1. Introduction of large deviation theory.

In the field of large deviations, people concern about asymptotic computation of small probabilities on an exponential scale. Since the remarkable works by Donsker-Varadhan (and others) in seventies and eighties, the field has been developed into a relatively complete system. There have been several "general" tricks that become standard approaches in dealing with large deviation problems. Perhaps the most useful is Gätner-Ellis Theorem.

We have no intension to state this theorem in its full generality. Let $\{Y_n\}$ be a sequence of non-negative random variables and let $\{b_n\}$ be a positive sequence such that $b_n \longrightarrow \infty$. The basic assumption is existence of the limt

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta b_n Y_n\Big\} \qquad \theta > 0 \tag{1.1}$$

Theorem 1.1. *(Gätner-Ellis). Under some regularity conditions on the function* $\Lambda(\cdot)$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \ge \lambda\} = \Lambda^*(\lambda) \qquad \lambda > 0 \tag{1.2}$$

where

$$\Lambda^*(\lambda) = \sup_{\theta>0} \Big\{\theta\lambda - \Lambda(\theta)\Big\}$$

If the exponential moment generating function

$$\mathbb{E} \exp\Big\{\theta b_n Y_n\Big\}$$

does not exist or, (1.1) is not in the right scale to describe the large deviation behavior of $\{Y_n\}$, we assume

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta b_n Y_n^{1/p}\Big\} \qquad \theta > 0 \tag{1.3}$$

where $p > 0$ is fixed.

Replacing Y_n by $Y_n^{1/p}$ in Theorem 1.1, we have

Theorem 1.2. *Under some regularity conditions on the function* $\Lambda(\cdot)$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \ge \lambda\} = \Lambda^*(\lambda^p) \qquad \lambda > 0 \tag{1.4}$$

By Taylor expansion,

$$\mathbb{E} \exp\Big\{\theta b_n Y_n^{1/p}\Big\} = \sum_{m=0}^{\infty} \frac{\theta^m}{m!} b_n^m \mathbb{E}\, Y_n^{m/p}$$

When establishing (1.3) by "standard" approaches becomes techniquely impossible, one may attempt to estimate

$$\mathbb{E}\, Y_n^{m/p}$$

When $p \neq 1$, there are some good reasons to feel unpleasent to face sometimes fractional power m/p.

Lemm 1.1. *The following two statements (1.5) and (1.6) are equivalent:*

$$\lim_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}\,Y_n^m\Big)^{1/p}=\Psi(\theta)\qquad \theta>0 \tag{1.5}$$

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta b_n Y_n^{1/p}\Big\}=p\Psi\Big(\frac{\theta}{p}\Big)\qquad \theta>0 \tag{1.6}$$

Proof. Due to similarity, we only show that (1.5) implies (1.6). Given $\epsilon>0$,

$$\begin{aligned}&\frac{1}{([p^{-1}m]+1)!}b_n^{[p^{-1}m]+1}\Big(\frac{(1+\epsilon)\theta}{p}\Big)^{[p^{-1}m]+1}\Big[\mathbb{E}\,Y_n^{[p^{-1}m]+1}\Big]^{1/p}\\ &\le\sum_{m=0}^{\infty}\frac{1}{m!}\Big(\frac{(1+\epsilon)\theta}{p}\Big)^m b_n^m\Big(\mathbb{E}\,Y_n^m\Big)^{1/p}\end{aligned}$$

By Jensen inequality,

$$\mathbb{E}\,Y_n^{m/p}\le\Big[\mathbb{E}\,Y_n^{[p^{-1}m]+1}\Big]^{\frac{p^{-1}m}{[p^{-1}m]+1}}$$

On the other hand, as

$$b_n^{[p^{-1}m]+1}\Big[\mathbb{E}\,Y_n^{[p^{-1}m]+1}\Big]^{1/p}\ge 1$$

we have

$$\Big(b_n^{p([p^{-1}m]+1)}\mathbb{E}\,Y_n^{[p^{-1}m]+1}\Big)^{\frac{p^{-1}m}{[p^{-1}m]+1}}\le b_n^{p([p^{-1}m]+1)}\mathbb{E}\,Y_n^{[p^{-1}m]+1}$$

Summerizing what we have,

$$b_n^m\mathbb{E}\,Y_n^{m/p}\le b_n^{p[p^{-1}m]+1}\mathbb{E}\,Y_n^{[p^{-1}m]+1}$$

Consequently,

$$\begin{aligned}&\frac{1}{\big(([p^{-1}m]+1)!\big)^p}b_n^m\Big(\frac{(1+\epsilon)\theta}{p}\Big)^{p([p^{-1}m]+1)}\mathbb{E}\,Y_n^{m/p}\\ &\le\Big(\sum_{m=0}^{\infty}\frac{1}{m!}\Big(\frac{(1+\epsilon)\theta}{p}\Big)^m b_n^m\Big(\mathbb{E}\,Y_n^m\Big)^{1/p}\Big)^p\end{aligned}$$

By Stirling formula, there are constants $C>0$ and $\delta>0$ such that

$$\frac{\theta^m}{m!}b_n^m\mathbb{E}\,Y_n^{m/p}\le C(1+\delta)^{-m}\Big(\sum_{m=0}^{\infty}\frac{1}{m!}\Big(\frac{(1+\epsilon)\theta}{p}\Big)^m b_n^m\Big(\mathbb{E}\,Y_n^m\Big)^{1/p}\Big)^p$$

Thus,

$$\mathbb{E}\,\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\le C\frac{1+\delta}{\delta}\Big(\sum_{m=0}^{\infty}\frac{1}{m!}\Big(\frac{(1+\epsilon)\theta}{p}\Big)^m b_n^m\Big(\mathbb{E}\,Y_n^m\Big)^{1/p}\Big)^p$$

Consequently,

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\le p\Psi\Big(\frac{(1+\epsilon)\theta}{p}\Big)$$

Letting $\epsilon\to 0^+$ on the right gives

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\le p\Psi\Big(\frac{\theta}{p}\Big)$$

On the other hand,

$$\mathbb{E}\,\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\ge\frac{\theta^{pm}}{(pm)!}b_n^{pm}\mathbb{E}\,Y_n^m$$

By Stirling formula again, for any $0<\delta<\epsilon$, there is $C>0$ such that

$$C(1+\delta)^{-m}\Big(\mathbb{E}\,\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\Big)^{1/p}\ge\frac{1}{m!}\Big(\frac{\theta}{(1+\epsilon)p}\Big)^m b_n^m\Big(\mathbb{E}\,Y_n^m\Big)^{1/p}$$

for all $m\ge 0$. Thus

$$C\frac{1+\delta}{\delta}\Big(\mathbb{E}\,\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\Big)^{1/p}\ge\sum_{m=0}^\infty\frac{1}{m!}\Big(\frac{\theta}{(1+\epsilon)p}\Big)^m b^m\Big(\mathbb{E}\,Y_n^m\Big)^{1/p}$$

By (1.5),

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\ge p\Psi\Big(\frac{\theta}{(1+\epsilon)p}\Big)\qquad\theta>0$$

Letting $\epsilon\to 0^+$ on the right,

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta b_n Y_n^{1/p}\Big\}\ge p\Psi\Big(\frac{\theta}{p}\Big)\qquad\theta>0$$

□

By Lemma 1.1 and Theorem 1.3, immediately we obtain

Theorem 1.4. *Under (1.5) and some regularity condition on* $\Psi(\cdot)$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\big\{Y_n\ge\lambda\big\}=-I(\lambda)\qquad(\lambda>0)\tag{1.7}$$

where

$$I(\lambda)=p\sup_{\theta>0}\Big\{\lambda^{1/p}-\Psi(\theta)\Big\}$$

We now apply Theorem 4 to a more special case. Let $Y\ge 0$ be a random variable such that

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^\gamma}\mathbb{E}\,Y^m=-\kappa\tag{1.8}$$

for some $\gamma>0$ and $\kappa\in\mathbb{R}$.

Theorem 1.5 (König and Morters (2002)). *Under (1.8)*

$$\lim_{t\to\infty} t^{-1/\gamma}\log\mathbb{P}\{Y\ge t\} = -\gamma e^{\kappa/\gamma}. \tag{1.9}$$

Proof. We only need to check the condition (1.5) with $Y_t = Y/t$, $b_t = t^{1/\gamma}$ and $p = 2\gamma$. Indeed, for any $\theta > 0$,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t^{1/\gamma}}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}t^{m/(2\gamma)}\Big(\mathbb{E}\,Y^m\Big)^{\frac{1}{2\gamma}}\\
&=\lim_{t\to\infty}\frac{1}{t^{1/\gamma}}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}t^{m/(2\gamma)}\Big((m!)^{\gamma}e^{-\kappa m}\Big)^{\frac{1}{2\gamma}}\\
&=\lim_{t\to\infty}\frac{1}{t^{1/\gamma}}\log\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\Big(\theta t^{\frac{1}{2\gamma}}e^{-\frac{\kappa}{2\gamma}}\Big)^{m}\\
&=\lim_{t\to\infty}\frac{1}{t^{1/\gamma}}\log\sum_{m=0}^{\infty}\frac{1}{\sqrt{2m!}}\Big(\theta t^{\frac{1}{2\gamma}}e^{-\frac{\kappa}{2\gamma}}\Big)^{2m}\\
&=\lim_{t\to\infty}\frac{1}{t^{1/\gamma}}\log\sum_{m=0}^{\infty}\frac{1}{2^m m!}\Big(\theta t^{\frac{1}{2\gamma}}e^{-\frac{\kappa}{2\gamma}}\Big)^{2m}\\
&=\frac{1}{2}\theta^2 e^{-\kappa/\gamma}
\end{aligned}$$

Hence,

$$I(\lambda) = 2\gamma\sup_{\theta>0}\Big\{\theta\lambda^{\frac{1}{2\gamma}} - \frac{1}{2}\theta^2 e^{-\kappa/\gamma}\Big\} = \lambda^{\frac{1}{2\gamma}}\gamma e^{\kappa/\gamma}$$

□

2. Large deviation for Brownian intersection local times.

Recall a d-dimensional Brownian motion $W(t)$ is a stochastic process in $\mathbb{R}^d$ with the following properties:

(1). For any $s < t$, the increment $W(t) - W(s)$ is independent of the history (up to the time s)

$$\Big\{W(u);\quad u\le s\Big\}$$

(2). For any $t > 0$, $W(t)$ is a normal random variable with mean $\mathbf{0}$ and covariance matrix $t\mathbf{I}_d$ (where $\mathbf{I}$ is the $d\times d$ identity matrix).

By convention, we usaully assume that $W(0) = 0$. When the fact that $W(t)$ is a Markov process is emphasized, however, we may allow $W(t)$ to start at any point $x\in\mathbb{R}$ (i.e., $W(0) = x$).

Let $W_1(t), \cdots, W_p(t)$ be independent d-dimensional Brownian motions. If we allow $W_j(\cdot)$ ran up to time t_j $(j = 1, \cdots, p)$, a natural question is to ask how much time is spent for the p independent trajectories $W_1(t), \cdots, W_p(t)$ to intersect. In other words, we are interested in the time set

$$\Big\{(s_1, \cdots, s_p) \in [0, t_1] \times \cdots \times [0, t_p]; \quad W_1(s_1) \approx \cdots \approx W_p(s_p)\Big\}$$

If properly defined, the Lebesgue measure of this et is called the intersection local time of $W_1(t), \cdots, W_p(t)$ and is denoted by $\alpha\big([0, t_1] \times \cdots \times [0, t_p]\big)$.

Theorem 2.1. *(Dvoretzky-Erdös-Kakutani (1950, 1954))*

$$W_1(0, \infty) \cap \cdots \cap W_p(0, \infty) \neq \phi$$

if and only if $p(d - 2) < d$.

In the rest of this section, we assume $p(d - 2) < d$.

There are two equivalent ways to construct Brownian intersection local time in the multi-dimensional case. The first approach (Geman, Horowitz and Rosen (1984)) corresponds to the notation

$$\begin{aligned} &\alpha\big([0, t_1] \times \cdots \times [0, t_p]\big) \\ &= \int_0^{t_1} \cdots \int_0^{t_p} \delta_0\big(W_1(s_1) - W_2(s_2)\big) \cdots \delta_0\big(W_{p-1}(s_{p-1}) - W_p(s_p)\big) ds_1 \cdots ds_p \end{aligned} \tag{2.1}$$

Geman, Horowitz and Rosen (1984) prove that $p(d-2) < d$, the occupation measure on $\mathbb{R}^{d(p-1)}$ given by

$$\mu_A(B) = \int_A 1_B\big(W_1(s_1) - W_2(s_2), \cdots, W_{p-1}(s_{p-1}) - W_p(s_p)\big) ds_1 \cdots ds_p \qquad B \subset \mathbb{R}^{d(p-1)}$$

is absolutely continuous, with probability 1, with respect to Lebesgue measure on $\mathbb{R}^{d(p-1)}$ for any Borel set $A \subset (\mathbb{R}^p)^+$ (in particular, for $A = [0, t_1] \times \cdots \times [0, t_p]$) and, the density $\alpha(x, A)$ of such measure can be chosen so that the function

$$(x, t_1, \cdots, t_p) \longmapsto \alpha\big(x, [0, t_1] \times \cdots \times [0, t_p]\big) \qquad x \in \mathbb{R}^{d(p-1)} \quad (t_1, \cdots, t_p) \in (\mathbb{R}^p)^+$$

is jointly continuous. The random measure $\alpha(\cdot)$ on $(\mathbb{R}^p)^+$ is defined as

$$\alpha(A) = \alpha(0, A) \qquad \forall \text{ Borel set } A \subset (\mathbb{R}^p)^+.$$

Another approach (Le Gall (1990)) constitutes the notation

$$\alpha\big([0, t_1] \times \cdots \times [0, t_p]\big) = \int_{\mathbb{R}^d} \Big[\prod_{j=1}^p \int_0^{t_j} \delta_x\big(W(s)\big) ds\Big] dx \tag{2.2}$$

Let $f(x)$ be a nice probability density function on R^d. Given $\epsilon > 0$, write $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ and define

$$\alpha_\epsilon\big([0,t_1]\times\cdots\times[0,t_p]\big) = \int_{\mathbb{R}^d}\left[\prod_{j=1}^p \int_0^{t_j} f_\epsilon\big(W(s)-x\big)ds\right]dx$$

Under $p(d-2)<d$, Le Gall (1990) shows that there is a random variable $\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)$ such that

$$\lim_{\epsilon\to 0^+}\alpha_\epsilon\big([0,t_1]\times\cdots\times[0,t_p]\big) = \alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)$$

holds in L^m-norm for any $m \geq 1$ and for any $t_1,\cdots,t_p > 0$.

In the special case $d=1$, let $L_1(t,x),\cdots,L_p(t,x)$ be the local times of $W_1,\cdots,W_p$, respectively. By the second construction, one can see that

$$\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big) = \int_{-\infty}^{\infty}\prod_{j=1}^p L_j(t_j,x)dx$$

By the scaling property of Brownian motions

$$\alpha\big([0,t]^p\big) \stackrel{d}{=} t^{\frac{2p-d(p-1)}{2}}\alpha\big([0,1]^p\big). \tag{2.3}$$

Our main theorem in this section is the following

Theorem 2.2. *Under $p(d-2)<d$,*

$$\lim_{t\to\infty} t^{-\frac{2}{d(p-1)}}\log\mathbb{P}\Big\{\alpha\big([0,1]^p\big)\geq t\Big\} = -\frac{p}{2}\kappa(d,p)^{-\frac{4p}{d(p-1)}} \tag{2.4}$$

where $\kappa(d,p)$ is the best constant of the Gagliardo-Nirenberg inequality

$$|f|_{2p} \leq C||\nabla f||_2^{\frac{d(p-1)}{2p}}||f||_2^{1-\frac{d(p-1)}{2p}} \qquad f\in W^{1,2}(\mathbb{R}^d)$$

Remark. We point out some facts about $\kappa(d,p)$ which will be used later. Let

$$\mathcal{F} = \Big\{f\in W^{1,2}(\mathbb{R}^d);\quad \int_{\mathbb{R}^d}|f(x)|^2 = 1\Big\}$$

Then

$$\begin{aligned}&\sup_{f\in\mathcal{F}}\left\{\Big(\int_{\mathbb{R}^d}|f(x)|^{2p}dx\Big)^{1/p} - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla f|^2dx\right\}\\ &= \frac{2p-d(p-1)}{2p}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p-d(p-1)}}\kappa(d,p)^{\frac{4p}{2p-d(p-1)}}\end{aligned} \tag{2.5}$$

The second fact is that

$$\rho = \Big(\frac{2p-d(p-1)}{2p}\Big)^{\frac{2p-d(p-1)}{2p}}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p}}\kappa(d,p)^2 \tag{2.6}$$

where

$$\rho = \sup_f \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x-y)f(x)f(y) \tag{2.7}$$

where the supremum is taken for all f on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} |f(x)|^{\frac{2p}{2p-1}}dx = 1$$

and where

$$G(x) = \int_0^\infty e^{-t}\frac{1}{(2\pi t)^{d/2}}\exp\Big\{-\frac{|x|^2}{2t}\Big\}dt \qquad x\in\mathbb{R}^d$$

We now discuss the proof our theorem. By Theorem 1.5 and by the relation (2.6) between $\kappa(d,p)$ and ρ, we need only to establish

$$\begin{aligned}&\lim_{m\to\infty}\frac{1}{m}\log(m!)^{-\frac{d(p-1)}{2}}\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\\ &= p\log\rho + \frac{2p-d(p-1)}{2}\log\frac{2p}{2p-d(p-1)}\end{aligned} \tag{2.8}$$

To calculate the moment of $\alpha\big([0,1]^p\big)$, notice that by (2.2) for any $t_1,\cdots,t_p>0$,

$$\begin{aligned}&\mathbb{E}\Big[\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big]\\ &= \mathbb{E}\Big[\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\prod_{j=1}^p\int_{[0,t_j]^m}ds_1\cdots ds_m\prod_{k=1}^m\delta_{x_k}\big(W_j(s_k)\big)\Big]\\ &= \int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\prod_{j=1}^p\int_{[0,t_j]^m}ds_1\cdots ds_m\mathbb{E}\prod_{k=1}^m\delta_{x_k}\big(W(s_k)\big)\end{aligned}$$

Let Σ_m be the permutation group on $\{1,\cdots,m\}$. By time rearrangement,

$$\begin{aligned}&\int_{[0,t_j]^m}ds_1\cdots ds_m\mathbb{E}\prod_{k=1}^m\delta_{x_k}\big(W(s_k)\big)\\ &= \sum_{\sigma\in\Sigma_m}\int_{\{0\le s_1\le\cdots,\le s_m\le t_j\}}ds_1\cdots ds_m\mathbb{E}\prod_{k=1}^m\delta_{x_{\sigma(k)}}\big(W(s_k)\big)\\ &= \sum_{\sigma\in\Sigma_m}\int_{\{0\le s_1\le\cdots,\le s_m\le t_j\}}ds_1\cdots ds_m\mathbb{E}\prod_{k=1}^m\delta_{x_{\sigma(k)}-x_{\sigma(k-1)}}\big(W(s_k)-W(s_{k-1})\big)\\ &= \sum_{\sigma\in\Sigma_m}\int_{\{0\le s_1\le\cdots,\le s_m\le t_j\}}ds_1\cdots ds_m\prod_{k=1}^m p_{s_k-s_{k-1}}(x_{\sigma(k)}-x_{\sigma(k-1)})\end{aligned}$$

where

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\Big\{ -\frac{|x|^2}{2t} \Big\} dt \qquad x \in \mathbb{R}^d$$

is the density function of $W(t)$ and, we follow the convention $s_0 = 0$, $x_{\sigma(0)} = 0$.

Therefore,

$$\begin{aligned} &\mathbb{E}\Big[\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big] \\ &= \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \prod_{j=1}^p \sum_{\sigma\in\Sigma_m} \int_{\{0\le s_1\le\cdots,\le s_m\le t_j\}} ds_1\cdots ds_m \\ &\times \prod_{k=1}^m p_{s_k-s_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)}) \end{aligned} \tag{2.9}$$

Let $\tau_1,\cdots,\tau_p$ be independnet exponential times with parameter 1. We assume the independence between $\{\tau_1,\cdots,\tau_p\}$ and $\{W_1(\cdot),\cdots,W_p(\cdot)\}$. Replacing $t_1,\cdots,t_p$ by $\tau_1,\cdots,\tau_p$ gives

$$\begin{aligned} &\mathbb{E}\Big[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big] \\ &= \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \Big[\sum_{\sigma\in\Sigma_m} \int_0^\infty dt e^{-t} \int_{\{0\le s_1\le\cdots,\le s_m\le t\}} \\ &\times ds_1\cdots ds_m \prod_{k=1}^m p_{s_k-s_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)})\Big]^p \\ &= \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \Big[\sum_{\sigma\in\Sigma_m} \prod_{k=1}^m \int_0^\infty e^{-t} p_t(x_{\sigma(k)} - x_{\sigma(k-1)}) dt\Big]^p \\ &= \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \Big[\sum_{\sigma\in\Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})\Big]^p \end{aligned} \tag{2.10}$$

where the second step follows from the identity

$$\begin{aligned} &\int_0^\infty dt e^{-t} \int_{\{0\le s_1\le\cdots,\le s_m\le t\}} ds_1\cdots ds_m \prod_{k=1}^m \varphi_k(s_k - s_{k-1}) \\ &= \prod_{k=1}^m \int_0^\infty e^{-t}\varphi_k(t) dt \end{aligned} \tag{2.11}$$

In the next section, we shall establish that

$$\lim_{m\to\infty} \frac{1}{m} \log \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \Big[\frac{1}{m!} \sum_{\sigma\in\Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})\Big]^p = p\log\rho \tag{2.12}$$

Or

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\mathbb{E}\left[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\right]=p\log\rho \tag{2.13}$$

We now prove the upper bound of (2.8). First notice that $\tau_{\min}=\min\{\tau_1,\cdots,\tau_p\}$ is exponential with parameter p. By (2.2),

$$\begin{aligned}&\mathbb{E}\left[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\right]\\ &\geq \mathbb{E}\left[\alpha\big([0,\tau_{\min}]^p\big)^m\right]=\mathbb{E}\,\tau_{\min}^{\frac{2p-d(p-1)}{2}m}\mathbb{E}\left[\alpha\big([0,1]^p\big)^m\right]\\ &=p^{-\frac{2p-d(p-1)}{2}m-1}\Gamma\Big(1+\frac{2p-d(p-1)}{2}m\Big)\mathbb{E}\,\alpha\big([0,1]^p\big)^m.\end{aligned}$$

Thus

$$\mathbb{E}\,\alpha\big([0,1]^p\big)^m\leq p^{\frac{2p-d(p-1)}{2}m+1}\Gamma\Big(1+\frac{2p-d(p-1)}{2}m\Big)^{-1}\mathbb{E}\left[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\right]$$

By Stirling formula and (2.13),

$$\begin{aligned}&\limsup_{m\to\infty}\frac{1}{m}\log(m!)^{-\frac{d(p-1)}{2}}\mathbb{E}\left[\alpha\big([0,1]^p\big)^m\right]\\ &\leq p\log\rho+\frac{2p-d(p-1)}{2}\log\frac{2p}{2p-d(p-1)}\end{aligned} \tag{2.14}$$

We now prove the lower bound of (2.8). Let $t_1,\cdots,t_p>0$. By (2.9)

$$\begin{aligned}&\mathbb{E}\left[\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\right]\\ &\leq\prod_{j=1}^p\bigg\{\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\bigg[\sum_{\sigma\in\Sigma_m}\int_{\{0\leq s_1\leq\cdots,\leq s_m\leq t_j\}}ds_1\cdots ds_m\\ &\times\prod_{k=1}^m p_{s_k-s_{k-1}}(x_{\sigma(k)}-x_{\sigma(k-1)})\bigg]^p\bigg\}^{1/p}\\ &=\prod_{j=1}^p\bigg(\mathbb{E}\left[\alpha\big([0,t_j]^p\big)^m\right]\bigg)^{1/p}\end{aligned}$$

So we have

$$
\begin{aligned}
&\mathbb{E}\Big[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big]\\
&=\int_0^\infty\cdots\int_0^\infty dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\mathbb{E}\Big[\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)^m\Big]\\
&\le\int_0^\infty\cdots\int_0^\infty dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\prod_{j=1}^p\bigg(\mathbb{E}\Big[\alpha\big([0,t_j]^p\big)^m\Big]\bigg)^{1/p}\\
&=\bigg\{\int_0^\infty e^{-t}\bigg(\mathbb{E}\Big[\alpha\big([0,t]^p\big)^m\Big]\bigg)^{1/p}dt\bigg\}^p\\
&=\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\bigg\{\int_0^\infty t^{\frac{2p-d(p-1)}{2p}m}e^{-t}dt\bigg\}^p\\
&=\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\Big[\Gamma\Big(\frac{2p-d(p-1)}{2p}m+1\Big)\Big]^p
\end{aligned}
$$

where the fourth step follows from (2.2). Consequently,

$$
\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\ge\Big[\Gamma\Big(\frac{2p-d(p-1)}{2p}m+1\Big)\Big]^{-p}\mathbb{E}\Big[\alpha\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big]
$$

By Stirling formula and (2.13),

$$
\begin{aligned}
&\liminf_{m\to\infty}\frac{1}{m}\log(m!)^{-\frac{d(p-1)}{2}}\mathbb{E}\Big[\alpha\big([0,1]^p\big)^m\Big]\\
&\ge p\log\rho+\frac{2p-d(p-1)}{2}\log\frac{2p}{2p-d(p-1)}
\end{aligned}
\tag{2.15}
$$

Finally, (2.8) follows from (2.14) and (2.15). □