The Voter Model in a Random Environment in $\mathbb{Z}^d$

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Outline

Joint work with Zhichao Shan, submitted to SPA.

1. a new result of the voter model
2. collision of two random walks

Introduction: the model

The voter model is an interacting particle system.

There is a voter in every site of V.

Every voter can have either of two political positions, denoted by 0 or 1, and constantly updates his political position.

The voter at x updates his political position at a random time, following the exponential distribution with parameter $\sum_z \mu_{xz}$.

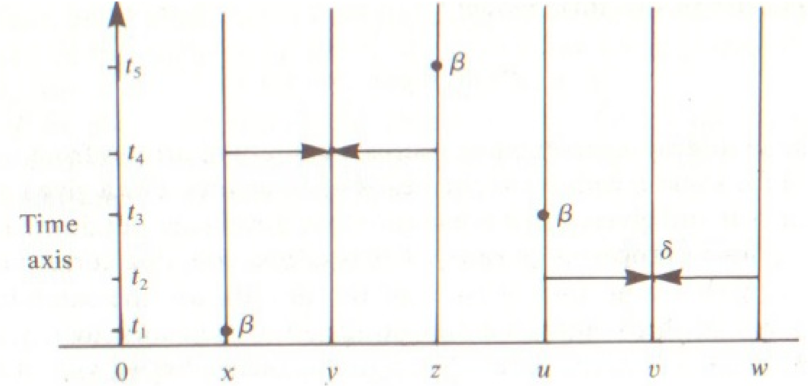
At the time of update the voter takes the position of his neighbor y with probability $\mu_{xy}/(\sum_z \mu_{xz})$.

Let $\eta(x)$ be the political position of voter x and the collection $\eta = \{\eta(x); x \in V\}$ be an element of $\{0,1\}^V$.

Introduction: construction

The voter model can be constructed either by the Markovian semigroup or by the graphical representation, see Liggett(85).

The second approach not only works for all positive μ_{xy}, but also clearly exhibits the duality relation.



Introduction: limit behavior

When the underlying graph is $\mathbb{Z}^d$ and $\mu_e \equiv 1$, this model is well studied.

There are two invariant measures δ_0 and δ_1, and if $d \leq 2$, all other invariant measures are linear combinations of δ_0 and δ_1.

New Result

The underlying graph is $\mathbb{Z}^d$ and
$\{\mu_e, e \in E_d\}$ are i.i.d. random variables satisfying $\mu_e \geq 1$.

The measures δ_0 and δ_1 of point mass are invariant.

Theorem

Let $d = 1$ or 2. Suppose that (μ_e) are i.i.d. and $\mu_e \geq 1$ $\mathbb{P}$-a.s. There exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$. For any $\omega \in \Omega_0$, the voter model has only two extremal invariant measures: δ_0 and δ_1.

Remark: I. Ferreira, The probability of survival for the biased voter model in a random environment, *Stochastic Processes and Their Appl.*, vol.34, (1990), 25–38.
I. Ferreira, Cluster for the Voter Model in a Random Environment and the probability of survival for the Biased Voter Model in a Random Environment, 1988

Duality Relation

For $\eta \in \{0,1\}^{\mathbb{Z}^2}$ and a finite set $A \subseteq \mathbb{Z}^2$, define

$$H(\eta, A) = \mathbf{1}_{\{\eta(z)=1 \text{ for all } z \in A\}} .$$

If there are two Markov processes, $\{\eta_t\}$ and $\{A_t\}$, such that

$$\mathbb{E}_\omega^\eta H(\eta_t, A) = \mathbb{E}_\omega^A H(\eta, A_t),$$

Then we say $\{\eta_t\}$ and $\{A_t\}$ are dual to one another.

can be a dual of the voter model.

taking values on the set of all finite sets of vertices of $\mathbb{Z}^d$.

Intuitively, image there is a particle at each $x \in A$ of the initial state. Each particle performs a variable speed random walk, independent of each other until they meet. Once two particles collide, they coalesce into one particle. Then A_t is the set of locations of all particles at time t.

$\{A_t\}$ and the voter model can be constructed by the same graphical representation.

$$\mathbb{P}^{\eta}_{\omega}(\eta_t(x) = 1 \text{ for all } x \in A) = \mathbb{P}^{A}_{\omega}(\eta(x) = 1 \text{ for all } x \in A_t)\ .$$

Reducing to the collision problem

If the initial state is a singleton and if singleton $\{x\}$ is identified with vertex x, then the coalescing Markov chain is exactly a continuous-time random walk in a random environment (or *variable speed random walk* or the *random conductance model*).

Theorem

Let $d = 2$. Suppose that $(\mu_e, e \in E_d)$ are i.i.d. and $\mu_e \geq 1$ $\mathbb{P}$-a.s. There exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$. Let $\omega \in \Omega_0$ and $\mathbb{P}_\omega$ denote the probability conditional on the environment. If $\{X_t\}$ and $\{Y_t\}$ are two independent variable speed random walks starting from x and y respectively, then $\mathbb{P}_\omega(X_t = Y_t \text{ for some } t \geq 1) = 1$.

$\Longrightarrow$ Starting from a doubleton (or a finite set), a coalescing Markov chain will eventually becomes a singleton.

$\Longrightarrow$ Any invariant measure of the voter model is a linear combinations of δ_0 and δ_1.

Collisions of Random Walks

The dual relation lead Liggett in 1974 to first consider collisions of two Markov chains, and to discover an example that two recurrent Markov chain may not necessarily meet each other.

Krishnapur and Peres (2004) found a simple example.

A recent paper by Barlow, Peres and Sousi.

Xinxing Chen and I also made contributions.

Many progresses, yet some questions remain open.

Part II

Collisions of Random Walks in a random environment

The Proof: The Main Lemma

Theorem

Let $d = 2$. Suppose that $(\mu_e, e \in E_d)$ are i.i.d. and $\mu_e \geq 1$ $\mathbb{P}$-a.s. There exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$. Let $\omega \in \Omega_0$ and $\mathbb{P}_\omega$ denote the probability conditional on the environment. If $\{X_t\}$ and $\{Y_t\}$ are two independent variable speed random walks starting from x and y respectively, then $\mathbb{P}_\omega(X_t = Y_t \text{ for some } t \geq 1) = 1$.

can be deduced by the 2nd Borel-Cantelli Lemma from

Lemma

Under the same assumption,

$$P_\omega(X_t = Y_t \text{ for some } t \geq 1) \geq \delta > 0,$$

where δ is a constant independent of ω, x and y.

The Proof: From Lemma to Theorem

Let $\delta > 0$ be defined as before. Fix $\omega \in \Omega_0$. By Lemma 4, there exists a function $f : V_2 \times V_2 \mapsto [1, \infty)$, such that for all x, $y \in V_2$,

$$P_\omega^{(x,y)}(X_t = Y_t \text{ for some } 1 < t \leq f(x,y)) \geq \frac{\delta}{2} \, . \qquad (1)$$

Set $x_0 = x$, $y_0 = y$ and $t_0 = 0$. Define x_i, y_i and t_i inductively for $i \geq 1$ as follows. Suppose that x_i, y_i and t_i are already defined. Let $\{\tilde{X}_t\}$ and $\{\tilde{Y}_t\}$ be two independent continuous-time random walks starting from x_i and y_i. Define

$$x_{i+1} := \tilde{X}(f(x_i, y_i)), \quad y_{i+1} := \tilde{Y}(f(x_i, y_i)), \text{ and } t_{i+1} := t_i + f(x_i, y_i).$$

Define $\mathcal{E}_i$ to be the event that $X_t = Y_t$ for some $t \in (t_i + 1, t_{i+1}]$ for $i \geq 0$. By (1) and the strong Markov property,

$$P_\omega(\mathcal{E}_i | X_t, Y_t, t \leq t_i) = P_\omega^{(x_i,y_i)}(\tilde{X}_t = \tilde{Y}_t \text{ for some } 1 < t \leq f(x_i, y_i)) \geq \frac{\delta}{2} \, .$$

By the second Borel-Cantelli lemma, $P_\omega(\mathcal{E}_i$ infinitely often$)=1$.

$$P_\omega(X_t = Y_t \text{ infinitely often}) \geq P_\omega(\mathcal{E}_i \text{ infinitely often}) = 1.$$

Proof of the Lemma based on another lemma

Define the random variable

$$H := \int_{t_0}^{T} \frac{1}{\mu(X_s)\mu(Y_s)} 1_{\{X_s = Y_s \in M(s^{1/2})\}} \, \mathrm{d}s \ .$$

where t_0 and T are constants to be specified later, as well as the subset $M(n)$.

Lemma

$E_\omega H \geq c_9 \log T$.
$E_\omega H^2 \leq (4\pi c_3^2 + 2\pi^2 c_3^4 / c_4)(\log T)^2$.

$$P_\omega(X_t = Y_t \ \text{ for some } t > 0) \geq P_\omega(H > 0) \geq \frac{(E_\omega H)^2}{E_\omega H^2}$$

$$\geq \frac{(c_9 \log T)^2}{(4\pi c_3^2 + 2\pi^2 c_3^4 / c_4)(\log T)^2} = \frac{c_9^2 c_4}{4\pi c_3^2 c_4 + 2\pi^2 c_3^4} > 0.$$

Key Ingredient: Heat Kernel Est. by Barlow & Deuschel

Theorem

Let $d \geq 2$ and $\sigma \in (0,1)$. There exist random variables S_x, $x \in Z^d$, such that

$$P(S_x(\omega) \geq n) \leq c_1 \exp(-c_2 n^\sigma), \tag{2}$$

and constants c_i (depending only on d and the distribution of μ_e) such that the following hold.
If $|x-y|^2 \vee t \geq S_x^2$, then

$q_t^\omega(x,y) \leq c_3 t^{-d/2} \mathrm{e}^{-c_4|x-y|^2/t}$ *when* $t \geq |x-y|$,
$q_t^\omega(x,y) \leq c_3 \exp(-c_4|x-y|(1 \vee \log(|x-y|/t)))$ *when* $t \leq |x-y|$.

If $t \geq S_x^2 \vee |x-y|^{1+\sigma}$, then

$$q_t^\omega(x,y) \geq c_5 t^{-d/2} \mathrm{e}^{-c_6|x-y|^2/t}. \tag{3}$$

Lemma

Let $A_n(\omega)$ be the random set defined by

$$A_n(\omega) = \{x : |x| \le n, S_x(\omega) \le 2\log n\}.$$

Then almost surely there exists a finite random variable $U(\omega)$ such that $|A_n(\omega)| \ge c_7 n^2$ for any $n \ge U(\omega)$.

For any $x, y \in \mathbb{Z}^2$ set

$$t_0 = [S_x(\omega) \vee S_y(\omega)]^2 + [U(\omega) + (|x| \vee |y|)(1 + 12\pi c_7^{-1})]^2,$$

and $T = \exp(\frac{2}{1+\sigma}\log t_0)$, where σ is given in the previous theorem.

$B_x(r) =$ disk of radius r centered at x,
$M_\omega(n) = B_x(n) \cap B_y(n) \cap A_n(\omega)$.

$|M_\omega(n)| > C_7 n^2/2$ for $n \ge U(\omega) + (|x| \vee |y|)(1 + 12\pi c_7^{-1})$.

$$\begin{aligned} \mathbb{E}_\omega H &= \int_{t_0}^{T} \mathbb{E}_\omega \frac{1}{\mu(X_s)\mu(Y_s)} \mathbf{1}_{\{X_s=Y_s\in M(s^{1/2})\}}\,\mathrm{d}s \\ &= \int_{t_0}^{T} \sum_{z\in M(s^{1/2})} \frac{1}{\mu_z^2}\mathbb{P}_\omega(X_s=z,\,Y_s=z)\,\mathrm{d}s \\ &= \int_{t_0}^{T} \sum_{z\in M(s^{1/2})} q_s^\omega(x,z)q_s^\omega(y,z)\,\mathrm{d}s\ . \end{aligned}$$

Since $z \in M(s^{1/2})$, we have $|x-z|^2 \le s \le T = \exp(\frac{2}{1+\sigma}\log t_0)$.
Thus $s \ge t_0 \ge S_x^2(\omega) \vee |x-z|^{1+\sigma}$.

Theorem

If $s \ge S_x^2 \vee |x-z|^{1+\sigma}$, then $q_s^\omega(x,z) \ge c_5 s^{-d/2} \mathrm{e}^{-c_6|x-z|^2/s}$.

Similarly $s \ge S_y^2(\omega) \vee |y-z|^{1+\sigma}$.

Lower bound of $E_\omega H$ (II)

$$\mathbb{E}_\omega H \geq \int_{t_0}^{T} \sum_{z \in M(s^{1/2})} c_5^2 s^{-2} \exp\left(-c_6 \frac{|x-z|^2}{s} - c_6 \frac{|y-z|^2}{s}\right) \mathrm{d}s$$

$$\geq c_5^2 \mathrm{e}^{-2c_6} \int_{t_0}^{T} \sum_{z \in M(s^{1/2})} s^{-2} \, \mathrm{d}s$$

$$\geq \frac{c_5^2 c_7 \mathrm{e}^{-2c_6}}{2} \int_{t_0}^{T} s^{-1} \, \mathrm{d}s \geq c_9 \log T \ .$$

The 2nd inequality is by the fact that $|x-z|^2 \leq s$ for $z \in M(s^{1/2})$.

and the 3rd inequality by the estimate that $|M(s^{1/2})| \geq c_7 s/2$ since $s^{1/2} \geq t_0^{1/2} \geq U(\omega) + (|x| \vee |y|)(1 + 12\pi c_7^{-1})$.

Upper bound of $E_\omega H^2$

$$
\begin{aligned}
&\mathbb{E}_\omega H^2 \\
=&2\mathbb{E}_\omega \int_{t_0}^{T} \mathrm{d}t \int_t^T \frac{\mathbf{1}_{\{X_t=Y_t\in M(t^{1/2})\}}}{\mu(X_t)\mu(Y_t)} \frac{\mathbf{1}_{\{X_s=Y_s\in M(s^{1/2})\}}}{\mu(X_s)\mu(Y_s)} \mathrm{d}s \\
=&2\int_{t_0}^{T} \mathrm{d}t \int_t^T \mathbb{E}_\omega \sum_{z\in M(t^{1/2})} \sum_{w\in M(s^{1/2})} \frac{1}{\mu_z^2}\mathbf{1}_{\{X_t=Y_t=z\}} \frac{1}{\mu_w^2}\mathbf{1}_{\{X_s=Y_s=w\}}\mathrm{d}s \\
=&2\int_{t_0}^{T} \mathrm{d}t \int_t^T \sum_{z\in M(t^{1/2})} \frac{\mathbb{P}_\omega^{(x,y)}(X_t=Y_t=z)}{\mu_z^2} \sum_{w\in M(s^{1/2})} \frac{\mathbb{P}_\omega^{(z,z)}(X_{s-t}=Y_{s-}}{\mu_w^2} \\
=&2\int_{t_0}^{T} \mathrm{d}t \int_t^T \sum_{z\in M(t^{1/2})} q_t^\omega(x,z)q_t^\omega(y,z) \sum_{w\in M(s^{1/2})} q_{s-t}^\omega(z,w)q_{s-t}^\omega(z,w)\mathrm{d}s \\
\leq&2\int_{t_0}^{T} \mathrm{d}t\Big[\sum_{z\in M(t^{1/2})} q_t^\omega(x,z)q_t^\omega(y,z) \int_0^T \sum_{w\in M((s+t)^{1/2})} (q_s^\omega(z,w))^2\mathrm{d}s\Big] .
\end{aligned}
$$

Upper bound of $E_\omega H^2$

$$\mathbb{E}_\omega H^2 \leq 2\int_{t_0}^{T} \mathrm{d}t \sum_{z\in M(t^{1/2})} q_t^\omega(x,z)q_t^\omega(y,z)\int_0^T \sum_{w\in M((s+t)^{1/2})} (q_s^\omega(z,w))^2\mathrm{d}$$

$$\leq 2\int_{t_0}^{T} \sum_{z\in M(t^{1/2})} (c_3^2 t^{-2})\left((2+\frac{\pi c_3^2}{c_4})\log T\right)\mathrm{d}t$$

$$\leq 2\int_{t_0}^{T} \frac{c_3^2\pi(2+\pi c_3^2/c_4)\log T}{t}\mathrm{d}t \leq (4\pi c_3^2 + \frac{2\pi^2 c_3^4}{c_4})(\log T)^2 \ .$$

if we verify that $q_t^\omega(x,z)q_t^\omega(y,z) \leq c_3^2 t^{-2}$ and

$$\int_0^T \sum_{w\in M((s+t)^{1/2})} (q_s^\omega(z,w))^2\mathrm{d}s \leq (2+\frac{\pi c_3^2}{c_4})\log T.$$

Upper bound of $E_\omega H^2$

To see that

$$q_t^\omega(x,z)q_t^\omega(y,z) \le c_3^2 t^{-2},$$

Notice that $z \in M(t^{1/2})$, $|x-z| \le t^{1/2} \le t$. Moreover $t \ge t_0 \ge [S_x \vee S_y]^2$.

Theorem

$$q_t^\omega(x,z) \le c_3 t^{-d/2} \mathrm{e}^{-c_4|x-z|^2/t} \le c_3 t^{-d/2} \qquad \textit{when } t \ge |x-z|.$$

Similarly $|y-z| \le t$.

Upper bound of $E_\omega H^2$

For the last inequality,

$$\int_0^T \sum_{w \in M((s+t)^{1/2})} (q_s^\omega(z,w))^2 \mathrm{d}s$$

$$\leq \log t + \int_{\log t}^T \sum_{w \in B_z(s)} (q_s^\omega(z,w))^2 \mathrm{d}s + \int_{\log t}^T \sum_{w \notin B_z(s)} (q_s^\omega(z,w))^2 \mathrm{d}s \ .$$

It is enough to show that

$$\int_{\log t}^T \sum_{w \in B_z(s)} (q_s^\omega(z,w))^2 \mathrm{d}s \leq \frac{c_3^2}{\log t} + \frac{\pi c_3^2}{c_4} \log T;$$

$$\int_{\log t}^T \sum_{w \notin B_z(s)} (q_s^\omega(z,w))^2 \mathrm{d}s \leq c_{10} \ .$$

Upper bound of $E_\omega H^2$

For $w \notin B_z(s)$, we have $S_z(\omega) \leq \log t \leq s \leq |z-w|$,

Theorem

$$q_t^\omega(x,y) \leq c_3 \exp(-c_4|x-y|(1 \vee \log(|x-y|/t)))$$
$$\leq c_3 \exp(-c_4|x-y|) \qquad \textit{when } t \leq |x-y|.$$

Hence

$$\int_{\log t}^{T} \sum_{v \notin B_z(s)} (q_s^\omega(z,w))^2 ds \leq \int_{\log t}^{T} \sum_{v \notin B_z(s)} c_3^2 \exp(-2c_4|z-w|)\,\mathrm{d}s$$
$$\leq \int_{\log t}^{T} \sum_{n=[s]}^{\infty} 2\pi n c_3^2 \exp(-2c_4 n)\,\mathrm{d}s \leq c_{10}\ .$$

Upper bound of $E_\omega H^2$

For $w \in B_z(s)$, $s \geq |z-w|$ and $s \geq S_z(\omega)$,

Theorem

$$q_t^\omega(x,y) \leq c_3 t^{-d/2} \mathrm{e}^{-c_4|x-y|^2/t} \text{ when } t \geq |x-y|,$$

Hence

$$\int_{\log t}^{T} \sum_{w \in B_z(s)} (q_s^\omega(z,w))^2 \,\mathrm{d}s$$

$$\leq \int_{\log t}^{T} \sum_{w \in B_z(s)} c_3^2 s^{-2} \exp\left(-2c_4|z-w|^2/s\right) \,\mathrm{d}s$$

$$\leq \int_{\log t}^{T} [c_3^2 s^{-2} + \sum_{n=1}^{[s]} c_3^2 2\pi n s^{-2} \exp\left(-2c_4 n^2/s\right)] \,\mathrm{d}s$$

Upper bound of $E_\omega H^2$

$$\le \int_{\log t}^{T} [c_3^2 s^{-2} + \sum_{n=1}^{[s]} c_3^2 2\pi n s^{-2} \exp\left(-2c_4 n^2/s\right)] \, \mathrm{d}s$$

$$\le c_3^2 \left(\frac{1}{\log t} - \frac{1}{T}\right) + 2\pi c_3^2 \sum_{n=1}^{[T]} n \int_{n}^{T} s^{-2} \exp\left(-2c_4 n^2/s\right) \, \mathrm{d}s$$

$$\le \frac{c_3^2}{\log t} + 2\pi c_3^2 \sum_{n=1}^{[T]} n \int_{T^{-1}}^{n^{-1}} \exp(-2c_4 n^2 u) \, \mathrm{d}u$$

$$\le \frac{c_3^2}{\log t} + \pi \frac{c_3^2}{c_4} \sum_{n=1}^{[T]} n^{-1} \le \frac{c_3^2}{\log t} + \frac{\pi c_3^2}{c_4} \log T \ .$$

References

1. Barlow, M.T. & Deuschel, J.-D. Invariance principle for the random conductance model with unbounded conductances, *The Annals of Probability*, vol.38,(2010), 234–276.
2. Barlow, M.T.,Peres, Yuval & Sousi, Perla, Collisions of random walks, 2010, *preprint*, available at http://arxiv.org/abs/1003.3255
3. D.Chen, B. Wei & F. Zhang, A note on the finite collision property of random walks, *Statistics and Probability Letters*, Vol. 78 (2008), 1742-1747.
4. Xinxing Chen & Dayue Chen, Two random walks on the open cluster of Z^2 meet infinitely often. *Science China Mathematics*, vol 53, (2010),1971–1978
5. –, Some sufficient conditions for infinite collisions of simple random walks on a wedge comb, *Electronic Journal of Probability*, vol.16, (2011), 1341–1355.
6. T. Delmotte & J.-D. Deuschel, On estimating the derivatives of symmetric diffusions in stationary random environment, with applications to $\nabla\phi$ interface model *Probab. Theory and Related Fields*, vol. 133, (2005), 358–390.
7. Manjunath Krishnapur & Yuval Peres, Recurrent graphs where two independent random walks collide finitely often *Elect. Comm. Probab.*, vol 9. (2004) 72–81.

8. Thomas M.Liggett, Interacting Particle Systems, Springer-Verlag, New York, 1985.

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