Two random walks on the open cluster of $\mathbb{Z}^2$ meet infinitely often

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August 14, 2009

Abstract

We prove that two independent continuous-time simple random walks on the infinite open cluster of a Bernoulli bond percolation in the lattice $\mathbb{Z}^2$ meet each other infinitely many times. An application to the voter model is also discussed.

2000 MR subject classification: 60K

Key words: random walk, percolation, infinite collision property.

1 Introduction

A (continuous-time) simple random walk on graph G is defined as the Markov chain that, waiting for an exponentially distributed random time, jumps from one vertex to a neighbor with equal probability. Will two independent (continuous-time) simple random walks $\{X_t\}, \{Y_t\}$ on G starting from the same vertex meet infinitely many times almost surely? We say $X_t = Y_t$ infinitely often if there is an infinite sequence $\{t_1, t_2, \cdots\}$ such that $t_1 < t_2 < \cdots$, $\lim_{i\to\infty} t_i = \infty$, and $X_{t_i} = Y_{t_i}$ for all $i \geq 1$.

When $G = \mathbb{Z}^d$, this problem was first investigated by Pólya [8], and is reduced to the recurrence of the simple random walk on a related graph which happens to be isomorphic to $\mathbb{Z}^d$ itself.

When G is not transitive, this problem can be complicated. One can not expect that a graph and its subgraph share the same property in this regard, as testified by an example in Krishnapur and Peres[5]. Additional elaborations are made in [2]. However, we believe that *the unique infinite open cluster of the Bernoulli bond percolation on G should resemble the original graph G*. This is true for a regular tree, since an infinite cluster of a regular tree is a Galton-Watson tree and the simple random walk on a Galton-Watson tree converges to a point in the boundary. This paper is devoted to verify this judgement when $G = \mathbb{Z}^d$.

Let us recall briefly the definition of the Bernoulli bond percolation. More details can be found in Grimmett[4]. For edges $e \in V$, we have i.i.d. Bernoulli r.v. η_e, with $\mathbf{P}_p(\eta_e = 1) = p \in [0, 1]$, defined on a probability space $(\Omega, \mathbf{P}_p)$. Edges e with $\eta_e = 1$ are called open and the open cluster $\mathcal{C}(x)$ that contain x is the set of y such that x and y are connected by an open path. It is well known that when $p > 1/2$ there is a unique infinite open cluster in $\mathbb{Z}^2$, which we denote $\mathcal{C}_\infty = \mathcal{C}_\infty(\omega)$.

Theorem 1.1 *Consider $\mathbb{Z}^2$ and let $p > 1/2$. There exists $\Omega_0 \subseteq \Omega$ with $\mathbf{P}_p(\Omega_0) = 1$. Let $\omega \in \Omega_0$ and $x \in \mathcal{C}_\infty(\omega)$. If $X = (X_t)$ and $Y = (Y_t)$ are two independent continuous-time simple random walks starting from x on $\mathcal{C}_\infty(\omega)$, then*

$$\mathbf{P}(X_t = Y_t \text{ infinitely often }) = 1.$$

The proof will be carried out in §3 after some preparations in §2. By the same argument we get the following result as a compliment to the above theorem.

Theorem 1.2 *Consider $\mathbb{Z}^d$, $d \geq 3$. Let $p > p_c(\mathbb{Z}^d)$ and $\omega \in \Omega_1$ and $x \in \mathcal{C}_\infty(\omega)$. If $X = (X_t)$ and $Y = (Y_t)$ are two independent continuous-time simple random walks starting from x on $\mathcal{C}_\infty(\omega)$, then $\mathbf{P}(X_t = Y_t \text{ infinitely often }) = 0$.*

The problem of collisions of two random walkers naturally arises in probing the invariant measures of the voter process on the graph. It is defined in (1.2), page 228 of Liggett [6], that

$$g(x, y) = P^{(x,y)}(X(t) = Y(t) \text{ for some } t \geq 0).$$

By Theorem 1.9, page 231 of Liggett [6], the property that $g(x,y) \equiv 1$ entails that the extreme invariant measures are δ_0 and δ_1 only.

Corollary 1.3 *Consider the voter model defined on the infinite cluster of the Bernoulli bond percolation on $\mathbb{Z}^2$. As time goes to infinite, a consensus will be reached among all voters. In other words, any limiting distribution is a linear combination of δ_0 and δ_1.*

The problem of collisions of two random walkers also appears in discussing the invariant measures of the exclusion process. Consider the exclusion process defined on the infinite cluster of the Bernoulli bond percolation on $\mathbb{Z}^2$. Note that the condition that $g(x,y) \equiv 1$ holds in this case and Theorem 1.1.2, page 369 of Liggett [6], holds consequently. As a matter of fact, to address the invariant measures of the exclusion process, Liggett [7] first constructed examples of recurrent Markov chain that $g(x,y) < 1$.

One may also ask the same question for two independent discrete-time random walks. It is stated in Barlow [1] that the same estimates of Lemma 2.1 below also hold in the discrete-time case. Thus we are sure that the same conclusions of Theorems 1.1 and 1.2 also hold in the discrete-time case. However we are unable to derive one from another directly.

Question 1: *Suppose that two independent (continuous-time) simple random walks on G starting from the same vertex meet infinitely many times almost surely. Will two independent discrete-time simple random walks on G meet infinitely many times almost surely? Or vice verse?*

One can consider two random walks on $G= (V,E)$ as a random walk on $V \times V$. But the neighbors are different for the discrete-time random walk and for the continuous-time random walk.

Question 2: *Suppose that a continuous-time simple random walk on graph G is transient. Will two independent simple random walks on G meet finitely many times almost surely? This is true if the simple random walk is symmetric, i.e. $p(x,y) = p(y,x)$. See Liggett [7], p202. The simple random walk on the infinite cluster of the*

Bernoulli bond percolation in $\mathbb{Z}^d$ is transient if $d \geq 3$. It would be nice if Theorem 1.2 can be derived from this fact. Note that this is false for a 1-dimensional random walk with a drift.

2 Three Lemmas

In the following discussions, we denote by $|A|$ the cardinality of set A, $x \vee y := \max\{x, y\}$ for any two real numbers x and y, and $|x|_\infty = |x_1| \vee |x_2|$ for $x = (x_1, x_2) \in \mathbb{Z}^2$. Let $[x]$ be the largest integer less than x for any real number x.

Lemma 2.1 *Let $p > p_c(\mathbb{Z}^d)$. There exists $\Omega_1 \subseteq \Omega$ with $\mathbf{P}_p(\Omega_1) = 1$ and r.v. V and $S_x, x \in \mathbb{Z}^d$, such that $V(\omega) < \infty$ and $S_x(\omega) < \infty$ for each $\omega \in \Omega_1, x \in \mathcal{C}_\infty(\omega)$. There exist constants $c_i = c_i(p, d)$ which is strictly positive and dependent only on p, such that for all $n \geq V(\omega)$,*

$$|\mathcal{C}_\infty(\omega) \cap [1, n]^d| \geq c_1 n^d; \tag{2.1}$$

and for all $x, y \in \mathcal{C}_\infty(\omega), t \geq 1$ with

$$S_x(\omega) \vee |x - y|_\infty \leq t. \tag{2.2}$$

the transition density $q_t^\omega(x, y)$ of a continuous-time simple random walk satisfies

$$c_2 t^{-d/2} \exp\{-c_3 \frac{|x - y|_\infty^2}{t}\} \leq q_t^\omega(x, y) \leq c_4 t^{-d/2} \exp\{-c_5 \frac{|x - y|_\infty^2}{t}\}. \tag{2.3}$$

Moreover, the tail of the random variable S_x satisfies

$$\mathbf{P}_p(x \in \mathcal{C}_\infty, S_x \geq n) \leq c_6 \exp\{-c_7 n^{c_8}\}. \tag{2.4}$$

Additionally, if $|x - y|_\infty > t$ then

$$q_t^\omega(x, y) \leq c_9 \exp\left\{-c_{10}|x - y|_\infty(1 + \log \frac{|x - y|_\infty}{t})\right\}. \tag{2.5}$$

Proof. Refer to (0.3),(0.4),(0.5) and (1.6) of Barlow[1]. □

Since $\{S_x; x \in \mathcal{C}_\infty(\omega)\}$ are not uniformly bounded above, we define random sets

$$A_n(\omega) := \{x : |x|_\infty \leq n, x \in \mathcal{C}_\infty, S_x \leq 2 \log n\}.$$

Lemma 2.2 *There exists a random variable U such that $U(\omega) < \infty$ a.s., and for any $n \geq U(\omega)$,*

$$|A_n(\omega)| \geq \frac{c_1}{8} n^2.$$

Proof. Set

$$X_n := \sum_{x \in [1,n]^2} \left(1_{\{x \in \mathcal{C}_\infty, S_x \leq 2 \log n\}} + 1_{\{x \notin \mathcal{C}_\infty\}} \right).$$

By (2.4),

$$\begin{aligned} \mathbf{E}_p X_{2^n} &= \sum_{x \in [1,2^n]^2} \mathbf{E}_p \left(1_{\{x \in \mathcal{C}_\infty, S_x \leq n \log 4\}} + 1_{\{x \notin \mathcal{C}_\infty\}} \right) \\ &= \sum_{x \in [1,2^n]^2} \left(1 - \mathbf{P}_p(x \in \mathcal{C}_\infty, S_x > n \log 4) \right) \\ &\geq 2^{2n} \left(1 - c_6 \exp\{-c_7 n^{c_8}\} \right). \end{aligned}$$

Since $0 \leq X_n \leq n^2$,

$$\mathbf{P}_p(X_{2^n} < 2^{2n} \left(1 - c_6 \exp\{-c_7 n^{c_8}/2\} \right)) \leq \exp\{-c_7 n^{c_8}/2\}.$$

Furthermore,

$$\sum_{n=1}^{\infty} \mathbf{P}_p(X_{2^n} < 2^{2n} \left(1 - c_6 \exp\{-c_7 n^{c_8}/2\} \right)) \leq \sum_{n=1}^{\infty} \exp\{-c_7 n^{c_8}/2\} < \infty.$$

By the Borel-Cantelli Lemma, $\{X_{2^n} < 2^{2n} \left(1 - c_6 \exp\{-c_7 n^{c_8}/2\} \right)\}$ almost surely occurs finite many times. There exists $\widetilde{\Omega} \subseteq \Omega$ with $\mathbf{P}_p(\widetilde{\Omega}) = 1$ and random variable $\widetilde{U}$ such that $\widetilde{U}(\omega) < \infty$ for all $\omega \in \widetilde{\Omega}$ and all $n \geq \widetilde{U}(\omega)$,

$$X_{2^n}(\omega) \geq 2^{2n} \left(1 - c_6 \exp\{-c_7 n^{c_8}/2\} \right). \tag{2.6}$$

If $\omega \in \Omega_1 \cap \widetilde{\Omega}$ and $n \geq V(\omega) + \widetilde{U}(\omega)$, then by (2.1) and (2.6)

$$\begin{aligned} |A_{2^n}(\omega)| =& X_{2^n}(\omega) - |[1, 2^n]^2 \setminus \mathcal{C}_\infty(\omega)| \\ \geq& 2^{2n} \left(1 - c_6 \exp\{-c_7 n^{c_8}/2\} \right) - (2^{2n} - c_1 2^{2n}) \\ =& 2^{2n} (c_1 - c_6 \exp\{-c_7 n^{c_8}/2\}) \end{aligned}$$

Set $U(\omega) > V(\omega) + \widetilde{U}(\omega)$ large enough, so that for all $n > U(\omega)$

$$c_1 - c_6 \exp\{-c_7 n^{c_8}/2\} \geq \frac{c_1}{2}.$$

Then $|A_{2^n}(\omega)| \geq c_1 2^{2n-1}$. For all $2^n < k < 2^{n+1}$,

$$|A_k(\omega)| \geq |A_{2^n}(\omega)| \geq c_1 2^{2n-1} \geq \frac{c_1}{8} k^2.$$

We have completed the proof. □

Set $\Omega_0 := \Omega_1 \cap \widetilde{\Omega}$. Then $\mathbf{P}_p(\Omega_0) = 1$. We shall show Ω_0 is just the set we want in Theorem 1.1.

Lemma 2.3 *Let $\omega \in \Omega_0$ and $x, y \in \mathcal{C}_\infty(\omega)$. Let $X = (X_t)$ be a continuous time simple random walk starting from x on $\mathcal{C}_\infty(\omega)$, $Y = (Y_t)$ a continuous time simple random walk starting from y. If X and Y are independent, then*

$$\mathbf{P}(X_t = Y_t \text{ for some } t > 1\ ) \geq \delta,$$

where δ is a strictly positive constant and dependent on p at most.

Proof. Fix $\omega \in \Omega_0$. For any n, set

$$\begin{aligned}
B_x(n) &= \{z : |z - x|_\infty \leq n\}, \\
M_n(\omega) &= B_x(n) \cap B_y(n) \cap A_n(\omega), \\
t_0 &= S_x(\omega) + S_y(\omega) + U(\omega) + (1 + 1000 c_1^{-1})(|x|_\infty \vee |y|_\infty) + 100 \\
T &= t_0^2 + \exp\left\{5 c_9^2 c_{10}^{-2} (1 - e^{-2c_{10}})^{-2} + c_3^2\right\}.
\end{aligned}$$

Define random valuable

$$H := \int_{t_0}^{T} 1_{\left\{X_s = Y_s \in M_{[s^{1/2}]}\right\}} ds.$$

We shall prove that

$$\mathbf{P}(H > 0) \geq \frac{c_1^2 c_2^4 e^{-4c_3}}{200000(c_3^2 + c_3^4 c_4^{-1})}. \tag{2.7}$$

Then the desired conclusion follows since $\mathbf{P}(X_t = Y_t \text{ for some } t > 1\ ) \geq \mathbf{P}(H > 0)$.

We now prove (2.7). For $n \geq U(\omega) + (1000c_1^{-1}+1)(|x|_\infty \vee |y|_\infty) + 100$,

$$\begin{aligned}
|M_n| &= |B_x(n)| + |B_y(n)| + |A_n(\omega)| - |B_x(n) \cup B_y(n)| - |A_n(\omega) \cup B_x(n)| \\
&\quad - |B_y(n) \cup A_n(\omega)| + |B_x(n) \cup B_y(n) \cup A_n(\omega)| \\
&\geq |B_x(n)| + |B_y(n)| + |A_n(\omega)| - 2|B_0(|x|_\infty \vee |y|_\infty + n)| \\
&\geq (2n+1)^2 + (2n+1)^2 + \frac{c_1}{8}n^2 - 2(2(|x|_\infty \vee |y|_\infty + n)+1)^2 \\
&= \frac{c_1}{8}n^2 - 8(|x|_\infty \vee |y|_\infty)((|x|_\infty \vee |y|_\infty) + 2n + 1) \\
&\geq \frac{c_1}{16}n^2 + \left(\frac{c_1}{48}n^2 - 8(|x|_\infty \vee |y|_\infty)^2\right) + \left(\frac{c_1}{48}n^2 - 8(|x|_\infty \vee |y|_\infty)\right) \\
&\quad + \left(\frac{c_1}{48}n^2 - 16n(|x|_\infty \vee |y|_\infty)\right) \\
&\geq \frac{c_1}{16}n^2 \geq \frac{c_1}{20}(n+1)^2.
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}H &= \int_{t_0}^{T} \mathbf{P}\left(X_s = Y_s \in M_{[s^{1/2}]}\right) ds \\
&= \int_{t_0}^{T} [\sum_{z \in M_{[s^{1/2}]}} \mathbf{P}\left(X_s = z, Y_s = z\right)] ds \\
&= \int_{t_0}^{T} [\sum_{z \in M_{[s^{1/2}]}} \mathbf{P}(X_s = z)\mathbf{P}(Y_s = z)] ds \\
&= \int_{t_0}^{T} [\sum_{z \in M_{[s^{1/2}]}} q_s^\omega(x,z) q_s^\omega(y,z)] ds \\
&\geq \int_{t_0}^{T} [\sum_{z \in M_{[s^{1/2}]}} c_2^2 s^{-2} \exp(-c_3 \frac{|x-z|_\infty^2}{s} - c_3 \frac{|y-z|_\infty^2}{s})] ds \\
&\geq \frac{c_1 c_2^2 e^{-2c_3}}{20} \int_{t_0}^{T} s^{-1} ds = \frac{c_1 c_2^2 e^{-2c_3}}{20} (\log T - \log t_0) \geq \frac{c_1 c_2^2 e^{-2c_3}}{40} \log T.
\end{aligned}$$

Here we get the first inequality above by (2.3), and the second inequality by the fact that $|x-z|_\infty^2 \leq s$ for $z \in M_{[s^{1/2}]}$ and $|M_{[s^{1/2}]}| \geq c_1 s/20$. Similarly,

$$\begin{aligned}
\mathbf{E}H^2 &= 2\mathbf{E}\left(\int_{t_0}^{T} dt \int_{t}^{T} 1_{\left\{X_t = Y_t \in M_{[t^{1/2}]}\right\}} 1_{\left\{X_s = Y_s \in M_{[s^{1/2}]}\right\}} ds\right) \\
&= 2\int_{t_0}^{T} dt \int_{t}^{T} \mathbf{P}(X_t = Y_t \in M_{[t^{1/2}]}, X_s = Y_s \in M_{[s^{1/2}]}) ds
\end{aligned}$$

$$= 2\int_{t_0}^{T} dt \int_t^T [\sum_{z\in M_{[t^{1/2}]}} \mathbf{P}(X_t = Y_t = z, X_s = Y_s \in M_{[s^{1/2}]})]ds$$

$$= 2\int_{t_0}^{T} dt \int_t^T [\sum_{z\in M_{[t^{1/2}]}} \mathbf{P}(X_t = Y_t = z)\mathbf{P}^{(z,z)}(X_{s-t} = Y_{s-t} \in M_{[s^{1/2}]})]ds$$

$$\leq 2\int_{t_0}^{T} dt[\sum_{z\in M_{[t^{1/2}]}} \mathbf{P}(X_t = Y_t = z)\int_0^T \mathbf{P}^{(z,z)}(X_s = Y_s \in M_{[(s+t)^{1/2}]})ds].$$

Now we estimate the right hand side of the above inequality. By (2.3)

$$\mathbf{P}(X_t = Y_t = z) = q_t^\omega(x,z)q_t^\omega(y,z) \leq c_3^2 t^{-2}$$

for $t > t_0$ and $z \in M_{[t^{1/2}]}$. If $z \in M_{[t^{1/2}]}$, then $z \in A_{[t^{1/2}]}(\omega)$, and consequently $S_z \leq \log t$ by the definition of A_n. Hence

$$\int_0^T \mathbf{P}^{(z,z)}\left(X_s = Y_s \in M_{[(s+t)^{1/2}]}\right) ds \leq \log t + \int_{\log t}^T \mathbf{P}^{(z,z)}\left(X_s = Y_s \in B_z([s])\right) ds$$
$$+ \int_{\log t}^T \mathbf{P}^{(z,z)}\left(X_s = Y_s \in \mathbb{Z}^2 \setminus B_z([s])\right) ds.$$

By (2.5)

$$\begin{aligned}
&\int_{\log t}^T \mathbf{P}^{(z,z)}\left(X_s = Y_s \in \mathbb{Z}^2 \setminus B_z([s])\right) ds \\
=&\int_{\log t}^T [\sum_{v\in\mathbb{Z}^2\setminus B_z([s])} (q_s^\omega(z,v))^2]ds \\
\leq&\int_{\log t}^T [\sum_{v\in\mathbb{Z}^2\setminus B_z([s])} c_9^2 \exp\left(-2c_{10}|z-v|_\infty\right)]ds \\
\leq&\int_{\log t}^T [\sum_{n=[s]}^{\infty} 12nc_9^2 \exp\left(-2c_{10}n\right)]ds \\
\leq&\frac{20c_9^2 e^{2c_{10}}}{(1-e^{-2c_{10}})^2}\int_{\log t}^T s\exp(-2c_{10}s)ds \leq \frac{5c_9^2 e^{2c_{10}} c_{10}^{-2}}{(1-e^{-2c_{10}})^2}.
\end{aligned}$$

By (2.3) again,

$$\int_{\log t}^T \mathbf{P}^{(z,z)}\left(X_s = Y_s \in B_z([s])\right) ds = \int_{\log t}^T [\sum_{v\in B_z([s])} (q_s^\omega(v,z))^2]ds$$

$$
\begin{aligned}
&\le \int_{\log t}^{T} [\sum_{v\in B_z([s])} c_3^2 s^{-2} \exp(-2c_4 \frac{|z-v|_\infty^2}{s})]ds \\
&\le \int_{\log t}^{T} [c_3^2 s^{-2} + \sum_{n=1}^{[s]} c_3^2 12 n s^{-2} \exp(-2c_4 n^2/s)]ds \\
&\le c_3^2(\frac{1}{\log t} - \frac{1}{T}) + 12c_3^2 \sum_{n=1}^{[T]} n \int_n^T s^{-2}\exp(-2c_4 n^2/s)ds \\
&\le \frac{c_3^2}{\log t} + 12c_3^2 \sum_{n=1}^{[T]} n \int_{T^{-1}}^{n^{-1}} \exp(-2c_4 n^2 u)du \\
&\le \frac{c_3^2}{\log t} + \frac{6c_3^2}{c_4}\sum_{n=1}^{[T]} n^{-1} \le \frac{c_3^2}{\log t} + \frac{6c_3^2}{c_4}\log T.
\end{aligned}
$$

Together,

$$
\begin{aligned}
\int_0^T \mathbf{P}^{(z,z)}(X_s = Y_s \in M_{[(s+t)^{1/2}]})ds &\le \log t + \frac{5c_9^2 c_{10}^{-2} e^{2c_{10}}}{(1-e^{-2c_{10}})^2} + \frac{c_3^2}{\log t} + \frac{6c_3^2}{c_4}\log T \\
&\le (2 + 6c_3^2/c_4)\log T.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}H^2 \le& 2\int_{t_0}^T (\sum_{z\in M_{[t^{1/2}]}} c_3^2 t^{-2}(2+6c_3^2/c_4)\log T)dt \\
\le& 2\int_{t_0}^T \frac{c_3^2(2+6c_3^2/c_4)\log T}{t} dt \le (40c_3^2 + 120c_3^4/c_4)(\log T)^2.
\end{aligned}
$$

By the Hӧlder inequality

$$
\mathbf{P}(H>0) \ge \frac{(\mathbf{E}H)^2}{\mathbf{E}H^2} \ge \frac{(\frac{c_1 c_2^2 e^{-2c_3}}{40}\log T)^2}{(40c_3^2 + 120c_3^4/c_4)(\log T)^2} \ge \frac{c_1^2 c_2^4 e^{-4c_3}}{200000(c_3^2 + c_3^4/c_4)}.
$$

This completes our proof. □

3 Proof of Theorems

Proof of Theorem 1.1. Let $\delta > 0$ defined in Lemma 2.3. Fix $\omega \in \Omega_0$. By Lemma 2.3, there exists a function $f : \mathbb{Z}^2 \times \mathbb{Z}^2 \mapsto [2, \infty)$, such that for all $x, y \in \mathcal{C}_\infty(\omega)$

$$
\mathbf{P}^{(x,y)}(\ X_t = Y_t \text{ for some } 1 < t \le f(x,y)\ ) \ge \frac{\delta}{2}. \tag{3.1}
$$

Let $\widetilde{X} = (\widetilde{X}_t)$ and $\widetilde{Y} = (\widetilde{Y}_t)$ be two independent continuous-time simple random walks starting from $x \in \mathcal{C}_\infty(\omega)$. Set $x_0 = y_0 = x$ and $t_0 = 0$. Define x_i, y_i and t_i inductively for $i \geq 1$ as follows:

$$t_i := t_{i-1} + f(x_{i-1}, y_{i-1}), \qquad x_i := \widetilde{X}_{t_i} \qquad \text{and} \qquad y_i := \widetilde{Y}_{t_i}.$$

Define E_i to be the event that $\widetilde{X}_t = \widetilde{Y}_t$ for some $t \in (t_i + 1, t_{i+1}]$ for $i \geq 0$. By (3.1) and the strong Markov property,

$$\mathbf{P}(E_i | \widetilde{X}_t, \widetilde{Y}_t, t \leq t_i) = \mathbf{P}^{(x_i, y_i)}(\ X_t = Y_t \text{ for some } 1 < t \leq f(x_i, y_i)\ ) \geq \frac{\delta}{2}.$$

By the Second Borel-Cantelli lemma (extended version, see page 237 of [3]),

$$\mathbf{P}(\ E_i \text{ infinitely often }) = 1.$$

Furthermore,

$$\mathbf{P}(\widetilde{X}_t = \widetilde{Y}_t \text{ infinitely often }) \geq \mathbf{P}(\ E_i \text{ infinitely often }) = 1.$$

Thus we complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. For any n, set $B_x(n) = \{z : |z - x|_\infty \leq n\}$. Define random variables

$$H_1 = \int_{S_x+1}^\infty 1_{\{X_t = Y_t \in B_x([t])\}} dt \quad \text{and} \quad H_2 = \int_{S_x+1}^\infty 1_{\{X_t = Y_t \in \mathbb{Z}^d \setminus B_x([t])\}} dt.$$

To prove Theorem 1.2, we need only to prove that $H_1 + H_2 < \infty$ *almost surely*. By (2.3) and $d \geq 3$,

$$\begin{aligned}
\mathbf{E}H_1 &= \int_{S_x+1}^\infty \mathbf{P}(X_t = Y_t \in B_x([t])) dt \\
&= \int_{S_x+1}^\infty \sum_{z \in B_x([t])} \mathbf{P}(X_t = Y_t = z) dt \\
&= \int_{S_x+1}^\infty \sum_{z \in B_x([t])} (q_t^\omega(x, z))^2\, dt \\
&\leq \int_{S_x+1}^\infty \sum_{z \in B_x([t])} c_4^2 t^{-d} \exp(-2c_5 |x - z|_1^2 / t) dt
\end{aligned}$$

$$
\begin{aligned}
&\le \int_1^\infty (c_4^2 t^{-d} + \sum_{n=1}^{[t]} c_4^2 t^{-d} (3n)^{d-1} 2d \exp(-2c_5 n^2/t))dt \\
&\le \frac{c_4^2}{d-1} + 3^d d c_4^2 \int_1^\infty (\sum_{n=1}^{[t]} t^{-d} n^{d-1} \exp(-2c_5 n^2/t))dt \\
&\le \frac{c_4^2}{d-1} + 3^d d c_4^2 \sum_{n=1}^\infty n^{d-1} \int_t^\infty t^{-d} \exp(-2c_5 n^2/t)dt \\
&\le \frac{c_4^2}{d-1} + 3^d d c_4^2 \sum_{n=1}^\infty n^{d-1} (c_5 n^2)^{-d+1} \int_0^\infty s^{d-2} e^{-s} ds \\
&\le \frac{c_4^2}{d-1} + 3^d d c_4^2 c_5^{-d+1} (d-2)! \sum_{n=1}^\infty n^{-d+1} \\
&\le \frac{c_4^2}{d-1} + 3^d c_4^2 c_5^{-d+1} (d-2)! \sum_{n=1}^\infty n^{-2} < \infty.
\end{aligned}
$$

By (2.5),

$$
\begin{aligned}
\mathbf{E} H_2 &= \int_{S_x+1}^\infty \mathbf{P}(X_t = Y_t \in \mathbb{Z}^d \setminus B_x([t]))dt \\
&= \int_{S_x+1}^\infty \sum_{z \in \mathbb{Z}^d \setminus B_x([t])} \mathbf{P}(X_t = Y_t = z)dt \\
&= \int_{S_x+1}^\infty \sum_{z \in \mathbb{Z}^d \setminus B_x([t])} (q_t^\omega(x,z))^2 \, dt \\
&\le \int_{S_x+1}^\infty \sum_{z \in \mathbb{Z}^d \setminus B_x([t])} c_9^2 \exp\{-2c_{10}|x-z|_\infty\} \, dt \\
&\le \int_1^\infty (\sum_{n=[t]}^\infty 3^d n^{d-1} d c_9^2 e^{-2c_{10} n})dt \\
&\le \sum_{n=1}^\infty 3^d n^{d-1} d c_9^2 e^{-2c_{10} n} \int_1^n dt \\
&\le 3^d d c_9^2 \sum_{n=1}^\infty n^d e^{-2c_{10} n} < \infty.
\end{aligned}
$$

□

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