdots,y_m\in\Omega}\phi_\mu(x_1)\cdots\phi_\mu(x_m)\prod_{k=1}^m K(y_{k-1},y_k)\\ &\ge\sum_{y_1,\cdots,y_m\in\Omega}\phi_\mu(x_1)\cdots\phi_\mu(x_m)1_{\{L_m^{\mathbf{y}}=\mu\}}\prod_{k=1}^m K(y_{k-1},y_k)\\ &=\Big(\prod_{x\in\Omega}\phi_\mu(x)^{m\mu(x)}\Big)\sum_{y_1,\cdots,y_m\in\Omega}1_{\{L_m^{\mathbf{y}}=\mu\}}\prod_{k=1}^m K(y_{k-1},y_k)\end{aligned}$$

where the last step follows from the fact that when $L_m^{\mathbf{y}}=\mu$, there are $m\mu(x)$ factors in the product $\phi_\mu(x_1)\cdots\phi_\mu(x_m)$ which are equal to $\phi_\mu(x)$ for any $x\in\Omega$.

Summarizing above steps,

$$\begin{aligned}&\sum_{x_1,\cdots,x_m\in\Omega}\pi(x_1)\cdots\pi(x_m)\bigg[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\bigg]^p\\ &=\sum_{x_1,\cdots,x_m\in\Omega}\pi(x_1)\cdots\pi(x_m)\bigg[\frac{1}{m!}\Big(\prod_{x\in\Omega}(m\mu(x))!\Big)\Big(\prod_{x\in\Omega}\phi_\mu(x)^{m\mu(x)}\Big)^{-1}\bigg]^p\\ &\times\bigg[\sum_{y_1,\cdots,y_m\in\Omega}\phi_\mu(x_1)\cdots\phi_\mu(x_m)\prod_{k=1}^m K(y_{k-1},y_k)\bigg]^p\end{aligned}\tag{3.8}$$

Notice that $\phi_\mu(x)\le 1$ for any $x\in\Omega$.

$$\begin{aligned}&\sum_{y_1,\cdots,y_m\in\Omega}\phi_\mu(x_1)\cdots\phi_\mu(x_m)\prod_{k=1}^m K(y_{k-1},y_k)\\ &\le\sup_{x,y\in\Omega}K(x,y)\sum_{y_1,\cdots,y_m\in\Omega}\prod_{k=2}^m\sqrt{\phi_\mu(x_{k-1})}K(y_{k-1},y_k)\sqrt{\phi_\mu(x_k)}\\ &=\#(\Omega)\sup_{x,y\in\Omega}K(x,y)\sum_{y_1,\cdots,y_m\in\Omega}g_0(x_1)\Big(\prod_{k=2}^m\sqrt{\phi_\mu(x_{k-1})}K(y_{k-1},y_k)\sqrt{\phi_\mu(x_k)}\Big)g_0(x_m)\end{aligned}\tag{3.9}$$

where $g_0(x) \equiv \#(\Omega)^{-1/2}$ $(x \in \Omega)$. Consider the Hilbert space $\mathcal{L}^2(\Omega)$ with norm defined as

$$|g|_2 = \Big(\sum_{x\in\Omega} g^2(x)\Big)^{1/2} \qquad g \in \mathcal{L}^2(\Omega)$$

If we define the linear operator T on $\mathcal{L}^2(\Omega)$ as

$$Tg(x) = \sqrt{\phi_\mu(x)} \sum_{y\in\Omega} K(x,y)\sqrt{\phi_\mu(y)}g(y) \qquad x \in \Omega$$

then T is self-adjoint: $\langle f, Tg\rangle = \langle Tf, g\rangle$ for any $f, g \in \mathcal{L}^2(\Omega, \pi)$. Further, the right hand side of (3.9) is equal to

$$\sup_{x,y\in\Omega} K(x,y)\langle g_0, T^{m-1}g_0\rangle$$

Notice that g_0 is a unit vector in $\mathcal{L}^2(\Omega)$. Thus,

$$\sum_{y_1,\cdots,y_m\in\Omega} \phi_\mu(x_1)\cdots\phi_\mu(x_m)\prod_{k=1}^m K(y_{k-1},y_k) \le \sup_{x,y\in\Omega} K(x,y)||T^{m-1}||$$
$$\le \sup_{x,y\in\Omega} K(x,y)||T||^{m-1} = \sup_{x,y\in\Omega} K(x,y) \sup_{|g|_2=1}\langle g, Tg\rangle^{m-1}$$
$$= \sup_{x,y\in\Omega} K(x,y)\Big(\sup_{|g|_2=1}\sum_{x,y\in\Omega} K(x,y)\sqrt{\phi_\mu(x)}g(x)\sqrt{\phi_\mu(y)}g(y)\Big)^{m-1}$$

Write

$$\sum_{x,y\in\Omega} K(x,y)\sqrt{\phi_\mu(x)}g(x)\sqrt{\phi_\mu(y)}g(y)$$
$$= \sum_{x,y\in\Omega} K(x,y)\Big(\sqrt{\phi_\mu(x)}g(x)\pi(x)^{-1}\Big)\Big(\sqrt{\phi_\mu(y)}g(y)\pi(y)^{-1}\Big)\pi(x)\pi(y)$$

and notice that

$$\sum_{x\in\Omega}\Big(\sqrt{\phi_\mu(x)}g(x)\pi(x)^{-1}\Big)^{\frac{2p}{2p-1}}\pi(x) = \sum_{x\in\Omega}\phi_\mu(x)^{\frac{p}{2p-1}}\pi(x)^{-\frac{1}{2p-1}}g(x)^{\frac{2p}{2p-1}}$$
$$\le \Big(\sum_{x\in\Omega}\phi_\mu(x)^q\pi(x)^{-\frac{1}{p-1}}\Big)^{\frac{p-1}{2p-1}}\Big(\sum_{x\in\Omega} g^2(x)\Big)^{\frac{p}{2p-1}} = \Big(\sum_{x\in\Omega}\mu(x)\Big)^{\frac{p-1}{2p-1}} = 1.$$

Thus,

$$\sum_{y_1,\cdots,y_m\in\Omega} \phi_\mu(x_1)\cdots\phi_\mu(x_m)\prod_{k=1}^m K(y_{k-1},y_k)$$
$$\le \sup_{x,y\in\Omega} K(x,y)\Big(\sup_f \sum_{x,y\in\Omega} K(x,y)f(x)f(y)\pi(x)\pi(y)\Big)^{m-1}$$

where the supremum is taken over all functions f on Ω satisfying

$$\sum_{x\in\Omega}|f(x)|^{\frac{2p}{2p-1}}\pi(x)=1$$

In view of (3.8), therefore, it remains to prove

$$\begin{aligned}&\lim_{m\to\infty}\frac{1}{m}\log\sum_{x_1,\cdots,x_m\in\Omega}\pi(x_1)\cdots\pi(x_m)\\&\times\left[\frac{1}{m!}\Big(\prod_{x\in\Omega}\big(m\mu(x)\big)!\Big)\Big(\prod_{x\in\Omega}\phi_\mu(x)^{m\mu(x)}\Big)^{-1}\right]^p=0\end{aligned}\tag{3.10}$$

Indeed, by Stirling formula, $m!\sim\sqrt{2\pi m}m^me^{-m}$ and

$$\big(m\mu(x)\big)!\le C\sqrt{m\mu(x)}\big(mu(x)\big)^{mu(x)}e^{mu(x)}\qquad x\in\Omega$$

Hence,

$$\frac{1}{m!}\Big(\prod_{x\in\Omega}\big(m\mu(x)\big)!\Big)\le Cm^{\#(\Omega)/2}\prod_{x\in\omega}\mu(x)^{m\mu(x)}$$

Recall that $\phi_\mu(x)=\mu(x)^{1/q}\pi(x)^{1/p}$ and

$$\mu(x)=\frac{1}{m}\sum_{k=1}^m\delta_{x_k}$$

Therefore,

$$\begin{aligned}&\sum_{x_1,\cdots,x_m\in\Omega}\pi(x_1)\cdots\pi(x_m)\left[\frac{1}{m!}\Big(\prod_{x\in\Omega}\big(m\mu(x)\big)!\Big)\Big(\prod_{x\in\Omega}\phi_\mu(x)^{m\mu(x)}\Big)^{-1}\right]^p\\&\le C^pm^{p\#(\Omega)/2}\sum_{x_1,\cdots,x_m\in\Omega}\pi(x_1)\cdots\pi(x_m)\prod_{x\in\Omega}\Big(\frac{\mu(x)}{\pi(x)}\Big)^{m\mu(x)}\\&=C^pm^{p\#(\Omega)/2}\sum_{x_1,\cdots,x_m\in\Omega}\mu(x_1)\cdots\mu(x_m)=C^pm^{p\#(\Omega)/2}\end{aligned}$$

which leads to (3.10). □

Our argument for the upper bound (3.6) substantially depends on the assumption that Ω is a finite set. The challenge we currently face is how to extend it to a more general setting. More precisely, compactification and discretization are two main issues as we try to apply the high moment asymptotics to some practically interesting models. By far our cabability are limited. In the next theorem, we extend Theorem 3.1 slightly by a trivil procedure of discritization.

Theorem 3.3. *Let $p \geq 1$ be a constant, let Ω be a compact set and let $\pi(\cdot)$ be a finite Borel measure on* **Ω** *Let $K\colon \Omega \times \Omega \longrightarrow \mathbb{R}^+$ be non-negative continuous function such that $K(x,y) = K(y,x)$ for any $x, y \in \Omega$. Then*

$$\limsup_{m\to\infty} \frac{1}{m} \log \int_{\Omega^m} \pi(dx_1)\cdots\pi(dx_m)\left[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)})\right]^p \\ \leq p \log \sup_f \int_{\Omega\times\Omega} K(x,y) f(x) f(y) \pi(dx)\pi(dy) \tag{3.11}$$

where the supremum is taken over all functions f on Ω satisfying

$$\int_\Omega |f(x)|^{\frac{2p}{2p-1}} \pi(dx) = 1 \tag{3.12}$$

Sketch of the proof for (3.1). We only need to prove the upper bound. By potential theory, as $p(d-2) < d$,

$$\int_{\mathbb{R}^d} G^p(x)dx < \infty$$

An easy estimate by Jessen's inequality gives

$$\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\left[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})\right]^p \\ \leq \frac{1}{m!}\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G^p(x_{\sigma(k)} - x_{\sigma(k-1)}) \\ = \left(\int_{\mathbb{R}^d} G^p(x)dx\right)^m$$

which leads to

$$\limsup_{m\to\infty}\frac{1}{m}\log\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\left[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})\right]^p \leq \int_{\mathbb{R}^d} G^p(x)dx$$

Unfortunately, this simple argument does not give us the right constant.

It is a classic facts that as $d = 1$, $G(x)$ is continuous on $\mathbb{R}^d$; and that as $d \geq 2$, $G(x)$ is ontinuous on $\mathbb{R}^d \setminus \{0\}$ but $\lim_{x\to 0} G(x) = \infty$.

Notice that for any $A \subset \mathbb{R}^d$, the following shift-invariance

$$\sup_f \int_{A\times A} f(x)f(y)\pi(x)dxdy = \sup_f \int_{(z+A)\times(z+A)} f(x)f(y)\pi(x)dxdy$$

suggests that an ordinary compactification by truncation does not work here. We carry out the compactification procedure as following: Let $M > 0$ be a large constant. Then

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \left[\frac{1}{m!} \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})\right]^p \\
&= \sum_{y_1, \cdots, y_m \in \mathbb{Z}^d} \int_{([-M,M]^d)^m} dx_1 \cdots dx_m \left[\frac{1}{m!} \sum_{\sigma \in \Sigma_m}\right. \\
&\times \left. \prod_{k=1}^m G\big((2My_{\sigma(k)} + x_{\sigma(k)}) - (2My_{\sigma(k-1)} + x_{\sigma(k-1)})\big)\right]^p \\
&\le \int_{([-M,M]^d)^m} dx_1 \cdots dx_m \left[\frac{1}{m!} \sum_{\sigma \in \Sigma_m} \sum_{y_1, \cdots, y_m \in \mathbb{Z}^d}\right. \\
&\times \left. \prod_{k=1}^m G\big(2M(y_{\sigma(k)} - y_{\sigma(k-1)}) + (x_{\sigma(k)}) - x_{\sigma(k-1)})\big)\right]^p \\
&= \int_{([-M,M]^d)^m} dx_1 \cdots dx_m \left[\frac{1}{m!} \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m \widetilde{G}(x_{\sigma(k)} - x_{\sigma(k-1)})\right]^p
\end{aligned} \tag{3.13}$$

where (**Why does the infinite series appearing on the right hand side of below converge?**)

$$\widetilde{G}(x) = \sum_{y \in \mathbb{Z}^d} G(2My + x)$$

To fix the problem that $\widetilde{G}$ discontinuous at 0, we try to replace $\widetilde{G}$ by the continuous function $\widetilde{G}_N(x) = \min\{\widetilde{G}(x), N\}$

Lemma 3.1. *(Lemma 3.3, König and Mörters (2002)). There is a constant $C > 0$ such that for all sufficiently large N and small $\eta > 0$, there are $\epsilon_\eta > 0$ and $\delta_N > 0$ such that*

$$\begin{aligned}
&\int_{([-M,M]^d)^m} dx_1 \cdots dx_m \left[\frac{1}{m!} \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m \widetilde{G}(x_{\sigma(k)} - x_{\sigma(k-1)})\right]^p \\
&\le p2^p \left\{ m(2C)^m \delta_N^{\eta m} + (1+\epsilon)^m \sum_{l=[m(1-p\eta)]}^m \right. \\
&\times \left. \int_{([-M,M]^d)^l} dx_1 \cdots dx_m \left[\frac{1}{l!} \sum_{\sigma \in \Sigma_l} \prod_{k=1}^l \widetilde{G}_N(x_{\sigma(k)} - x_{\sigma(k-1)})\right]^p \right\}
\end{aligned}$$

where $\lim_{N\to\infty} \delta_N = \lim_{\eta \to 0} \epsilon_\eta = 0$.

The proof given by König and Mörters (2002) is technical. So we omit it here. By

Theorem 3.3 we have

$$\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{m}\log\int_{([-M,M]^d)^m}dx_1\cdots dx_m\bigg[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m\widetilde{G}_N(x_{\sigma(k)}-x_{\sigma(k-1)})\bigg]^p\\
&\le p\log\sup_f\int_{[-M,M]^d\times[-M,M]^d}\widetilde{G}_N(x-y)f(x)f(y)dxdy\\
&\le p\log\sup_f\int_{[-M,M]^d\times[-M,M]^d}\widetilde{G}(x-y)f(x)f(y)dxdy
\end{aligned}$$

By (3.13) and Lemma 3.1,

$$\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{m}\log\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\bigg[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\bigg]^p\\
&\le p\log\sup_f\int_{[-M,M]^d\times[-M,M]^d}\widetilde{G}(x-y)f(x)f(y)dxdy
\end{aligned}$$

Finally, the desired upper bound follows from the analytic fact (proof omitted here) that

$$\limsup_{M\to\infty}\sup_f\int_{[-M,M]^d\times[-M,M]^d}\widetilde{G}(x-y)f(x)f(y)dxdy\le\rho$$

□

4. Large deviations for local times of additive Brownian motions.

In this section, we provide another successful story in high moment asymptotics. We come back to independent d-dimensional Brownian motions $W_1(t),\cdots,W_p(t)$. The following multi-parameter process

$$W_1(t_1)+\cdots+W_p(t_p)\qquad t_1,\cdots,t_p\ge 0$$

is called an additive Brownian motion. The local time of this process is formally given as

$$\eta^x(I)=\int_I\delta_x\big(W_1(s_1)+\cdots+W_p(s_p)\big)ds_1\cdots ds_p\qquad x\in\mathbb{R}^d,\quad I\subset(\mathbb{R}^+)^p.$$

We rely on two recent papers by Khoshnevisan, Xiao and Zhong (2003a, b) for the constructions of the local time $\eta^x(I)$. In their papers, Khoshnevisan, Xiao and Zhong (2003a, b) consider a more general multi-parameter random field named additive Lévy process, which is generated by independent Lévy processes. In their construction, $\eta^x(I)$ is defined as the density function of the occupation measure μ_I:

$$\mu_I(A)=\int_I\delta_{W_1(s_1)+\cdots+W_p(s_p)}(A)ds_1\cdots ds_p\qquad A\subset\mathbb{R}^d$$

in the case when μ_I is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. Applying Theorem 1.1 in Khoshnevisan, Xiao and Zhong (2003a) to our setting, the local time $\eta^x(I)$ exists for every super interval $I \subset (\mathbb{R}^+)^p$ if and only if

$$d < 2p.$$

Further, (1.2) also implies that almost surely, the local time

$$\eta^x\big([0,t]^p\big) \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+$$

is jointly continuous in (x,t) (Corollary 3.3, Khoshnevisan, Xiao and Zhong (2003b)).

Write $\eta(I) = \eta^0(I)$. We have

$$\eta\big([0,t]^p\big) \overset{d}{=} t^{\frac{2p-d}{2}} \eta\big([0,1]^p\big) \tag{4.1}$$

We are interested in the large deviation for $\eta\big([0,1]^p\big)$. Notice that as $p=2$,

$$\eta\big([0,t]^2\big) \overset{d}{=} \alpha\big([0,t]^2\big)$$

Theorem 4.1. *Under $d < 2p$,*

$$\lim_{t\to\infty} t^{-2/d} \log \mathbb{P}\Big\{\eta\big([0,1]^p\big) \geq t\Big\} = -(2\pi)^2 \frac{d}{2}\Big(1 - \frac{d}{2p}\Big)^{\frac{2p-d}{d}} \rho^{-2/d} \tag{4.2}$$

where ρ is given as

$$\rho = \sup_{||f||_2=1} \int_{\mathbb{R}^d} \bigg[\int_{\mathbb{R}^d} \frac{f(\lambda+\gamma)f(\gamma)}{\sqrt{1+2^{-1}|\lambda+\gamma|^2}\sqrt{1+2^{-1}|\gamma|^2}} d\gamma\bigg]^p d\lambda \tag{4.3}$$

Remark. It can be proved that

$$\rho \leq \int_{\mathbb{R}^d} \frac{1}{(1+2^{-1}|\gamma|^2)^p} d\lambda < \infty$$

We now discuss the proof of Theorem 4.1. By Theorem 1.5 we need only to prove

$$\lim_{n\to\infty} \frac{1}{n} \log (n!)^{-d/2} \mathbb{E}\Big[\eta([0,1]^p)^n\Big] = \log\Big(\frac{2p}{2p-d}\Big)^{\frac{2p-d}{2}} + \log \frac{\rho}{(2\pi)^d}. \tag{4.4}$$

By Fourier transform, for any $t_1, \cdots, t_p \geq 0$,

$$\begin{aligned}
&\eta([0,t_1] \times \cdots \times [0,t_p]) \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \bigg[\int_{\mathbb{R}^d} \eta^x([0,t_1]\times\cdots\times[0,t_p]) e^{i\lambda\cdot x} dx\bigg] d\lambda \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\lambda \int_0^{t_1} \cdots \int_0^{t_p} \exp\Big\{i\lambda\cdot\big(W_1(s_1) + \cdots + W_p(s_p)\big)\Big\} ds_1 \cdots ds_p \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\lambda \prod_{j=1}^p \int_0^{t_j} \exp\Big\{i\lambda\cdot W_j(s)\Big\} ds
\end{aligned}$$

where the second step follows from the definition of the local times as the density of occupation measures. Hence, for any integer $n \geq 1$,

$$\begin{aligned}
&\mathbb{E}\Big[\eta([0,t_1]\times\cdots\times[0,t_p])^n\Big]\\
&=\frac{1}{(2\pi)^{dn}}\int_{(\mathbb{R}^d)^n} d\lambda_1\cdots d\lambda_n\prod_{j=1}^p\int_{[0,t_j]^n}\mathbb{E}\exp\Big\{i\sum_{k=1}^n\lambda_k\cdot W(s_k)\Big\}ds_1\cdots ds_n.
\end{aligned}$$

By time rearrangement,

$$\begin{aligned}
&\int_{[0,t_j]^n}\mathbb{E}\exp\Big\{i\sum_{k=1}^n\lambda_k\cdot W(s_k)\Big\}ds_1\cdots ds_n\\
&=\sum_{\sigma\in\Sigma_n}\int_{\{0\leq s_1\leq\cdots s_n\leq t_j\}}\mathbb{E}\exp\Big\{i\sum_{k=1}^n\lambda_{\sigma(k)}\cdot W(s_k)\Big\}ds_1\cdots ds_n\\
&=\sum_{\sigma\in\Sigma_n}\int_{\{0\leq s_1\leq\cdots\leq s_n\leq t_j\}}\mathbb{E}\exp\Big\{i\sum_{k=1}^n\Big(\sum_{j=k}^n\lambda_{\sigma(j)}\Big)\cdot\big(W(s_k)-W(s_{k-1})\big)\Big\}ds_1\cdots ds_n\\
&=\sum_{\sigma\in\Sigma_n}\int_{\{0\leq s_1\leq\cdots\leq s_n\leq t_j\}}\prod_{k=1}^n\exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big|\sum_{j=k}^n\lambda_{\sigma(j)}\Big|^2\Big\}ds_1\cdots ds_n.\\
&=\sum_{\sigma\in\Sigma_n}\int_{\{0\leq s_1\leq\cdots\leq s_n\leq t_j\}}\prod_{k=1}^n\exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big|\sum_{j=1}^k\lambda_{\sigma(j)}\Big|^2\Big\}ds_1\cdots ds_n.
\end{aligned}$$

where the last step follows from the bijection $j\mapsto n-j$ and permutation invariance and we adopt the convention that $s_0=0$. where we adopt the convention that $s_0=0$. Thus

$$\begin{aligned}
&\mathbb{E}\Big[\eta^0([0,t_1]\times\cdots\times[0,t_p])^n\Big]\\
&=\frac{1}{(2\pi)^{dn}}\int_{(\mathbb{R}^d)^n} d\lambda_1\cdots d\lambda_n\prod_{j=1}^p\sum_{\sigma\in\Sigma_n}\\
&\times\int_{\{0\leq s_1\leq\cdots\leq s_n\leq t_j\}}\prod_{k=1}^n\exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big|\sum_{j=1}^k\lambda_{\sigma(j)}\Big|^2\Big\}ds_1\cdots ds_n.
\end{aligned}\tag{4.5}$$

To simplify the above representation, we replace $t_1 \cdots, t_p$ by $\tau_1, \cdots, \tau_p$.

$$
\begin{aligned}
&\mathbb{E}\Big[\eta([0,\tau_1]\times\cdots\times[0,\tau_p])^n\Big]\\
&=\frac{1}{(2\pi)^{dn}}\int_{(\mathbb{R}^d)^n} d\lambda_1\cdots d\lambda_n\bigg[\sum_{\sigma\in\Sigma_n}\int_0^\infty e^{-t}dt\\
&\times\int_{\{0\le s_1\le\cdots s_n\le t\}}\prod_{k=1}^n \exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big|\sum_{j=1}^k\lambda_{\sigma(j)}\Big|^2 ds_1\cdots ds_n\bigg]^p \qquad (4.6)\\
&=\frac{1}{(2\pi)^{dn}}\int_{(\mathbb{R}^d)^n} d\lambda_1\cdots d\lambda_n\bigg[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n\int_0^\infty e^{-t}\exp\Big\{-\frac{t}{2}\Big|\sum_{j=1}^k\lambda_{\sigma(j)}\Big|^2\Big\}dt\bigg]^p\\
&=\frac{1}{(2\pi)^{dn}}\int_{(\mathbb{R}^d)^n} d\lambda_1\cdots d\lambda_n\bigg[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\bigg]^p
\end{aligned}
$$

where $Q(\lambda) = (1 + 2^{-1}|\lambda|^2)^{-1}$.

Lemma 4.1. *Under $d < 2p$,*

$$
\lim_{n\to\infty}\frac{1}{n}\log\int_{(\mathbb{R}^d)^n} d\lambda_1\cdots d\lambda_n\bigg[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\bigg]^p=\log\rho \qquad (4.7)
$$

or, equivalently

$$
\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^p}\mathbb{E}\Big[\eta^0([0,\tau_1]\times\cdots\times[0,\tau_p])^n\Big]=\log\frac{\rho}{(2\pi)^d} \qquad (4.8)
$$

We now prove Theorem 4.1 based on Theorem 4.1. Let $t_1, \cdots, t_p \ge 0$. In view of (4.5), by Hölder inequality,

$$
\begin{aligned}
&\mathbb{E}\Big[\eta([0,t_1]\times\cdots\times[0,t_p])^n\Big]\\
&\le\prod_{j=1}^p\Big\{\mathbb{E}\Big[\eta\big([0,t_j]^p\big)^n\Big]\Big\}^{1/p}=(t_1\cdots t_p)^{\frac{2p-d}{2p}n}\mathbb{E}\Big[\eta^0\big([0,1]^p\big)^n\Big]
\end{aligned}
$$

where the last step follows from (4.1). Thus,

$$
\begin{aligned}
&\mathbb{E}\Big[\eta([0,\tau_1]\times\cdots\times[0,\tau_p])^n\Big]\\
&=\int_0^\infty\cdots\int_0^\infty e^{-(t_1+\cdots+t_p)}\mathbb{E}\Big[\eta([0,t_1]\times\cdots\times[0,t_p])^n\Big]dt_1\cdots dt_p\\
&\le\mathbb{E}\Big[\eta([0,1]^p)^n\Big]\int_0^\infty\cdots\int_0^\infty(t_1\cdots t_p)^{\frac{2p-d}{2p}n}e^{-(t_1+\cdots+t_p)}dt_1\cdots dt_p\\
&=\mathbb{E}\Big[\eta([0,1]^p)^n\Big]\Big[\Gamma\Big(\frac{2p-d}{2p}n+1\Big)\Big]^p.
\end{aligned}
$$

By Lemma 4.1 and Stirling formula,

$$\liminf_{n\to\infty}\frac{1}{n}\log(n!)^{-d/2}\mathbb{E}\Big[\eta([0,1]^p)^n\Big]\ge\log\Big(\frac{2p}{\alpha p-d}\Big)^{\frac{2p-d}{2}}+\log\frac{\rho}{(2\pi)^d}. \tag{4.9}$$

On the other hand, notice that $\bar\tau\equiv\min\{\tau_1,\cdots,\tau_p\}$ has the exponential distribution with the parameter p. Hence,

$$\begin{aligned}&\mathbb{E}\Big[\eta\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)\Big]^n\ge\mathbb{E}\Big[\eta\big([0,\bar\tau]^p\big)^n\Big]=\mathbb{E}\,\bar\tau^{\frac{2p-d}{2}n}\mathbb{E}\Big[\eta\big([0,1]^p\big)^n\Big]\\&=p^{-\frac{2p-d}{2}n-1}\Gamma\Big(1+\frac{2p-d}{2}n\Big)\mathbb{E}\Big[\eta\big([0,1]^p\big)^n\Big]\end{aligned}$$

where the second step follows from (4.1). By Stirling formula and Lemma 4.1 we have

$$\limsup_{n\to\infty}\frac{1}{n}\log(n!)^{-d/2}\mathbb{E}\Big[\eta([0,1]^p)^n\Big]\le\log\Big(\frac{2p}{2p-d}\Big)^{\frac{2p-d}{2}}+\log\frac{\rho}{(2\pi)^d}. \tag{4.10}$$

Combining (4.9) and (4.10) gives (4.4). □

Sketch of the proof of Lemma 4.1. As before, the proof of the lower bound is the easy side. Let $q>1$ be the conjugate number of p defined by $p^{-1}+q^{-1}=1$ and let f be a positive continuous function on $\mathbb{R}^d$ with $f(-\lambda)=f(\lambda)$ and $||f||_q=1$. We have

$$\begin{aligned}&\bigg(\int_{(\mathbb{R}^d)^n}d\lambda_1\cdots d\lambda_n\Big[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^nQ\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\Big]^p\bigg)^{1/p}\\&\ge\int_{(\mathbb{R}^d)^n}d\lambda_1\cdots d\lambda_n\Big(\prod_{k=1}^nf(\lambda_k)\Big)\sum_{\sigma\in\Sigma_n}\prod_{k=1}^nQ\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\\&=n!\int_{(\mathbb{R}^d)^n}d\lambda_1\cdots d\lambda_n\Big(\prod_{k=1}^nf(\lambda_k)\Big)\prod_{k=1}^nQ\Big(\sum_{j=1}^k\lambda_j\Big)\\&=n!\int_{(\mathbb{R}^d)^n}d\lambda_1\cdots d\lambda_n\prod_{k=1}^nf(\lambda_k-\lambda_{k-1})Q(\lambda_k)\end{aligned}$$

where we follow the convention that $\lambda_0=0$.

Define the linear operator T on $\mathcal{L}^2(\mathbb{R}^d)$ as

$$Tg(\lambda)=\sqrt{Q(\lambda)}\int_{\mathbb{R}^d}f(\gamma-\lambda)\sqrt{Q(\gamma)}g(\gamma)d\gamma\qquad g\in\mathcal{L}^2(\mathbb{R}^d).$$

One can prove that there is a constant $C>0$ such that

$$\langle h,Tg\rangle\le C||g||_2||h||_2\qquad g,h\in\mathcal{L}^2(\mathbb{R}^d).$$

So T is a continuous operator.

In addition, one can see that $\langle h, Tg\rangle = \langle g, Th\rangle$ for any $g, h \in \mathcal{L}^2(\mathbb{R}^d)$. It means that T is self adjoint. We now let g be a bounded and locally supported function on $\mathbb{R}^d$ with $||g||_2 = 1$. Then there is $\delta > 0$ such that $f \geq \delta$ and $Q \geq \delta$ on the support of g. In addition, notice that $Q \leq 1$. Thus,

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^n} d\lambda_1 \cdots d\lambda_n \prod_{k=1}^n f(\lambda_k - \lambda_{k-1}) Q(\lambda_k) \\
&\geq \delta^2 ||g|_\infty^{-2} \int_{(\mathbb{R}^d)^n} d\lambda_1 \cdots d\lambda_n g(\lambda_1) \Big(\prod_{k=2}^n \sqrt{Q(\lambda_{k-1})} f(\lambda_k - \lambda_{k-1}) \sqrt{Q(\lambda_k)} \Big) g(\lambda_n) \\
&= \delta^2 ||g||_\infty^{-2} \langle g, T^{n-1} g\rangle.
\end{aligned}$$

Consider the spectral representation of the self-adjoint operator T:

$$\langle g, Tg\rangle = \int_{-\infty}^{\infty} \theta \mu_g(d\theta)$$

where $\mu_g(d\theta)$ is a probability measure on $\mathbb{R}$. By the mapping theorem,

$$\langle g, T^{n-1} g\rangle = \int_{-\infty}^{\infty} \theta^{n-1} \mu_g(d\theta) \geq \Big(\int_{-\infty}^{\infty} \theta \mu_g(d\theta) \Big)^{n-1} = \langle g, Tg\rangle^{n-1}$$

where the second step follows from Jensen's inequality.

Hence,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \Big(\int_{(\mathbb{R}^d)^n} d\lambda_1 \cdots d\lambda_n \Big[\sum_{\sigma\in\Sigma_n} \prod_{k=1}^n Q\Big(\sum_{j=1}^k \lambda_{\sigma(j)} \Big) \Big]^p \Big)^{1/p} \\
&\geq \log\langle g, Tg\rangle = \log \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(\gamma - \lambda) \sqrt{Q(\lambda)} \sqrt{Q(\gamma)} g(\lambda) g(\gamma) d\lambda d\gamma \\
&= \log \int_{\mathbb{R}^d} f(\lambda) \Big[\int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)} \sqrt{Q(\gamma)} g(\lambda+\gamma) g(\gamma) d\gamma \Big] d\lambda.
\end{aligned}$$

Notice that the set of all bounded, locally supported g is dense in $\mathcal{L}^2(\mathbb{R}^d)$. Taking supremum over g on the right hand sides gives

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{n!} \Big(\int_{(\mathbb{R}^d)^n} d\lambda_1 \cdots d\lambda_n \Big[\sum_{\sigma\in\Sigma_n} \prod_{k=1}^n Q\Big(\sum_{j=1}^k \lambda_{\sigma(j)} \Big) \Big]^p \Big)^{1/p} \\
&\geq \log \sup_{|g|_2=1} \int_{\mathbb{R}^d} f(\lambda) \Big[\int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)} \sqrt{Q(\gamma)} g(\lambda+\gamma) g(\gamma) d\gamma \Big] d\lambda.
\end{aligned} \tag{4.11}$$

Since for any g, the function

$$H(\lambda) = \int_{\mathbb{R}^d} \sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)}g(\lambda+\gamma)g(\gamma)d\gamma$$

is even: $H(-\lambda) = H(\lambda)$. Hence, taking supremum over all positive, continuous and even functions f with $||f||_q = 1$ on the right hand side of (4.11) gives

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{n!}\bigg(\int_{(\mathbb{R}^d)^n} d\lambda_1\cdots d\lambda_n \Big[\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k \lambda_{\sigma(j)}\Big)\Big]^p\bigg)^{1/p}\\
&\geq \frac{1}{p}\log \sup_{|g|_2=1}\int_{\mathbb{R}^d}\bigg[\int_{\mathbb{R}^d}\sqrt{Q(\lambda+\gamma)}\sqrt{Q(\gamma)}g(\lambda+\gamma)g(\gamma)d\gamma\bigg]^p d\lambda\\
&= \frac{1}{p}\log\rho.
\end{aligned}$$

The upper bound is an indirect consequence of the next lemma (proof omitted). □

Lemma 4.2. *Let $\pi(x)$ and $Q(x)$ be two non-negative functions on $\mathbb{Z}^d$ such that π is locally supported, $\pi(-x) = \pi(x)$ for all $x \in \mathbb{Z}^d$, and that*

$$\lim_{|x|\to\infty} Q(x) = 0. \tag{4.12}$$

Then

$$\lim_{n\to\infty}\frac{1}{n}\log \sum_{x_1,\cdots,x_n\in\mathbb{Z}^d}\Big(\prod_{k=1}^n \pi(x_k)\Big)\Big[\frac{1}{n!}\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n Q\Big(\sum_{j=1}^k x_{\sigma(j)}\Big)\Big]^p = \log\tilde{\rho} \tag{4.13}$$

where

$$\tilde{\rho} = \sup_{|f|_2=1}\sum_{x\in\mathbb{Z}^d}\pi(x)\Big[\sum_{y\in\mathbb{Z}^d}\sqrt{Q(x+y)}\sqrt{Q(y)}f(x+y)f(y)\Big]^p$$

and

$$|f|_2 = \Big(\sum_{x\in\mathbb{Z}^d} f^2(x)\Big)^{1/2}.$$

5. Remaining problems.

One of interesting problem is to study the large deviation for intersection local time of Markov processes. Under reasonable condition, the intersection local time (assume existence) run by p i.i.d. Markov processes and stopped by the i.i.d. exponential times $\tau_1, \cdots, \tau_p$, can be written in the form as

$$\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)})\Big]^p$$

The main difficulty is on the upper bound.

Another open problem is the intersection in the supercrtical case defined by

$$p(d-2) > d \tag{5.1}$$

Let $\{S_1(n)\}, \cdots, \{S_p(n)\}$ be i.i.d. symmetric random walk on $\mathbb{Z}^d$. It is a well known fact that under (5.1),

$$I_\infty = \sum_{k_1,\cdots,k_p=1}^{\infty} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}} < \infty \qquad a.s.$$

$$J_\infty = \#\Big\{S_1[1,\infty)\cap\cdots\cap S_p[1,\infty)\Big\} < \infty \qquad a.s.$$

In their very influetial peper, Khanin, Mazel, Shlosman and Sinai (1994) claim that in the special case $d \geq 5$ and $p = 2$,

$$\exp\{-c_1 t^{1/2}\} \leq \mathbb{P}\{I_\infty \geq t\} \leq \exp\{-c_2 t^{1/2}\}$$

$$\exp\Big\{-t^{\frac{d-2}{d}+\delta}\Big\} \leq \mathbb{P}\{J_\infty \geq t\} \leq \exp\Big\{-t^{\frac{d-2}{d}-\delta}\Big\}$$