The Reversible Nearest Particle System on a Finite Set *

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Abstract: In this paper we study the one-parameter family of attractive reversible nearest particle systems on $\{1, 2, \cdots, N\}$. Denote by σ_N the time that the system first hits the empty set. Then, σ_N has a logarithmic increasing rate as the parameter λ is small enough, but an exponential increasing rate as λ is large enough. Especially, it has a polynomial increasing rate in the critical case, i.e. $\lambda = 1$.

Keywords: nearest particle system, first hitting time.

1 Introduction

A nearest particle system (or NPS for short) on $S \subset \mathbb{Z}$ is a spin system taking values in subsets of S, and with flip rates for any finite $A \subset S$

$$\begin{aligned} &q(A, A \setminus \{x\}) = 1 && \text{if } x \in A, \\ &q(A, A \cup \{x\}) = \beta(l_x(A), r_x(A)) && \text{if } x \in S \setminus A, \\ &q(A, B) = 0 && \text{otherwise,} \end{aligned}$$

where $l_x(A)$ and $r_x(A)$ are the distances from x to the nearest points in A to the left and right respectively. See Chapter 7 of [4] for more details.

Suppose the system is *reversible*, equivalently, by Theorem VI.1.2 of [4]

$$\beta(l, r) = \frac{\beta(l)\beta(r)}{\beta(l + r)}, \quad \beta(l, \infty) = \beta(\infty, l) = \beta(l),$$

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where $\sum_{l=1}^{\infty}\beta(l)<\infty$. In this paper we consider the one-parameter family $\beta_\lambda(l)=\lambda\psi(l)$, where $\psi(\cdot)$ is strictly positive and satisfies $\sum_{n=1}^{\infty}\psi(n)=1$. Suppose that $\psi(n)/\psi(n+1)$ decreases to 1 as $n\to\infty$, which ensures that the process is Feller and attractive.

Denote by $\{\xi_t^N: t\geq 0\}$ the NPS on $\{1,2,\cdots,N\}$ starting from all sites occupied, and by σ_N the first time it hits the empty set. We estimate σ_N, and the results read as follows.

Theorem 1.1 *Suppose*

$$M\triangleq\sup_n\sum_{l+r=n}\frac{\psi(l)\psi(r)}{\psi(n)}<\infty. \tag{1}$$

Then, for any C_N such that $\lim_{N\to\infty}C_N=\infty$,

$$\lim_{N\to\infty}P_\lambda\big(\sigma_N\leq C_Nf_\lambda(N)\big)=1,$$

where

$$f_\lambda(N)=\begin{cases}\log N, & \text{if } \lambda M<1,\\ N\log N, & \text{if } \lambda M=1,\\ (\lambda M)^N, & \text{if } \lambda M>1.\end{cases}$$

Theorem 1.2 *Suppose there exists n_0 such that*

$$\frac{\lambda}{\lambda_c'}>\max\left\{\frac{2\psi(3n_0)}{\sum_{l=n_0}^{2n_0}\psi(l)\psi(3n_0-l)},\quad\frac{1}{\sum_{n_0}^{2n_0}\psi(l)}\right\}, \tag{2}$$

where λ_c' is the critical value for the contact process on $\mathbb{Z}$. Then there is a constant $\gamma>0$ such that

$$\lim_{N\to\infty}P\left(\sigma_N\geq e^{\gamma N}\right)=1.$$

Theorem 1.3 *Suppose $\lambda=1$, and the initial distribution of the NPS follows the renewal measure $Ren(\beta)$ whenever the initial state is not $\emptyset$. Suppose C_N and C_N' are two sequences of constants such that $\lim_{N\to\infty}C_N=\infty$ and $\lim_{N\to\infty}C_N'=0$. Then*

$$\lim_{N\to+\infty}P\left(C_N'N\leq\sigma_N\leq C_NN^2\right)=1.$$

The initial distribution $Ren(\beta)$ of the previous theorem will be further elaborated in the beginning part of Section 4. The inequality (1) is not very restrictive. For example, let $\psi(n) = cn^{-\alpha}$, where $\alpha > 1$. A standard coupling shows that

$$\lim_{N\to\infty} P\big(\sigma_N > (1-\varepsilon)\log N\big) = 1, \quad \forall\, \varepsilon > 0. \tag{3}$$

Theorem 1.1 and (3) imply that σ_N has a logarithmic increasing rate as λ is small enough. By Theorem 1.1 and Theorem 1.2, σ_N has an exponential increasing rate as λ is large enough. Theorem 1.3 tells us that σ has a polynomial increasing rate as $\lambda = 1$, the critical point of the NPS on $\mathbb{Z}$.

This study is inspired by a series of papers by R. Durrett *et al* [1, 2, 3], in which the contact process is concerned. Namely, let $\{\zeta_t^N : t \geq 0\}$ be the contact process on $\{1, 2, \cdots, N\}$ with the parameter λ' starting from all sites occupied, and τ_N be the first time it hits the empty set. Denote by λ_c' the critical value of the contact process on $\mathbb{Z}$.

Theorem 1.4 (i) *If $\lambda' > \lambda_c'$, then there is a constant $\gamma_1(\lambda') > 0$ so that as $N \to \infty$, $\tau_N / (\log N) \to 1/\gamma_1(\lambda')$ in probability* ([1], Theorem 1).

(ii) *If $\lambda' > \lambda_c'$, then there is a constant $\gamma_2(\lambda') > 0$ so that as $N \to \infty$, $\log \tau_N / N \to \gamma_2(\lambda')$ in probability* ([2],Theorem 2).

(iii) *If $\lambda' = \lambda_c'$ and $a, b \in (0, \infty)$, then $P\left(aN \leq \tau_N \leq bN^4\right) \to 1$ as $N \to \infty$* ([3], Theorem 1.6).

The contact process is a non-reversible NPS. We believe the same conclusion is also true for the reversible NPS. However, we are only able to show it for very small λ and very large λ. Moreover, the parameters in the lower estimate and the upper estimate should be amended to the same.

Theorems 1.1, 1.2 and 1.3 are proved in Sections 2, 3 and 4 in turn. In Section 3, we give a proof of (3) for the completeness.

2 Upper Estimate of σ_N

To get an upper bound of σ_N, we compare the evolution of $\{|\xi_t^N| : t \geq 0\}$ with a birth and death process on $\{0, 1, \cdots, N\}$. On one hand, for any configuration

$|\xi| = i$, there are at most $i+1$ intervals of consecutive vacant sites, which do not intersect mutually; in each interval, the rate that a new particle is born is no more than λM. Hence the rate that $|\xi_t^N|$ increases 1 is not more than $(i+1)\lambda M$. On the other hand, when $|\xi_t^N| = i$, the rate that $|\xi_t^N|$ decreases 1 equals i, the total rate that there is a particle dying.

Let $\{X_t : t \geq 0\}$ be the birth and death process on $\{0, 1, \cdots, N\}$ with death rate $a_i = i$, for any $i = 1, \cdots, N$; and birth rate $b_i = (i+1)\lambda M$, for any $i = 0, \cdots, N-1$. If initially $X_0 = x \geq |\xi^N|$, there is a coupling of $\{X_t : t \geq 0\}$ and $\{\xi_t^N : t \geq 0\}$ such that

$$P^{x,\xi^N}\left(X_t \geq |\xi_t^N|,\ \forall\, t \geq 0\right) = 1, \tag{4}$$

where P^{x,ξ^N} is the conditional distribution of the initial state (x, ξ^N).

Proof of Theorem 1.1. Let $\tau = \inf\{t > 0 : X_t = 0\}$ be the first time that $\{X_t : t \geq 0\}$ hits 0. Let P^i be the conditional probability distribution on the initial state i, and E^i be the expectation with respect to P^i, where $i = 0, \cdots, N$. By (4), σ_N is stochastically dominated by τ if $X_0 \geq |\xi_0^N|$. Therefore, for any $t \geq 0$,

$$P(\sigma_N \geq t) \leq P^N(\tau \geq t). \tag{5}$$

By the Chebyshev inequality, for any $c > 0$,

$$P^N\left(\tau \geq cE^N\tau\right) \leq \frac{E^N\tau^2}{\left(cE^N\tau\right)^2}. \tag{6}$$

It is shown in [6] that

$$E^N\tau = \sum_{i=1}^N e_i, \quad E^N\tau^2 = \sum_{i=1}^N \varepsilon_i,$$

where

$$\begin{aligned} e_i &= \frac{1}{a_i} + \sum_{k=0}^{N-2-i} \frac{b_i b_{i+1}\cdots b_{i+k}}{a_i a_{i+1}\cdots a_{i+k}a_{i+k+1}} + \frac{b_i b_{i+1}\cdots b_{N-1}}{a_i a_{i+1}\cdots a_{N-1}a_N}, \\ \varepsilon_i &= \frac{2m_i}{a_i} + \sum_{k=0}^{N-2-i} \frac{2b_i b_{i+1}\cdots b_{i+k}m_{i+k+1}}{a_i a_{i+1}\cdots a_{i+k}a_{i+k+1}} + \frac{2b_i b_{i+1}\cdots b_{N-1}m_N}{a_i a_{i+1}\cdots a_{N-1}a_N}, \end{aligned} \tag{7}$$

and $m_i = E^i\tau$ for $i = 1, \cdots, N$. Notice that $m_i \leq m_N$ for any $i \leq N$. It follows that $\varepsilon_i \leq 2m_N e_i$. Therefore,

$$E^N\tau^2 = \sum_{i=1}^N \varepsilon_i \leq 2m_N \sum_{i=1}^N e_i \leq 2m_N E^N\tau = 2\left(E^N\tau\right)^2.$$

This together with (5) and (6) yields that

$$P(\sigma_N \geq c_N E^N \tau) \leq 2c_N^{-2}. \tag{8}$$

Therefore, an upper estimate of σ_N can be taken as $c_N E^N \tau$. Suppose $C_N \to \infty$ as $N \to \infty$. Let $c_N = C_N/C$, where C is the constant given in Lemma 2.1. Then the result holds by (8) and the next lemma. □

Lemma 2.1 *There is a constant C such that for large N,*

$$E^N \tau \leq \begin{cases} C \log N & \text{if } \lambda M < 1, \\ CN \log N & \text{if } \lambda M = 1, \\ C(\lambda M)^N & \text{if } \lambda M > 1. \end{cases}$$

Proof. By (7), for $i = 1, \cdots, N$,

$$e_i \leq \begin{cases} \left(1 - (\lambda M)^{N-i+1}\right) / \left((1 - \lambda M)\ i\right), & \text{if} \lambda M \neq 1; \\ (N - i + 1)/i, & \text{if} \lambda M = 1. \end{cases}$$

Hence, if $\lambda M < 1$, there is a constant C so that

$$E^N \tau = \sum_{i=1}^{N} e_i \leq (1 - \lambda M)^{-1} \sum_{i=1}^{N} i^{-1} \leq C \log N;$$

if $\lambda M > 1$, there is a constant C so that

$$E^N \tau = \sum_{i=1}^{N} e_i \leq (\lambda M - 1)^{-1} \sum_{i=1}^{N} (\lambda M)^{N-i+1} \leq C(\lambda M)^N;$$

and if $\lambda M = 1$, there is a constant C so that

$$E^N \tau = \sum_{i=1}^{N} e_i \leq (N + 1) \sum_{i=1}^{N} i^{-1} - N \leq CN \log N.$$

□

3 Lower Estimate of σ_N

We begin this section with recalling the monotone property of spin systems. Let $\{\xi_t : t \geq 0\}$ and $\{\zeta_t : t \geq 0\}$ be two spin systems with the same state space. Suppose that whenever $\xi \leq \zeta$,

$$c_1(x, \xi) \leq c_2(x, \zeta) \quad \text{if } \xi(x) = \zeta(x) = 0,$$

and

$$c_1(x,\xi) \geq c_2(x,\zeta) \text{ if } \xi(x) = \zeta(x) = 1.$$

Then by Theorem III.1.5 of [4] there is a coupling such that $P^{\xi,\zeta}(\xi_t \leq \zeta_t) = 1$ for all $\xi \leq \zeta$ and all $t \geq 0$. This together with Theorem 1.4 enlightens us to compare a NPS with a contact process by the *renormalization* argument. In other words, we divide $\{1, \cdots, N\}$ into some subintervals and consider the existence of particles in each interval rather than at each site.

Proof of Theorem 1.2. Given n_0, let $L = [N/n_0]$ be the integer part of N/n_0, and divide $\{1, 2, \cdots, Ln_0\}$ into subintervals

$$I_k = \big\{(k-1)n_0 + 1, (k-1)n_0 + 2, \cdots, kn_0\big\}, \quad k = 1, 2, \cdots, L.$$

We compare $\{\xi_t^N : t \geq 0\}$ with a contact process $\{\varsigma_t^L : t \geq 0\}$ on $\{1, \cdots, L\}$, whose initial state is

$$\varsigma_0^L(k) = \begin{cases} 1 & \text{if } \sum_{x\in I_k} \xi_0^N(x) \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

We claim that, by choosing carefully the infection parameter of ς_t^L,

$$\sum_{x\in I_k} \xi_t^N(x) \geq \varsigma_t^L(k), \ \forall t \geq 0, k = 1, \cdots, L. \tag{9}$$

This can be violated only when $\sum_{x\in I_k} \xi_t^N(x) = \varsigma_t^L(k)$. So ξ_t^N and ς_t^L evolve independently until $\sum_{x\in I_k} \xi_t^N(x) = \varsigma_t^L(k)$ for some k and $t > 0$. A coupling is then needed to preserve the inequality 9. There are two cases.

Case 1. $\sum_{x\in I_k} \xi^N(x) = \varsigma^L(k) = 1$. Equivalently, there is only one particle in the k-th subinterval in the configuration ξ of the NPS, and the individual at site k is infected in the configuration ς of the contact process. Because both death rates are 1, we let both particles die at the same time.

Case 2. $\sum_{x\in I_k} \xi^N(x) = \varsigma^L(k) = 0$. Equivalently, there are no particles in I_k and the individual at site k of $\varsigma$is healthy. Consider birth rates of both processes.

If $k = 1$, by attractiveness of the NPS, the total birth rate in I_1 is at least $\lambda \sum_{l=n_0}^{2n_0} \psi(l)$ if there are particles in I_2. The case $k = L$ is similar. If $1 < k < L$, the total birth rate in I_k is at least $\lambda \sum_{l=n_0}^{2n_0} \psi(l)$ if there is at least one particle in I_{k-1} and no particle in I_{k+1}, or vice versa. If there are particles in both I_{k-1} and I_{k+1}, then the total birth rate in I_k is at least $\lambda \sum_{l=n_0}^{2n_0} \psi(l)\psi(3n_0 - l)/\psi(3n_0)$. Assumption (2) implies that we can choose the infection rate λ' of the contact process ς_t^L to satisfy the following inequality.

$$\lambda_c' < \lambda' \leq \min\left\{\lambda \sum_{l=n_0}^{2n_0} \frac{\psi(l)\psi(3n_0 - l)}{2\psi(3n_0)},\ \sum_{l=n_0}^{2n_0} \lambda\psi(l)\right\}.$$

Then there is $P_{\lambda'}$, a coupling of $\{\xi_t^N : t \geq 0\}$ and $\{\varsigma_t^{[N/n_0]} : t \geq 0\}$, such that for any $t \geq 0$,

$$P_{\lambda'}\left(\sum_{x\in I_k} \xi_t^N(x) \geq \varsigma_t^{[N/n_0]}(k),\ \forall\ k = 1, \cdots, [N/n_0]\right) = 1. \tag{10}$$

For any $t \geq 0$,

$$P(\sigma_N \geq t) = P\left(\sum_{x\in I_k} \xi_t^N(x) \neq \emptyset\right) \geq P\left(\varsigma_t^L \neq \emptyset\right) \geq P\left(\tau_L \geq t\right),$$

where $\tau_L = \inf\{t : \zeta_t^L = \emptyset\}$. This together with part (ii) of Theorem 1.4 implies that

$$\liminf_{N\to\infty} P(\sigma_N \geq e^{\gamma(\lambda')L/2}) \geq \lim_{N\to\infty} P\left(\tau_L \geq e^{\gamma(\lambda')L/2}\right) = 1.$$

Let $\gamma = \gamma(\lambda')/4n_0$, then the result follows. □

Proof of (3). To be self-contained, we give a proof of (3). Let $\{\gamma_t^N : t \geq 0\}$ be a spin system on $\{1, 2, \cdots, N\}$ starting from all sites occupied, in which particles die independently with rate 1 and no new particles are born. Then there is a coupling such that $P\left(\gamma_t^N \leq \xi_t^N,\ \forall\ t > 0\right) = 1$. This implies that

$$P(\xi_t^N \neq \emptyset) \geq P(\gamma_t^N \neq \emptyset) \quad \forall\ t \geq 0.$$

Notice that $P\left(\gamma_t^N(x) = 1\right) \geq e^{-t}$ for any $x = 1, \cdots, N$, and $\gamma_t^N(x)$ are mutually independent. So

$$P\big(\sigma_N \geq \alpha(N)\big) \geq 1 - \left(1 - e^{-\alpha(N)}\right)^N, \quad \forall\ \alpha(N) \geq 0.$$

Choose $\alpha(N)$ such that $\left(1 - e^{-\alpha(N)}\right)^N$ converges to zero as $N \to \infty$. This gives the lower estimate of σ_N. Especially, let $\alpha(N) = (1 - \varepsilon)\log N$, where $\varepsilon > 0$. Then (3) follows. □

4 The Critical Case

Theorem 1.3 can be divided into two separate statements:

$$\lim_{N\to\infty} P\left(\sigma_N \le C_N N^2\right) = 1; \tag{11}$$

and

$$\lim_{N\to\infty} P\left(C'_N N \le \sigma_N\right) = 1. \tag{12}$$

The two statements will be proved by two distinct approaches. We shall compare the critical NPS $\{\xi_t^N : t \ge 0\}$ with a critical NPS on $\mathbb{Z}$ to show (11), and compare it with a modified process to prove (12).

For any $A = \{x_0, x_1, \cdots, x_k\} \subset \{1, 2, \cdots, N\}$, define

$$\nu_\beta(A) = \beta(x_1 - x_0)\beta(x_2 - x_1)\cdots\beta(x_k - x_{k-1}) \sum_{l=x_0}^{\infty} \beta(l) \sum_{r=N+1-x_k}^{\infty} \beta(r).$$

Let $K_N = \sum_{A\in\mathcal{S}_N\setminus\{\emptyset\}} \nu_\beta(A)$ and $\pi(A) = \nu_\beta(A)/K_N$. Then π is the induced probability measure of the renewal measure $Ren(\beta)$ restricted on $\{1, 2, \cdots, N\}$.

The critical NPS $\{\xi_t^N\}$ is a Markov process taking values in $\mathcal{S}_N$ with jump rate

$$q(A,B) = \begin{cases} 1 & \text{if } x \in A,\ B = A \setminus \{x\},, \\ \beta(l)\beta(r)/\beta(l+r) & \text{if } x \notin A,\ B = A \cup \{x\}; \\ 0 & \text{otherwise,} \end{cases}$$

and reversible with respect to π in the sense that $\pi(A)q(A,B) = \pi(B)q(B,A)$ for $A, B \in \mathcal{S}_N$ $A \ne \emptyset, B \ne \emptyset$. Throughout this section we take π to be the initial distribution of $\{\xi_t^N\}$.

Let $\{\widetilde{\xi_t^N} : t \ge 0\}$ be a Markov process on $\mathcal{S}_N$, which has the same transition rates as $\{\xi_t^N : t \ge 0\}$ except that particles can be born from the empty set. Namely, denote by $\tilde{q}$ and q respectively the transition rates of $\{\widetilde{\xi_t^N} : t \ge 0\}$ and $\{\xi_t^N : t \ge 0\}$, then

$$\tilde{q}(A,B) = \begin{cases} q(A,B) & \text{if } A \ne \emptyset, \\ q & \text{if } A = \emptyset \text{ and } |B| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $q > 0$ is a constant. Let

$$\nu_\beta\left(\{\emptyset\}\right) = q^{-1}, \qquad \nu_\beta|_{\emptyset^c} = Ren(\beta)|_{\emptyset^c}.$$

Then $\tilde{\pi} = \nu_\beta / \left(K_N + q^{-1}\right)$ is a reversible distribution of $\{\widetilde{\xi_t^N} : t \geq 0\}$.

Proof of Equation (12). Let

$$\tau = \inf\{t \geq 0, \widetilde{\xi_t^N} = \emptyset\},$$

$\tilde{P}$ be the distribution of $\{\widetilde{\xi_t^N} : t \geq 0\}$ with initial distribution π, and $\tilde{E}$ be the expectation with respect to $\tilde{P}$. Notice that $\{\widetilde{\xi_t^N} : t \geq 0\}$ is stationary under $\tilde{P}$. For any $t > 0$,

$$2t\pi(\{\emptyset\}) = \tilde{E} \int_0^{2t} 1_{\{\widetilde{\xi_s^N} = \emptyset\}} ds.$$

By the Strong Markovian Property, the right side above equals

$$\begin{aligned} &\tilde{E}\tilde{E}\left(\int_0^{2t} 1_{\{\widetilde{\xi_s^N}=\emptyset\}} ds \middle| \mathcal{F}_\tau \right) \geq \tilde{E}\tilde{E}\left(1_{\{\tau<t\}} \int_0^{2t} 1_{\{\widetilde{\xi_s^N}=\emptyset\}} ds \middle| \mathcal{F}_\tau \right) \\ \geq\ & \tilde{E}\tilde{E}\left(1_{\{\tau<t\}} \int_\tau^{\tau+t} 1_{\{\widetilde{\xi_s^N}=\emptyset\}} ds \middle| \mathcal{F}_\tau \right) = \tilde{P}(\tau < t)\tilde{E}\left(\int_0^t 1_{\{\widetilde{\xi_s^N}=\emptyset\}} ds \middle| \widetilde{\xi_0^N} = \emptyset \right) \end{aligned}$$

Denote by σ the first time $\{\widetilde{\xi_t^N} : t \geq 0\}$ jumps. Then

$$\tilde{E}\left(\int_0^t 1_{\{\widetilde{\xi_s^N}=\emptyset\}} ds \middle| \widetilde{\xi_0^N} = \emptyset \right) \ \geq\ \tilde{E}\left(\sigma 1_{\{\sigma \leq t\}} | \widetilde{\xi_0^N} = \emptyset \right) = \int_0^t s\tilde{q}_\emptyset e^{-\tilde{q}_\emptyset s} ds,$$

where $\tilde{q}_\emptyset = \sum_\xi \tilde{q}(\emptyset, \xi) = Nq$. Hence

$$\tilde{P}(\tau < t) \leq 2t\pi(\{\emptyset\}) \left(\int_0^t s\tilde{q}_\emptyset e^{-\tilde{q}_\emptyset s} ds \right)^{-1} = \frac{2tq^{-1}}{K_N + q^{-1}} \left(\int_0^t Nqse^{-Nqs} ds \right)^{-1}. \quad (13)$$

Notice that

$$\begin{aligned} \tilde{P}(\tau < t) \ &\geq\ \tilde{P}\left(\tau < t, \widetilde{\xi_0^N} \neq \emptyset\right) = \tilde{P}\left(\widetilde{\xi_0^N} \neq \emptyset\right) \tilde{P}\left(\tau < t | \widetilde{\xi_0^N} \neq \emptyset\right) \\ &=\ P(\sigma_N < t) K_N / \left(K_N + q^{-1}\right). \end{aligned}$$

This together with (13) yields that

$$P\left(\sigma_N < t\right) \leq 2tK_N^{-1} \left(q \int_0^t Nqse^{-Nqs} ds \right)^{-1} = 2NtK_N^{-1} \left(\int_0^{Nqt} se^{-s} ds \right)^{-1},$$

Notice that the right side does not depend on q, which is arbitrary. Let $q \to \infty$, then it follows that

$$P\left(\sigma_N < t\right) \leq 2NtK_N^{-1}, \quad \forall\ t > 0.$$

By (14), the lower estimate of σ_N is such t_N that t_N/N converges to zero, which implies the result. □

Lemma 4.1 *Let*

$$S_N(x,y) = \left\{\xi^N : \xi^N(x) = \xi^N(y) = 1, \xi^N(z) = 0, \ \forall\ z < x, \text{ or } z > y\right\}.$$

Then $\nu_\beta(S_N(x,y)) \leq 1$ *and there is a constant* $C > 0$ *such that* $\nu_\beta(S_N(x,y)) \geq C$ *whenever* $y - x$ *is large enough.*

On the other hand, when N is large enough,

$$K_N = \sum_{\xi \in \mathcal{S}_N \setminus \{\emptyset\}} \nu_\beta(\xi) \geq \sum_{x=0}^{[N/3]} \sum_{y=[2N/3]}^{N} \nu_\beta\left(S_N(x,y)\right) \geq CN^2. \tag{14}$$

where $\mathcal{S}_N = \{0,1\}^{\{1,2,\cdots,N\}}$.

Proof of Lemma 4.1. Let X_n be the time until the first renewal $\geq n$. Then $\{X_n : n \geq 0\}$ is a Markov chain with transition probability $p(0,n) = \beta(n+1)$, $p(n+1,n) = 1$ for all $n \geq 0$. Since $\mu := \sum_{n=1}^{\infty} n\beta(n) < \infty$, $\{X_n : n \geq 0\}$ has an invariant distribution π, and $\pi(0) = 1/\mu$. Thus $P(X_n = 0)$ converges to $1/\mu$ as $n \to +\infty$. Notice that $X_n = 0$ if and only if n is a renewal time. Hence, there exists $n_0 > 0$ such that $P(X_n = 0) > (2\mu)^{-1}$ for any $n \geq n_0$. Then

$$\nu_\beta(S_N(x,y)) = P(X_{y-x} = 0) > (2\mu)^{-1}, \quad \text{if } y - x > n_0.$$

It is not difficult to check $\nu_\beta(S_N(x,y)) = P(X_{y-x} = 0) \leq 1$. □

Let $\{\eta_t : t \geq 0\}$ be a reversible nearest particle system on $\mathbb{Z}$, and r_t the rightmost particle in $\{\eta_t : t \geq 0\}$, i.e.

$$r_t := \sup\{x : \eta_t(x) = 1\}.$$

The properties of r_t of the critical NPS are studied in [5].

Lemma 4.2 ([5], Theorem 1) *Let* $\{\eta_t : t \geq 0\}$ *be the critical reversible nearest particle system on* $\mathbb{Z}$. *Suppose the initial configurations have a particle at the origin and no particle on the left of the origin, and follows the renewal measure* $Ren(\beta)$ *with density* $\beta(\cdot)$. *Then, as* $a \to \infty$, r_{a^2t}/a *converges in distribution to a Brownian motion with diffusion constant* $D > 0$ *in the Skorohod space.*

Proof of Equation (11). Since the transition mechanism of $\{\eta_t : t \geq 0\}$ is translation invariant, we can regard the NPS on $[0,N]$ as the NPS on $[n, N+n]$ for any n. So we do not distinguish them in symbols.

To use Lemma 4.2, we partition the configurations of $\{0,1\}^N$ by the position of the rightmost particle. Namely, let A_x be the set of configurations whose rightmost particle is at x, i.e.

$$A_x = \left\{\xi \in \{0,1\}^{\{0,\cdots,N\}} : \xi(x) = 1, \xi(y) = 0, \ \forall\, y > x.\right\}$$

Denote by P be the distribution of $\{\xi_t^N : t \geq 0\}$ with initial distribution in Theorem 1.3, and by $P_{N,x}$ the distribution of the NPS on $[-N, \cdots, 0]$ whose initial configurations have a particle at x, no particle to the right of x, and follows the renewal measure $Ren(\beta)$. Then

$$P = \sum_{x=0}^{N} P(A_x) P_{N,x}. \tag{15}$$

Denote by $\mathbf{P}$ the distribution of the NPS on $\mathbb{Z}$ with the initial distribution in Lemma 4.2. Now regard $\mathbf{P}_{N,x}$ the distribution of the NPS on $[-x, N-x]$. Thanks to the attractive property, there is a coupling of $\mathbf{P}$ and $\tilde{P}_{N,x}$ such that for all $t > 0$,

$$\xi_t^N(i) \leq \eta_t(i), \quad -x \leq i \leq N - x \tag{16}$$

almost surely if $\eta_0|_{[-x,0]} = \xi_0^N$. By (16), $\xi_t^N \equiv \emptyset$ once $r_t < -x$, hence $\sigma_N \leq \inf\{t : r_t < -x\}$ almost surely.

Suppose $\lim_{N\to\infty} C_N = \infty$. For any $C > 0$,

$$\begin{aligned} & P_{N,x}\left(\sigma_N \leq C_N N^2\right) \geq P_{N,x}\left(\sigma_N \leq C_N x^2\right) \geq \mathbf{P}\left(\exists\, t \leq C_N x^2 \text{ s.t. } r_t < -x\right) \\ \geq\ & \mathbf{P}\left(\exists\, t \leq C_N \text{ s.t. } r_{x^2 t} < -x\right) \geq \mathbf{P}\left(\exists\, t \leq C \text{ s.t. } r_{x^2 t}/x < -1\right), \end{aligned}$$

whenever N is large. This together with Lemma 4.2 implies that

$$\liminf_{N,x\to+\infty} P_{N,x}\left(\sigma_N \leq C_N N^2\right) \geq \mathbf{P}\left(\exists\, t \leq C \text{ s.t. } B_t < -1\right), \quad \forall\, C > 0,$$

where $\{B_t : t \geq 0\}$ is a Brownian motion with diffusion constant D. Let $C \to +\infty$, the right side of the above equation converges to 1. Hence

$$\lim_{N,x\to+\infty} P_{N,x}\left(\sigma_N \leq C_N N^2\right) = 1.$$

Consequently, for any $\varepsilon > 0$, there exists $N_0 > 0$ such that

$$P_{N,x}\left(\sigma_N \leq C_N N^2\right) > 1 - \varepsilon.$$

for any $N \geq x \geq N_0$. This together with (15) implies that

$$P\left(\sigma_N \leq C_N N^2\right) = \sum_{x=0}^{N} P(A_x) P_{N,x}\left(\sigma_N \leq C_N N^2\right) \geq (1-\varepsilon) \sum_{x=N_0}^{N} P(A_x). \tag{17}$$

By Lemma 4.1, on one hand,

$$\sum_{x=0}^{N_0-1} \nu_\beta(A_x) \leq \sum_{x=0}^{N_0-1} \sum_{y=0}^{x} \nu_\beta(S_N(y,x)) \leq N_0^2.$$

Therefore, as $N \to \infty$,

$$\sum_{x=N_0}^{N} P(A_x) \geq 1 - N_0^2/(CN^2) \to 1.$$

This together with (17) implies that

$$\liminf_{N\to\infty} P\left(\sigma_N \leq C_N N^2\right) \geq (1-\varepsilon).$$

Let $\varepsilon \to 0$ and the result follows. □

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