Workshop on Stochastic Analysis and Related Topics

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The Motion of a Tagged Particle in the Simple Exclusion Process

Dayue Chen

Peking University

1. The model
2. basic results
3. new results

The exclusion process is an interacting particle system.

Underlying space S, usually is the set of vertices of a graph (V, E).

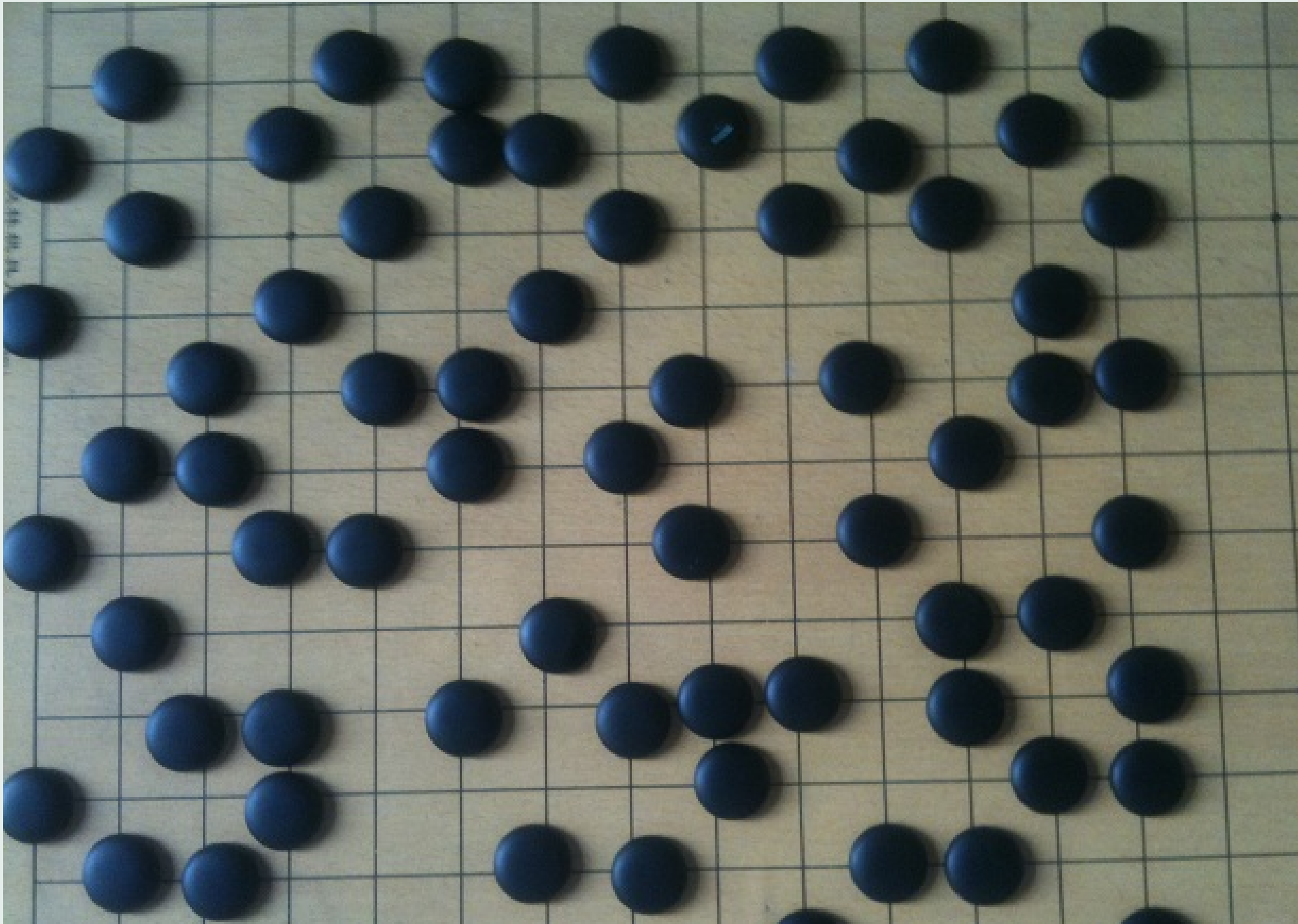
The default choice is the lattice Z^d

A configuration η is a point of $\{0,1\}^S$.

$\eta = \{\eta(x); x \in S\}$.

There is a particle at x if $\eta(x)$ = 1

and site x is unoccupied if $\eta(x)$=0.



Transition mechanism of particles.

1. There is at most particle in every site of S.

2. A particle at x waits for an exponential time and attempts to jump to another site y with probability $p(x, y)$.

3. If y is vacant, particle moves to y; if y is occupied, then particle stays in x and the attempt is suspended.

$p(x, y)$ is the transition probability of a Markov chain on S.

Extra assumptions on $p(x, y)$ are made usually.

E.g. $p(x, y) = 1/d_x$ if $|x - y| = 1$ and $p(x, y) = 0$ if $|x - y| \neq 1$.

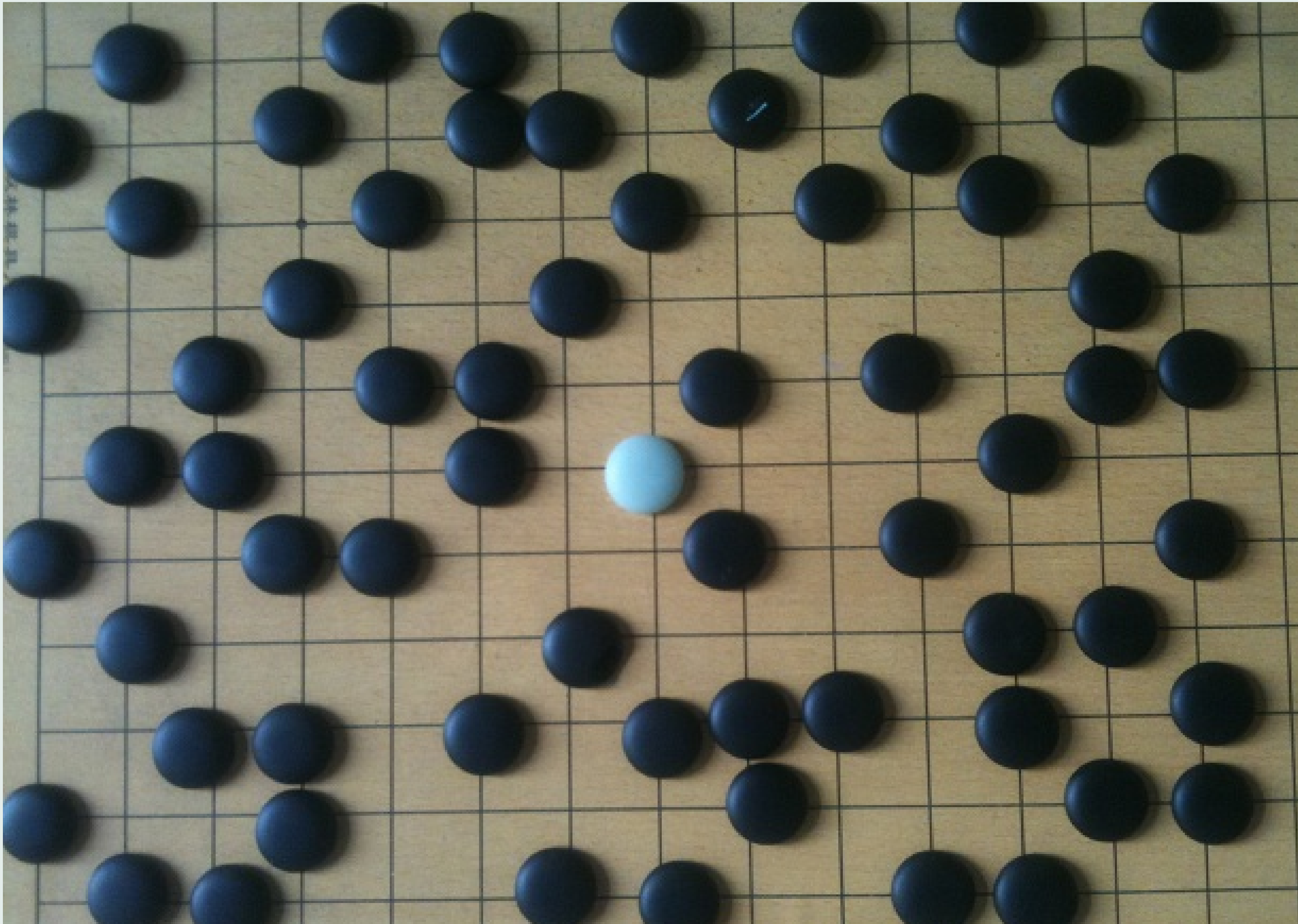
There is no birth and death, the density of particles is preserved.

The Bernoulli product measure μ_ρ is invariant (and ergodic).

No easy to identify all invariant measures. symmetric or Z^1 nearest neighbor, or Z^1 mean zero.

From now on we work with the exclusion process that $p(x, y) = 1/d_x$ if $|x - y| = 1$ and $p(x, y) = 0$ if $|x - y| \neq 1$, and the initial measure is the Bernoulli product measure μ_ρ.

(althoughthe conclusions could be valid in a more general setting.)



Mark a particle (called the tagged particle).

Goal: to study the motion $X(t)$ of the tagged particle.

$X(t)$ behaves very much like a random walk on S,

except some suspensions due to collision with other particles.

If the initial measure is the Bernoulli product measure μ_ρ,

an attempt to jump will be suspended with probability ρ.

$$\lim_{t\to\infty} \frac{X(t)}{t} = (1-\rho)\sum_y y p(0,y) \qquad a.s.$$

Basic Results

1. LLN

2. CLT

3. Invariance principle

$$\lim_{t\to\infty} \frac{X(t)}{t} = (1-\rho)\sum_y y p(0,y) \qquad a.s.$$

was first verified in two cases:

1) $S = Z^1$ and $p(x, x+1) = 1$ (totally asymmetric).

2) $S = Z^1$ and $p(x, x+1) = p(x, x-1) = 1/2$

However, it is not trivial at all. In the second case above

$$EX_t^2 \approx c\sqrt{t}.$$

$X(t)$ is not a random walk with a certain rate of suspension. It is subdiffusive.

Key step: to verify that the environment viewed from the tagged particle is stationary and ergodic.

Ergodicity of μ_ρ can not be inherited automatically when μ_ρ is conditioned on $\eta(0) = 1$.

Assuming translational invariance $p(x, y) = p(0, y-x)$ for all x, y, this was done by E. Saada in the following cases:

1) Z^d, $d \geq 2$,

2) Z^1, $p(x, x+1) + p(x, x-1) < 1$.

and by P.A. Ferrari

3) Z^1, $p(x, x+1) + p(x, x-1) = 1$.

CLT (Kipnis 85, Kipnis & Varadhan 85).

$$Z_t = \frac{X_t - EX_t}{\sqrt{t}}$$

is asymptotically normal if

1) $S = Z^1$, $p(x, x+1) + p(x, x-1) = 1$; or

2) $S = Z^d$, $p(x, y) = p(y, x) = p(0, y-x)$, irreducibility of the random walk and $\sum_x |x|^2 p(0, x) < \infty$.

But both excludes the case that $S = Z^1$, $p(x, x+1) = p(x, x-1) = 1/2$.

3) $S = Z^d$, $\sum_y y p(0, y) = 0$. (Varadhan 1995)

4) $S = Z^d$, $d \geq 3$, $\sum_y y p(0, y) \neq 0$. (Sethuraman, Varadhan and Yau 1999)

Theorem (Arratia 83): If $S = Z^1$, $p(x, x+1) = p(x, x-1) = 1/2$ and the initial distribution is the Bernoulli product measure μ_ρ conditioned on $\eta(0) = 1$. $X(t)$ is the position of the tagged particle initially at the origin. Then $X_t/t^{1/4}$ converges in distribution to the normal law with mean zero and variance $\sqrt{2/\pi}(1-\rho)/\rho$. Furthermore

$$\lim_t \frac{var(X_t)}{\sqrt{t}} = \sqrt{\frac{2}{\pi}} \frac{1-\rho}{\rho}.$$

Invariance Principle: As $N \to \infty$, $Z_t^N = Z_{Nt}$ converges to a Brownian motion with a none-degenerated coefficient.

the exceptional case

Let $\sigma_X^2 = \sqrt{2/\pi}(1-\alpha)/\alpha$.

$$\frac{X(\lambda t)}{\sigma_X \lambda^{1/4}} \Rightarrow B_{1/4}(t),$$

where $B_{1/4}(t)$ is the standard fractional Brownian motion with parameter 1/4. M. Peligrad, S. Sethuraman. *On fractional Brownian motion limits in one dimensional nearest-neighbor symmetric simple exclusion.* ALEA. **4** (2008), 245–255.

Some new results

1. random environments

2. with a stirring

3. on a regular tree.

I: random environment (RWRE, slow down)

$S = Z^1$, $\{\omega_i, i \in Z^1\}$ are i.i.d. random variables, $0 < c < \omega < c^{-1}$. Fix the environment, then run an exclusion process.

A particle at site i attempts to jump to $i+1$ at rate ω_i and attempts to jump to $i-1$ at rate ω_{i-1}.

A particle at site i waits for an exponential time with parameter $\omega_{i-1} + \omega_i$. When the clock rings the particle attempts to jump with transition probability

$$p(i, i+1) = \frac{\omega_i}{\omega_{i-1} + \omega_i}, \qquad p(i, i-1) = \frac{\omega_{i-1}}{\omega_{i-1} + \omega_i}.$$

Theorem (Jara and Landim (2008)). Let X_t be the position of a tagged particle at time t. Initially the tagged particle is at the origin. Particles are assigned to sites other than the origin independently with probability ρ. For almost all environment ω,

$$\frac{X(t)}{t^{1/4}} \Longrightarrow Y, \quad \text{and} \quad Y \sim N(0, \frac{2(1-\rho)}{\rho\sqrt{\alpha\pi}})$$

where $\alpha = E\omega_i^{-1}$.

Remark: When $\omega_i = 1/2$, this is reduced to the result of Arratia. Thus to make a comparisopn, we may assume that $E\omega_i = 1/2$. Then $\alpha = E\omega_i^{-1} \geq 1/E\omega_i = 2$. Thus the randomness of the environment further slows down the motion of the tagged particle.

A different approach is to estimate

$$J(t) = J(t)^{+} - J(t)^{-}$$

the net left-to-right particle current across the origin up to time t.

Theorem (P. Chen). For almost all environment ω,

$$\frac{J(t)}{t^{1/4}} \Longrightarrow Z, \quad \text{and} \quad Z \sim N(0, \frac{2(1-\rho)\rho}{\sqrt{\alpha\pi}})$$

where $\alpha = E\omega_i^{-1}$.

II, "Stirring-Exclusion" process on $\mathbb{Z}^d$.

$p(x, y)$ and $p_{st}(x, y)$: the probability transition functions for two discrete time Markov chains on $\mathbb{Z}^d$.

An intuitive description of the model:

Associate each ordered pair (x, y) in $\mathbb{Z}^d$ with a rate $p(x, y)$ Poisson process, denoted by $\mathscr{N}^{x,y}$.

Associate each unordered pair $\{x, y\}$ in $\mathbb{Z}^d$ with a rate $rp_{st}(x, y)$ Poisson process, for some constant $r \in (0, 1)$, denoted by $\mathscr{N}^{\{x,y\}}$.

All these Poisson processes are mutually independent.

"exclusion dynamics". At each event time of $\mathscr{N}^{x,y}$, the particle at the site x, if there's any, jumps to the site y if in addition y is empty; otherwise, nothing happens.

"stirring dynamics" . At each event time of $\mathscr{N}^{\{x,y\}}$, if x and y are both occupied, then interchange the positions of the particles at sites x and y; otherwise, nothing happens.

Assumption A:

$p(x,y)$, irreducible, translational-invariant, finite range and $p(0,0) = 0$. $p(x,y)$ is not the nearest-neighbor in one dimension.

$p_{st}(x,y)$, symmetric, translational-invariant, finite range and $p_{st}(0,0)$ = 0.

Theorem (LLN & CLT): Fix any constant $\rho \in [0, 1]$. Under Assumption A and the measure $\mathbb{P}_{\nu_\rho^*}$, then

$$\lim_{t\to\infty} \frac{X_t}{t} = (1-\rho)m \quad a.s.$$

When $m \neq 0$, $X_t/t^{1/2}$ converges in distribution, as $t \uparrow \infty$, to a mean zero Gaussian random vector with covariance matrix denoted by $D(\rho)$.

P. Chen & F. Zhang, Limit Theorems for the Position of a Tagged Particle in the Stirring-Exclusion Process, Preprint

III: Simple exclusion process on tree.

$T_d =$ regular tree of degree $d + 1$.

Consider the simple random walk on T_d.

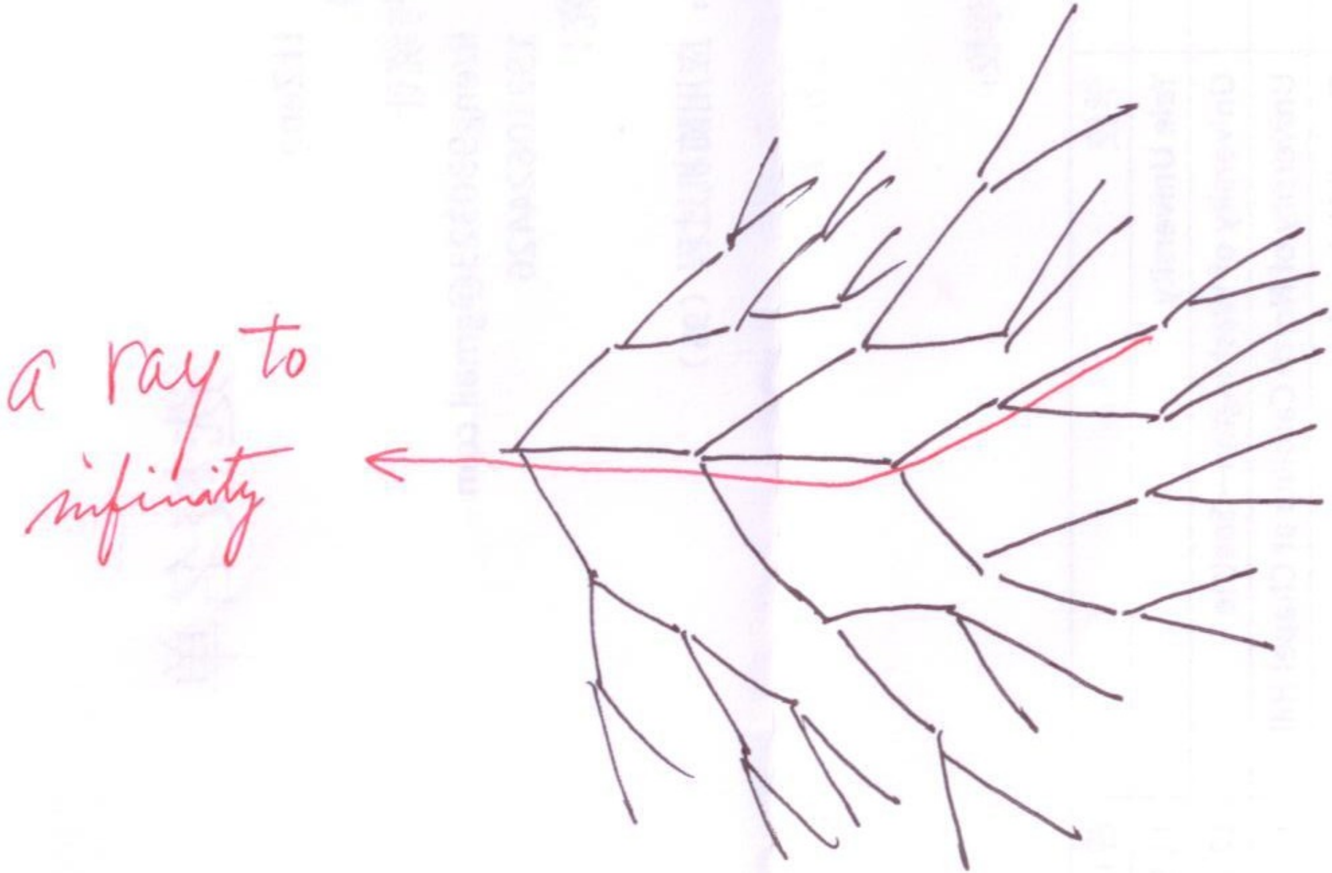
Fix a site as the root and the walker is at the root initially.

The walk waits for an exponential time with parameter 1, and moves to a neighboring site with probability $1/(d+1)$ when the clock rings.

$Y(t) =$ distance between the walker and the root at time t.

Fix a ray from the root to infinity γ.

At each jump, the walker moves towards the infinity or away from the infinity along the ray γ by one unit.


a ray to
infinity

Let ξ_k be i.i.d. random variables with

$$P(\xi = 1) = \frac{d}{d+1}, \qquad P(\xi = -1) = \frac{1}{d+1}.$$

$\{K(t); t \geq 0\}$ be a Poisson process with parameter 1.

$$Z_t = \sum_{k=1}^{K(t)} \xi_k.$$

Then $\lim_t (Z(t) - Y(t))$ exists and

$$\lim_{t\to\infty} \frac{Y(t)}{t} = \lim_{t\to\infty} \frac{Z(t)}{t} = \frac{d-1}{d+1}.$$

The simple exclusion process on tree.

Each particle performs a random walk, with any possible collision being suspended.

The initial distribution is the Bernoulli product measure μ_ρ.

μ_ρ is invariant.

Suppose there is a particle at the root initially.

$X(t) =$ distance between the tagged particle and the root at time t.

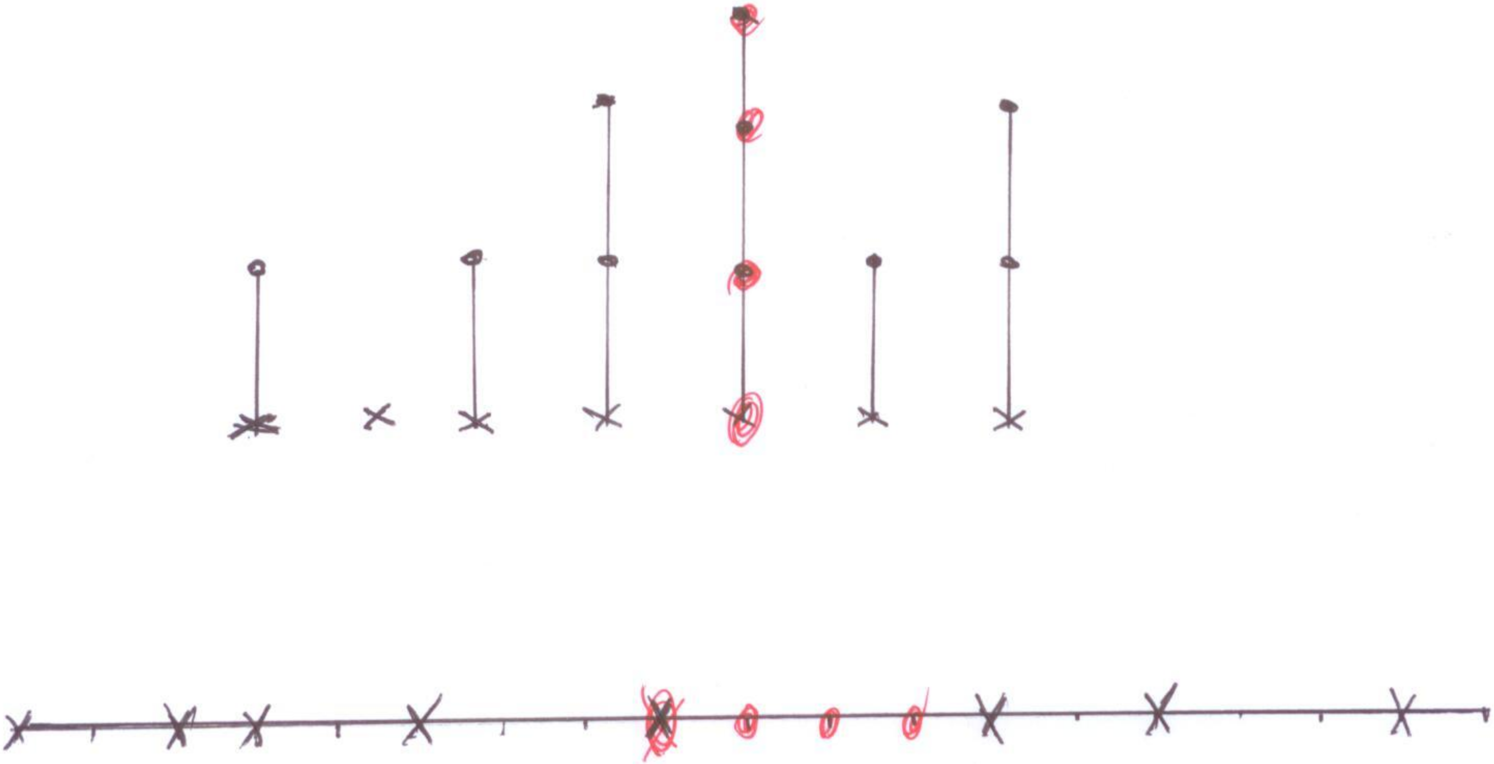
Then

$$\lim_{t\to\infty} \frac{X(t)}{t} = \lim_{t\to\infty} (1-\rho)\frac{Y(t)}{t} = (1-\rho)\frac{d-1}{d+1}.$$

A technical assumption: $(1-\rho)d \leq 1$.

Viewed from the tagged particle, there are $d+1$ neighboring sites. A neighboring site is either occupied by a particle, or vacant adjacent by more vacant sites, forming a Galton-Watson tree.

The assumption that $(1-\rho)d \leq 1$ ensure any Galton-Watson tree is finite. The environment is stationary and ergodic.



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Thank You

E-mail: dayue@pku.edu.cn