11th Workshop on Stochastic Analysis on Large Scale Interactive Systems

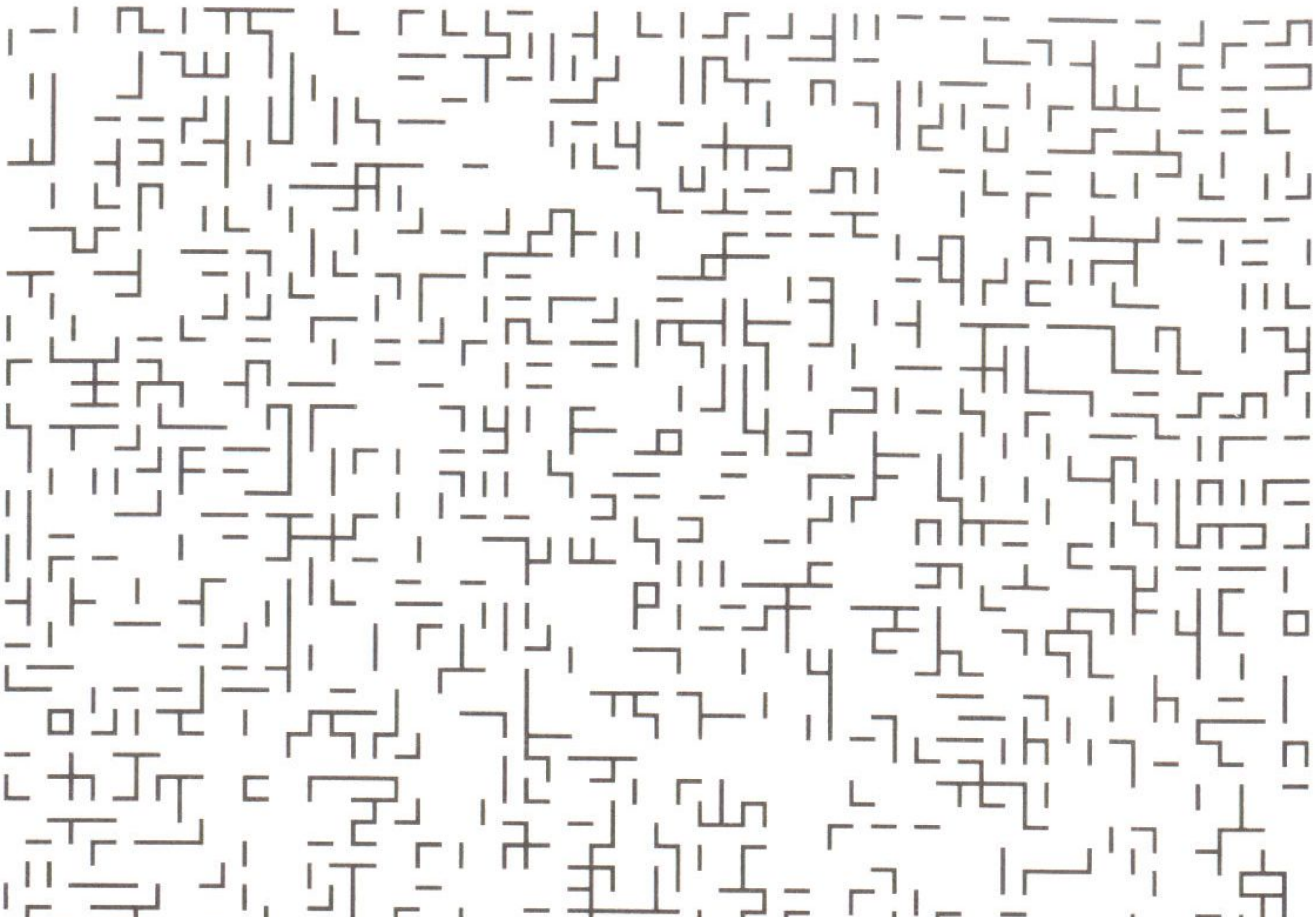
Tokyo University, July 6, 2012

A Survey of Random Walks on a Percolation Cluster

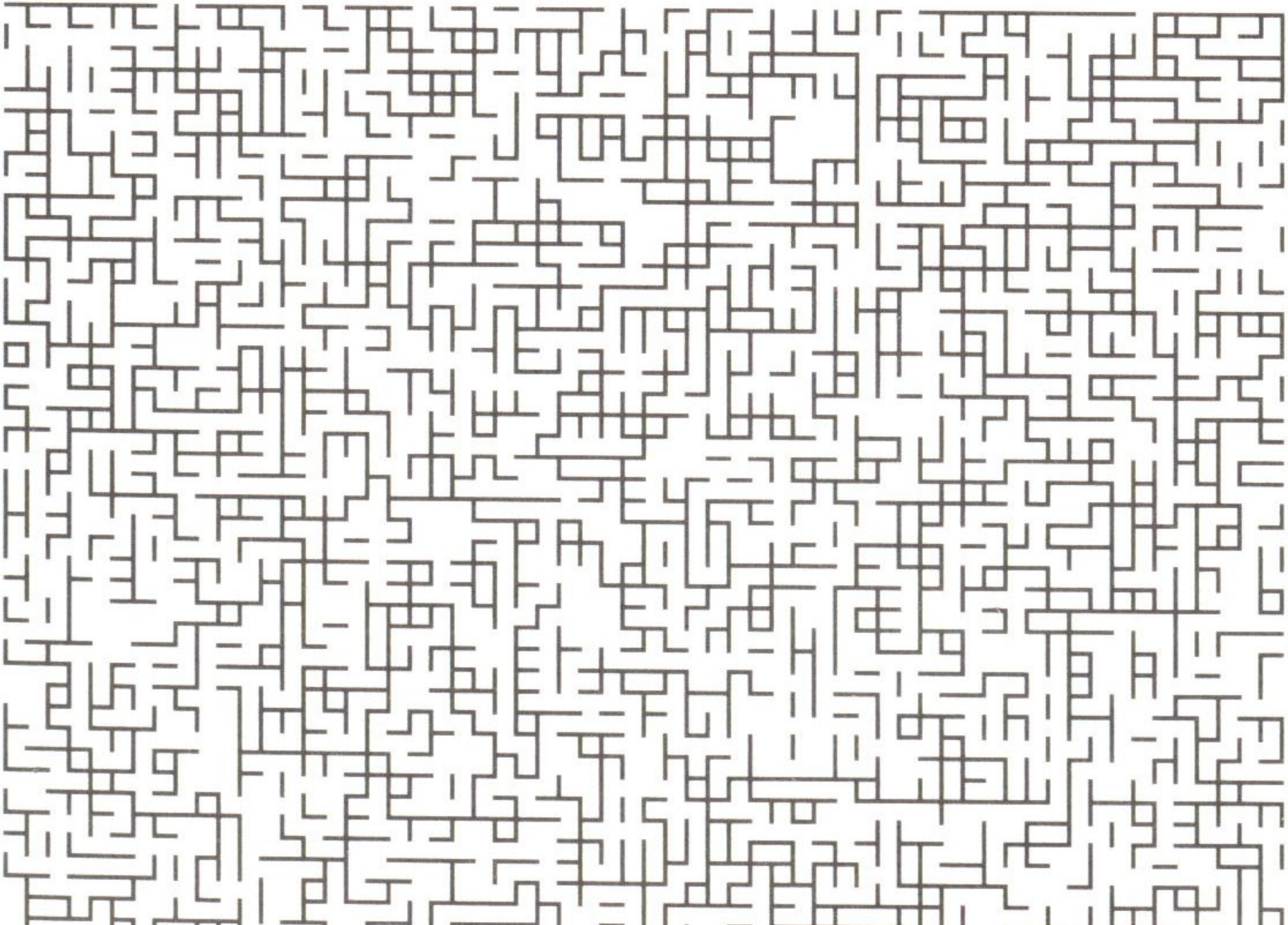
Dayue Chen

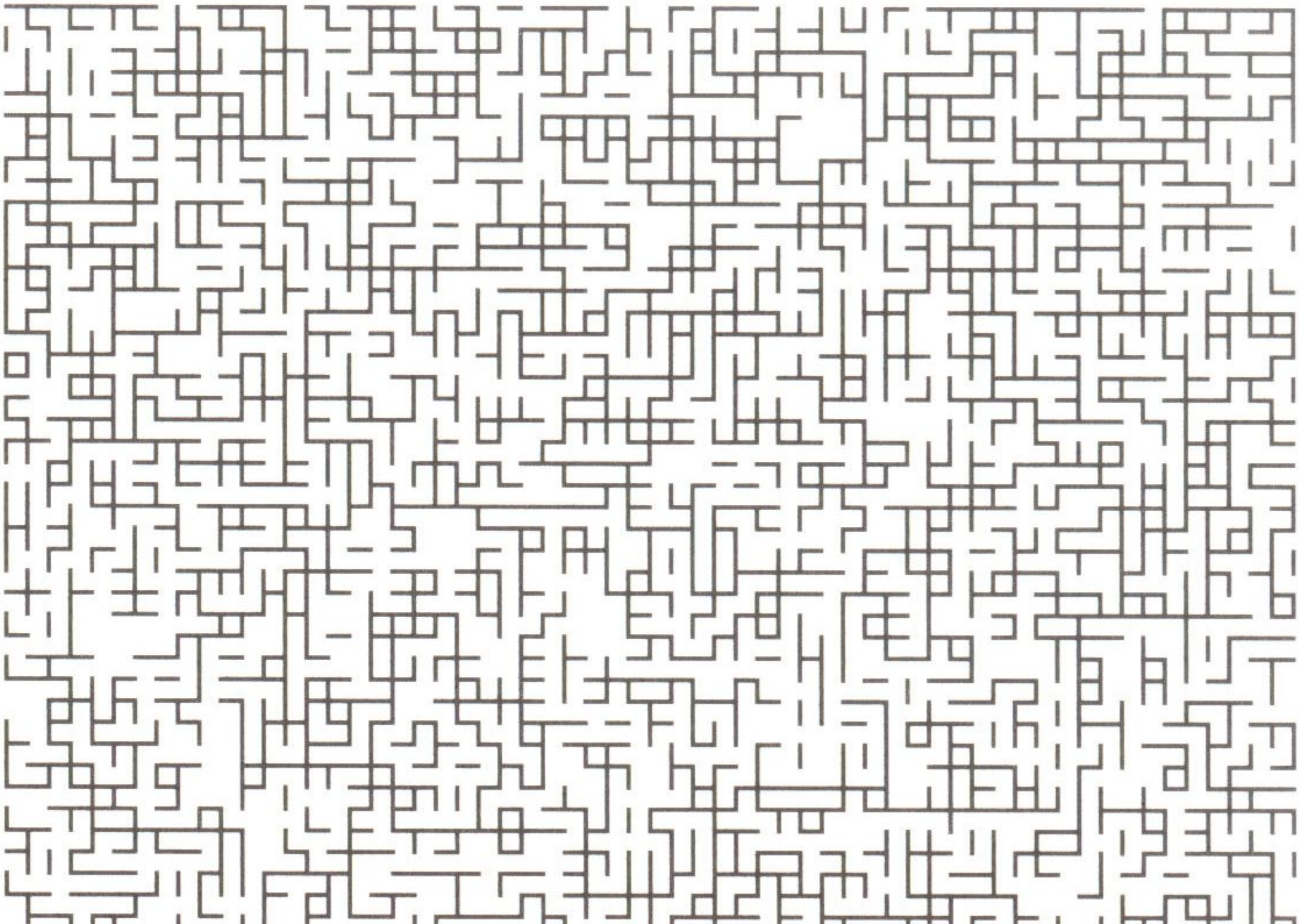
Peking University

Bernoulli bond percolation. Let $\mathcal{G} = (V, E)$ be an infinite graph. Each edge of $\mathcal{G}$ is independently declared *open* with probability p and *closed* with probability $1 - p$.



(a) $p = 0.25$





All open bonds, together with all the vertices, consist of a subgraph.

A connected component is called an open cluster.

$P(\mathcal{C}$ is infinite$)$ is increasing in p.

Critical value $p_c = \inf\{p, P(\mathcal{C}$ is infinite$) > 0\}$.

The critical probability of bond percolation on the square lattice equals 1/2,

H. Kesten, Comm. Math. Phys. 74, (1980) 41-59..

Suppose that $p > p_c$ and that $\mathcal{C}$ is infinite.

What properties are shared by $\mathcal{G}$ and $\mathcal{C}$?

Benjamini, Lyons and Schramm (1999) initiated a systematic study of the properties of a transitive graph $\mathcal{G}$ that are preserved under random perturbations.

Run simple random walks to explore similarities between $\mathcal{G}$ and $\mathcal{C}$.

The (simple) random walk on a graph is a Markov chain, taking values on the vertices of the graph, with equal transition probabilities among adjacent vertices.

Outline:

1. Transience and Recurrence
2. Collisions of two independent walks
3. Speed
4. Anchored Expansion Constant

Part I: Transience and Recurrence

The simple random walk on the infinite cluster of $\mathbb{Z}^{\mathbf{3}}$

The simple random walk on the infinite cluster of $\mathbb{Z}^{\mathbf{3}}$ is transient for sufficiently large $\boldsymbol{p}$.

The simple random walk on the infinite cluster of $\mathbb{Z}^3$ is transient for any $p > p_c$.

G. Grimmett, H. Kesten, and Y. Zhang, **PTRF**, Vol 96 (1993), 33–44.

We say graph $\mathcal{G}$ is recurrent (transient) if the simple random walk on $\mathcal{G}$ is recurrent (transient).

The infinite cluster of $\mathbb{Z}^3$ is transient.

Question: $\mathcal{G}$ is transient $\Longrightarrow$ $\mathcal{C}$ is transient?

Example: Wedge of $\mathbb{Z}^3$ = subgraph induced by the vertices of $\mathbb{Z}^3$

$$\{(x, y, z) | x \geq 0 \text{ and } |z| \leq h(x)\}.$$

Conclusion: The wedge is transient if and only if

$$\sum_j \frac{1}{jh(j)} < \infty.$$

So is the infinite cluster of the wedge for any $p > p_c$.

T. Lyons, **Ann. Probab.**, Vol. 11, (1983) 393-402.

O. Häggerström and E. Mossel, **Ann. Probab.**, 26,(1998), 1212–1231.

O. Angel, I. Benjamini, N. Berger, Y.Peres, **E. J. of Probab.**, Vol.11, (2006).

Example: The Scherk's graph

subgraph of $\mathbb{Z}^3$, same set of vertices, some edges removed.

$(x, y, z) \sim (x', y', z')$ if

either $x = x'$ and $|y - y'| + |z - z'| = 1$,

or $z = z' = 0$ and $|x - x'| + |y - y'| = 1$.

Conclusion: The Scherk's graph is transient.

So is the infinite cluster when $p > 1/2$;

However, the infinite cluster is recurrent when $p_c < p < 1/2$.

S. Markvorsen, S. McGuinness & C. Thomassen, **Proc. London Math. Soc.**, Vol. 64, (1992) 1-20.

D. Chen, **J. Applied Probab.,** Vol.38, No.4, (2001), 828-940.

Proving (i) by constructing a transient subgraph and

(ii) by the Dirichlet Principle. D. Griffeath & T.M. Liggett, **Ann. Probab**. Vol.10, (1982), 881-895,.

Lemma: Let network $\mathbb{H}$ be obtained by modifying the half-line $\mathbb{Z}^+$ as follows. The vertices of $\mathbb{H}$ are positive integers. For all $n \geq 1$ there is an edge joining n and $n+1$ with weight 1, and an edge joining n to $2n$ with weight $n^{-\alpha}$ for some $\alpha < 1$. Then network $\mathbb{H}$ is transient.

Question: Is there a transitive graph which exhibits the dichotomy above the critical point p_c?

Part II: Collisions of Two Random Walks

Question: Will two independent simple random walks on G starting from the same vertex meet infinitely many times a.s. ?

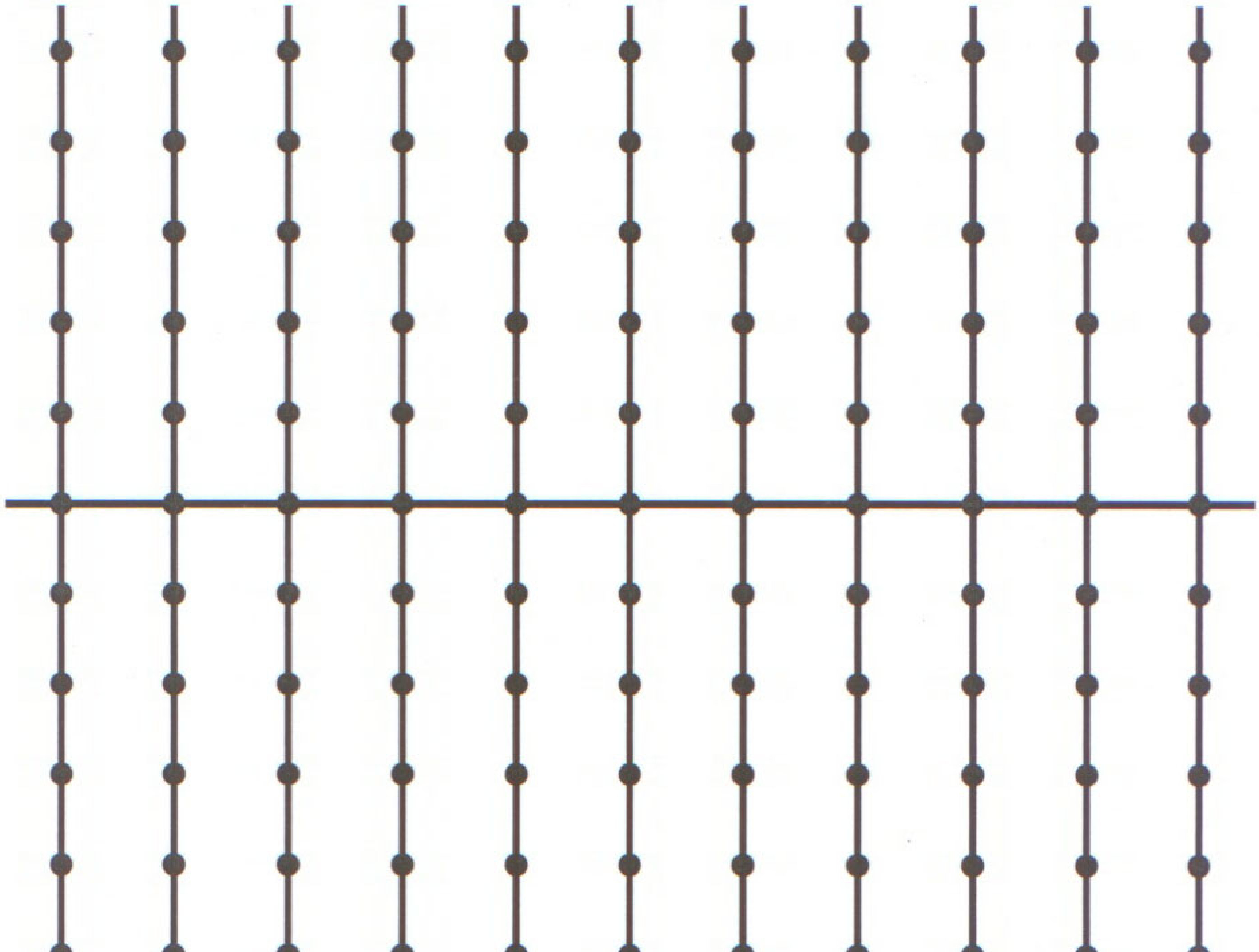
NO for transient (symmetric) random walks.

NO for some recurrent Markov chains.

The dual process of the voter process is a coalescing random walk.

T.M. Liggett, **Trans Amer. Math. Society**, Vol. 198, 201-213, (1974).

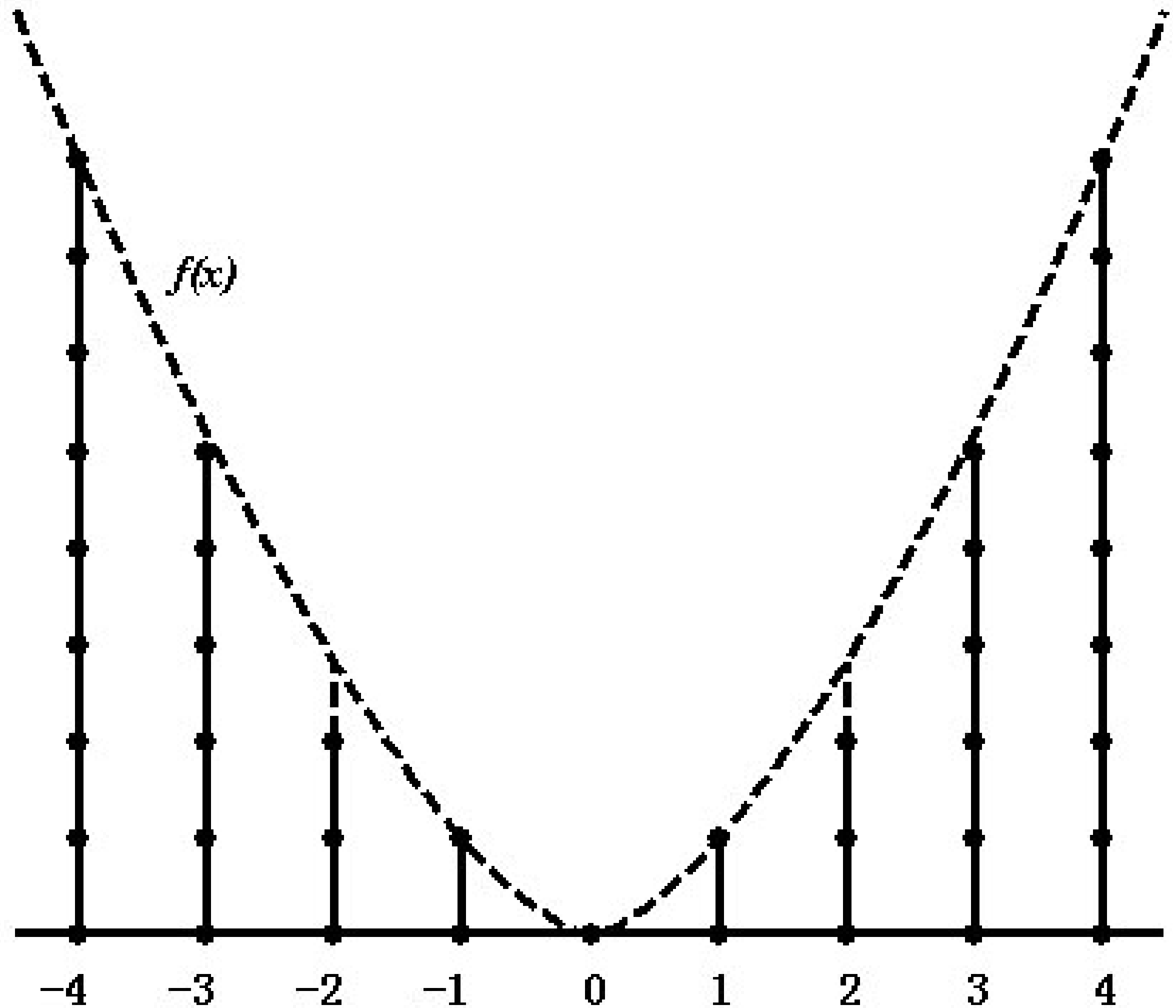
Krishnapur and Peres, **Electronic Comm. in Prob.**, Vol. 9, 72-81,(2004).



Monotonicity fails. $\mathbb{Z} \subset$ Comb $\subset \mathbb{Z}^{\mathbf{2}}$.

A subgraph of recurrent graph is still recurrent.

Could the monotonicity be true in a more restrictive class?



f(x)
−4
−3
−2
−1
0
1
2
3
4

Wedge Comb $\mathcal{G}$ with profile f is the graph with

vertex set $V = \{(x, y) \in Z^2, 0 \leq y \leq f(x)\}$ and

edge set $E = \{[(x, n), (x, m)] : |n - m| = 1, n, m \leq f(x)\}$

$$\cup\{[(x, 0), (y, 0)] : |x - y| = 1\}.$$

Presumably, a phase transition is expected to occur:

$\mathcal{G}$ has the infinite collision property if $f(x)$ increases slowly in x;

$\mathcal{G}$ has the finite collision property otherwise.

Theorem. Let $\mathcal{G}$ be a wedge comb with profile $f(x) = |x|^{\alpha}$.

When $\alpha \leq 1$, two independent simple random walks on $\mathcal{G}$ with continuous time parameter will meet infinitely often.

When $\alpha > 1$, two independent simple random walks on $\mathcal{G}$ with continuous time parameter will meet finitely many times.

Martin Barlow, Yuval Peres & Perla Sousi, *Collisions of Random Walks*, Preprint, 2010.

The case that $\alpha < 1/5$ was investigated earlier.

D. Chen, B. Wei and Fuxi Zhang, **Stat. and Prob. Letters**, Vol. 78, 1742-1747, (2008).

Improvement

$f(x) \leq x \log x \Longrightarrow$ infinitely often *a.s.*

$f(x) \geq x(\log x)^2 \Longrightarrow$ finitely many times *a.s.*

Theorem. Let $\mathcal{G}$ be a wedge comb with profile $f(x)$. If

$$\sum_n \frac{1}{\hat{f}(n)} = \infty,$$

where

$$\hat{f}(n) = 1 \vee \max\{f(i), -n \leq i \leq n\},$$

then two independent simple random walks on G with continuous time parameter will meet infinitely often.

Remark: $f(x)$ is not required to be increasing in x.

Xinxing Chen & D. Chen, **Electronic J. of Probab.**, vol.16, 1341–1355, (2011).

Random Media!

We believe that an open cluster of the Bernoulli bond percolation on $\mathcal{G}$ should resemble the original graph $\mathcal{G}$.

Fact: the simple random walk on the infinite cluster of the Bernoulli bond percolation in $\mathbb{Z}^d$ is transient if $d \geq 3$ (Grimmett, Kesten & Zhang).

Theorem A. Consider $\mathbb{Z}^2$ and let $p > 1/2$. There exists $\Omega_0 \subseteq \Omega$ with $\mathrm{P}_p(\Omega_0) = 1$. Let $\omega \in \Omega_0$ and $x \in \mathcal{C}_\infty(\omega)$. If $X = (X_t)$ and $Y = (Y_t)$ are two independent continuous-time simple random walks starting from x on $\mathcal{C}_\infty(\omega)$, then

$$\mathrm{P}(X_t = Y_t \text{ infinitely often }) = 1.$$

Namely, there is an infinite sequence $\{t_1, s_1, t_2, s_2, \cdots\}$ such that $t_1 < s_1 < t_2 < s_2 < \cdots$, $X_{t_i} = Y_{t_i}$ and $X_{s_i} \neq Y_{s_i}$ for all $i \geq 1$.

Martin Barlow, Yuval Peres & Perla Sousi, *Collisions of Random Walks*, Preprint, 2010.

X. Chen and D. Chen, *Science China Mathematics*, vol **53**, 1971–1978, (2010).

Another formulation: random conductance. $(\mu_e, e \in E_d)$ are i.i.d.

Bernoulli bond percolation if $P(\mu_e = 1) = 1 - P(\mu_e = 0) = p$.

Theorem B. Let $d = 2$. Suppose that $(\mu_e, e \in E_d)$ are i.i.d. and $\mu_e \geq 1$ P-a.s. There exists $\Omega_0 \subseteq \Omega$ with $\mathrm{P}(\Omega_0) = 1$. Let $\omega \in \Omega_0$ and $\mathbb{P}_\omega$ denote the probability conditional on the environment. If $\{X_t\}$ and $\{Y_t\}$ are two independent variable speed random walks starting from x and y respectively, then

$$\mathbb{P}_\omega(X_t = Y_t \text{ for some } t \geq 1) = 1.$$

The proof of both theorems is based on the following key lemma.

Lemma: Let $\omega \in \Omega_0$ and $x, y \in \mathcal{C}_\infty(\omega)$. Let $X = (X_t)$ be a continuous time simple random walk starting from x on $\mathcal{C}_\infty(\omega)$, $Y = (Y_t)$ a continuous time simple random walk starting from x. If X and Y are independent, then

$$\mathrm{P}(X_t = Y_t \text{ for some } t > 1) \geq \delta,$$

where δ is a strictly positive constant and dependent on p at most.

This lemma is in turn proved by the second moment method. Define

$$H := \int_{t_0}^{T} 1_{\left\{X_s = Y_s \in M_{[s^{1/2}]}\right\}} ds.$$

Then

$$\mathrm{P}(X_t = Y_t \text{ for some } t > 1) \geq \mathrm{P}(H > 0) \geq \frac{(\mathrm{E}H)^2}{\mathrm{E}H^2}.$$

Need to show

$$\mathbf{E}H = \int_{t_0}^{T} \mathrm{P}\left(X_s = Y_s \in M_{[s^{1/2}]}\right) ds \geq \frac{c_1 c_2^2 e^{-2c_3}}{32} \log T.$$

$$\begin{aligned}
\mathbf{E}H^2 &\leq 2 \int_{t_0}^{T} \left(\sum_{z \in M_{[t^{1/2}]}} c_3^2 t^{-2} (2 + 4c_3^2 c_4^{-1}) \log T \right) dt \\
&\leq 2 \int_{t_0}^{T} \left(c_3^2 t^{-1} (2 + 4c_3^2 c_4^{-1}) \log T \right) dt \\
&\leq (4c_3^2 + 8c_3^4 c_4^{-1}) (\log T)^2.
\end{aligned}$$

$$\frac{(\mathbf{E}H)^2}{\mathbf{E}H^2} \geq \frac{c_1^2 c_2^4 e^{-4c_3}}{10000(c_3^2 + c_3^4 c_4^{-1})}.$$

Theorem Let $p > p_c$. There exits $\Omega_1 \subset \Omega$ with $P_p(\Omega_1) = 1$, and random variables $\{S_x, x \in \mathbb{Z}^d\}$, such that $S_x(\omega) < \infty$ for each $\omega \in \Omega_1, x \in \mathcal{C}_\infty(\omega)$. There exist constants $c_i = c_i(d,p)$ such that for $x, y \in \mathcal{C}_\infty(\omega), t \geq 1$ with

$$S_x(\omega) \vee |x - y|_1 \leq t,$$

the transition density $q_t^\omega(x,y) = P_x(Y_t = y)/\mu(y)$ of Y satisfies:

$$c_1 t^{-d/2} e^{-c_2 |x-y|_1^2/t} \leq q_t^\omega(x,y) \leq c_3 t^{-d/2} e^{-c_4 |x-y|_1^2/t}.$$

M.T. Barlow. **Ann. Prob.**, vol. 32, 3024-3084, (2004).

M.T. Barlow & J.-D. Deuschel, **Ann. Prob.**, vol.38, 234–276,(2010).

Theorem. Let $d \geq 2$ and $\sigma \in (0,1)$. There exist random variables S_x, $x \in Z^d$, such that

$$P(S_x(\omega) \geq n) \leq c_1 \exp(-c_2 n^{\sigma}), \tag{1}$$

and constants c_i (depending only on d and the distribution of μ_e) such that the following hold.

If $|x-y|^2 \vee t \geq S_x^2$, then

$q_t^{\omega}(x,y) \leq c_3 t^{-d/2} \mathrm{e}^{-c_4 |x-y|^2/t}$ when $t \geq |x-y|$,

$q_t^{\omega}(x,y) \leq c_3 \exp(-c_4 |x-y|(1 \vee \log(|x-y|/t)))$ when $t \leq |x -$

If $t \geq S_x^2 \vee |x-y|^{1+\sigma}$, then

$$q_t^{\omega}(x,y) \geq c_5 t^{-d/2} \mathrm{e}^{-c_6 |x-y|^2/t}. \tag{2}$$

Application to the Voter Model.

The underlying graph is $\mathbb{Z}^d$ and

The measures δ_0 and δ_1 of point mass are invariant.

Theorem. Let $d = 1$ or 2. Suppose that (μ_e) are i.i.d. and $\mu_e \geq 1$ $\mathbf{P}$-a.s. There exists $\Omega_0 \subseteq \Omega$ with $\mathbf{P}(\Omega_0) = 1$. For any $\omega \in \Omega_0$, the voter model has only two extremal invariant measures: δ_0 and δ_1.

Remark: I. Ferreira, *The probability of survival for the biased voter model in a random environment*, **Stochastic Processes and Their Appl.**, vol.34, (1990), 25–38.

Part III: Speed

$$\lim_{n\to\infty} \frac{|X_n|}{n}$$

if exists, is called the **speed** of SRW $\{X_n\}$,

where $|x|$ is the graphic distance from x to o.

Example: SRW on $\mathbb{Z}^d$ has zero speed.

Theorem. The speed of the simple random walk on an infinite cluster of a transitive graph $\mathcal{G}$ exits, and is positive if Cheeger constant of $\mathcal{G}$ is positive.

I. Benjamini, R. Lyons and O. Schramm (1999), *Percolation perturbation in potential theory and random walk.* In Random Walks and Discrete Potential Theory, 56-84, Cambridge Univ. Press.

Cheeger constant

$$\iota(\mathcal{G}) = \inf \frac{|\partial S|}{|S|}$$

where the infinium is over all finite connected subsets $S \subset V(\mathcal{G})$, $|S|$ the cardinality of S. ∂S the set of boundary edges.

Conjecture (BLS). If $\mathcal{G}$ is a Cayley graph on which simple random walk has positive (zero) speed, then a.s., simple random walk on each infinite cluster of p-Bernoulli percolation has positive (zero) speed.

1. Cheeger constant > 0;
2. sub-exponential growth ($\Longrightarrow$ Cheeger constant $= 0$);
3. Exponential growth and the Cheeger constant $= 0$.

Cheeger constant $> 0 \Longrightarrow$ speed > 0.

H. Kesten, **Trans. AMS**, Vol. 92, 336-354 (1959).

The Case of sub-exponential growth.

$$\limsup |\{x \in V(\mathcal{G}) : |x| \leq n\}|^{1/n} = 1$$

Theorem. If a graph $\mathcal{G}$ has sub-exponential growth, then simple random walk on $\mathcal{G}$ has zero speed.

N. Th. Varopoulos,**Bull. Sci. Math.**, Vol. 109, 225-252,(1985).

Corollary. The SRW on any open cluster of a graph with sub-exponential growth has zero speed.

How about an Cayley graph with exponential growth and $\iota(G) = 0$?

e.g. **Lamplighter groups $\mathcal{G}_d$**

A vertex of $\mathcal{G}_d$ can be identified as

$$(m, \eta) \in \mathbb{Z}^d \times \{\text{finite subsets of } \mathbb{Z}^d\}.$$

Heuristically, $\mathbb{Z}^d$ is the set of lamps, η is the set of lamps which are on, and m is the position of the lamplighter, or "marker".

Each vertex of $\mathcal{G}_d$ has degree $2d + 1$.

Example. $d = 1$, the neighbors of (m, η) are

$$(m+1, \eta), (m-1, \eta) \text{ and } (m, \eta\Delta\{m\}),$$

where $\eta\Delta\{m\}$ is $\eta \setminus \{m\}$ if $m \in \eta$, and is $\eta \cup \{m\}$ if $m \notin \eta$.

Theorem. The simple random walk on the Cayley graph $\mathcal{G}_d$ of the lamplighter group has speed zero for $d = 1, 2$ and has positive speed for $d \geq 3$.

V.A. Kaimanovich and A.M. Vershik, **Ann. Probab.**, Vol.11, 457-490, (1983).

Theorem.

(1) Let $d = 1$ or 2. Then the simple random walk on the infinite cluster of $\mathcal{G}_d$ has zero speed, a.s.

(2) Suppose that $d \geq 3$. If $p > p_c(\mathbb{Z}^d) > p_c(\mathcal{G}_d)$, then the simple random walk on the infinite cluster of Bernoulli bond percolation in $\mathcal{G}_d$ has positive speed $a.s.$ on the event that o is in the infinite cluster.

D. Chen & Y. Peres, with an appendix by Gabor Pete, **Ann. Prob.**, Vol.32, No.4, (2004), 2978-2995.

Partially verifies the BLS Conjecture that the positivity of the speed is preserved.

Generalization: Replace $\mathbb{Z}^d$ by the Cayley graph of a finitely generated infinite group G. Replace $\{0,1\}$ by the Cayley graph of a finite group F.

$W = G \ltimes \sum_{x \in G} F$ is a semi-direct product of G with the direct sum of copies of F indexed by G.

Vertices of W are identified as $\{(m,\eta) : m \in V(G), \eta \in \sum_{x \in G} F\}$.

(m,η) and (m_1,ξ), are neighbors if either

(i) $m = m_1$, $\eta(x) = \xi(x)$ for all $x \neq m$, and $\eta(m)$ is a neighbor of $\xi(m)$ in F, or

(ii) $\eta = \xi$, m and m_1 are neighbors in G.

Theorem 1'. Suppose that G is a recurrent Cayley graph and that F is the Cayley graph of a finite group. Then the simple random walk on the infinite cluster of supercritical Bernoulli bond percolation in $W = G \ltimes \sum_{x \in G} F$ has zero speed a.s.

Theorem 2'. Let $0 < p < 1$. Suppose that the infinite cluster of p-Bernoulli bond percolation on the Cayley graph G is transient and that F is the Cayley graph of a finite group. Then the simple random walk on the infinite cluster of p-Bernoulli bond percolation in $W = G \ltimes \sum_{x \in G} F$ has positive speed a.s.

D. Chen & Y. Peres, with an appendix by Gabor Pete, **Ann. Prob.**, Vol.32, No.4, (2004), 2978-2995.

Question (posed by Yueyun Hu):

Is the speed of the simple random walk on an infinite cluster of a (transitive) graph increasing in p?

No for the binary tree with pipes. The speed can be calculated and is not monotone.

$$\frac{1}{3}\,\frac{(2p-1)^2}{p^2+(1-p)^2}\,\frac{1-p}{(2p^3-6p^2+3p+3)}.$$

Yes for regular trees, and for Galton-Watson tree.

The question remains largely unanswered. $\mathbb{Z}\times T_d$?

Galton-Watson tree is a sample point of a Galton-Watson process, which is uniquely determined by the offspring distribution $\{p_0, p_1, p_2, \ldots\}$. Let q be the extinction probability, i.e., $q = \sum_k p_k q^k$. Then

$$\text{Speed } = \sum_{k=0}^{\infty} p_k \frac{k-1}{k+1} \frac{1-q^{k+1}}{1-q^2}.$$

Lyons, R., Pemantle, R. & Peres, Y., **Ergodic Theory Dynamical Systems**, vol. 15, 593–619, (1995).

Theorem. The speed of the simple random walk on an infinite cluster of a Galton-Watson tree is increasing in p. Furthermore it is differentiable in $(1/m, 1)$.

D. Chen & Fuxi Zhang, **Acta Mathematica Sinica,** English Series, Vol. 23, 1949-1954, (2007).

Part IV: Anchored Expansion Constant

Cheeger constant

$$\iota(\mathcal{G}) := \inf\left\{\frac{|\partial S|}{|S|} : S \subset V(\mathcal{G}),\ S \text{ is connected},\ 1 \le |S| < \infty\right\}$$

Bad News: The Cheeger constant of an open cluster is 0.

Anchored Expansion Constant $\iota^*_E(\mathcal{G})$

$$\liminf_{n\to\infty}\left\{\frac{|\partial S|}{|S|} : o \in S \subset V(\mathcal{G}),\ S \text{ is connected},\ n \le |S| < \infty\right\}$$

Independent of the choice of the basepoint o and $\iota_E(\mathcal{G}) \le \iota^*_E(\mathcal{G})$.

Theorem.

Let $\mathcal{G}$ be a bounded degree graph with $\iota_E^*(\mathcal{G}) > 0$. Then the simple random walk $\{X_n\}$ in $\mathcal{G}$, started at o, satisfies $\liminf_{n\to\infty} |X_n|/n > 0$ a.s. and there exists $C > 0$ such that

$P[X_n = o] \leq \exp(-Cn^{1/3})$ for all $n \geq 1$.

B. Virág, **Geom. Funct. Anal.**, Vol. 10, 1588-1605, (2000).

Theorem. Consider p-Bernoulli percolation on a graph G with $\iota_E^*(G) > 0$. If $p < 1$ is sufficiently close to 1, then almost surely on the event that the open cluster $\mathcal{C}$ containing o is infinite, we have $\iota_E^*(\mathcal{C}) > 0$.

Our proof shows the conclusion holds for all $p > 1 - h/(1+h)^{1+\frac{1}{h}}$ where $h = \iota_E^*(\mathcal{G})$.

A refinement of the argument by Gabor Pete shows the conclusion holds for all $p > 1/(\iota_E^*(\mathcal{G}) + 1)$.

Remark: Theorem 2 of Benjamini and Schramm (1996) states that $p_c(\mathcal{G}) \leq 1/(\iota_E(\mathcal{G}) + 1)$,

but their proof yields the stronger inequality $p_c(\mathcal{G}) \leq 1/(\iota_E^*(\mathcal{G}) + 1)$.

$\partial^V S$ = the set of vertices in S^c having a neighbor in S.

The vertex version of anchored expansion constant

$$\iota_V^*(\mathcal{G}) := \lim_{n\to\infty} \inf\left\{\frac{|\partial^V S|}{|S|} : o \in S \subset V(\mathcal{G}),\ S \text{ is connected},\ n \le |S| < \infty\right\}.$$

Theorem (Gabor Pete). *Suppose that $\iota_V^*(\mathcal{G}) > 0$. Consider p-Bernoulli site percolation on a graph G. If $p > 1/(\iota_V^*(\mathcal{G}) + 1)$, then almost surely on the event that the open cluster $\mathcal{C}$ containing o is infinite, it satisfies $\iota_V^*(\mathcal{C}) > 0$.*

Then the corresponding form of the arguments needs no modification.

Questions: Suppose that $\iota^*_E(\mathcal{G}) > 0$.

Is the anchored expansion constant of a cluster positive for all $p > p_c$?

Yes for regular trees.

Is the anchored expansion constant of a cluster monotone in p?

No answer even for regular trees.

Thank You

E-mail: dayue@pku.edu.cn