从随机游动谈起

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抛投硬币

- 连续抛投均匀硬币（骰子），记录结果：ξ_1，ξ_2， ξ_3， ...，ξ_n， ...... 独立同分布(i.i.d.)
- 蒲丰 4040 (2048)
- 皮尔逊 12000 (6019)
- 皮尔逊 24000 (12012)
- John Kerrich 10,000

- 抛币定胜负，赌博ξ=1或-1，
- 累计输赢$S_n = \xi_1+\xi_2+\xi_3+\dots+\xi_n$，

随机游动的定义

- 第一种定义. 设ξ_1, ξ_2, ξ_3, … , ξ_n, …… 独立同分布(i.i.d.).

  $$S_n = \xi_1+\xi_2+\xi_3+\dots+\xi_n$$

  称随机变量序列$\{S_n\}$为(Z^1上的简单)**随机游动**,

- Z^2, Z^d上的简单随机游动.
- Pearson, K. The problem of the random walk. *Nature*. **72**, (1905),294.

K. Pearson, (1857-1936).

另一种定义

- 想象有一位醉汉在街头漫无目的游荡，前进/后退, 向左/向右. 一维简单随机游动

- 醉汉在城里乱逛，每到一个十字街头就等可能选取一个路口，继续瞎逛，二维简单随机游动

- 更一般的, 给定图G= (V, E). 可以定义图G上的简单随机游动.

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dered, which we will
d in Figure 5.1.

2-dimensional lattice

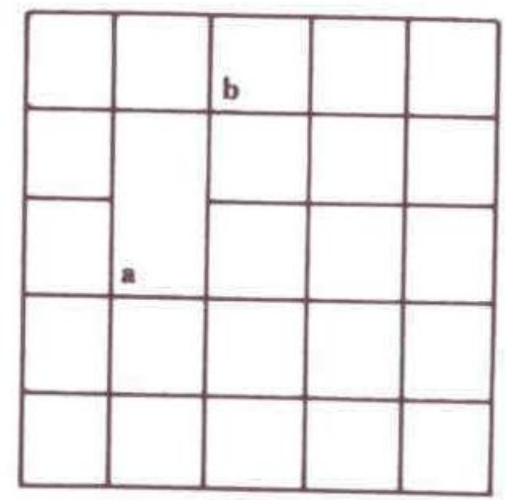


FIGURE 4.2

赌徒破产问题

- $S_0=x$ 初始赌本.
- $S_n = S_0 + \xi_1+\xi_2+ \xi_3+\dots + \xi_n$ =时刻n的赌资
- $\tau_0= \min\{ n, S_n =0\}$, $=\infty$?
- $\tau_N= \min\{ n, S_n =N\}$, 理智? 克制?
- $\tau_0 \neq \tau_N$
- $\tau_0 < \tau_N$ 输光(之前不曾有过N元) ;
- $\tau_0 > \tau_N$ (在输光之前曾)有过N元.
- $\sigma= \min\{\tau_N, \tau_N\}$. 关于σ你能说什么?

一维情形

- σ与$S_0=x$有关.
- 记$f(x) = P_x(\sigma = \tau_0) = P_x(\tau_0 < \tau_N)$全输光的可能性
- $f(0)=1,\ f(N) = 0,\ f(x) = \frac{1}{2} f(x+1) + \frac{1}{2} f(x-1)$.
- $f(x) = 1 - \frac{x}{N}$.
- 赌博很公平,赢得N之前全输光的可能性与初始赌本成反比.
- $N \longrightarrow \infty,\ f(x) \longrightarrow 0$
- 如果N越大,贪心越大,输光的可能性也越大.
- $N=\infty$, 则 $f(x)= 0$. 百分之百迟早输光.

- 记$g(x) = E_x \sigma$ 停止赌博的平均时间.

- $g(0)= g(N) = 0$;
- $g(x) = 1 + 0.5\, g(x+1) + 0.5\, g(x-1)$.

- $g(x) = x\,(N-x)$.

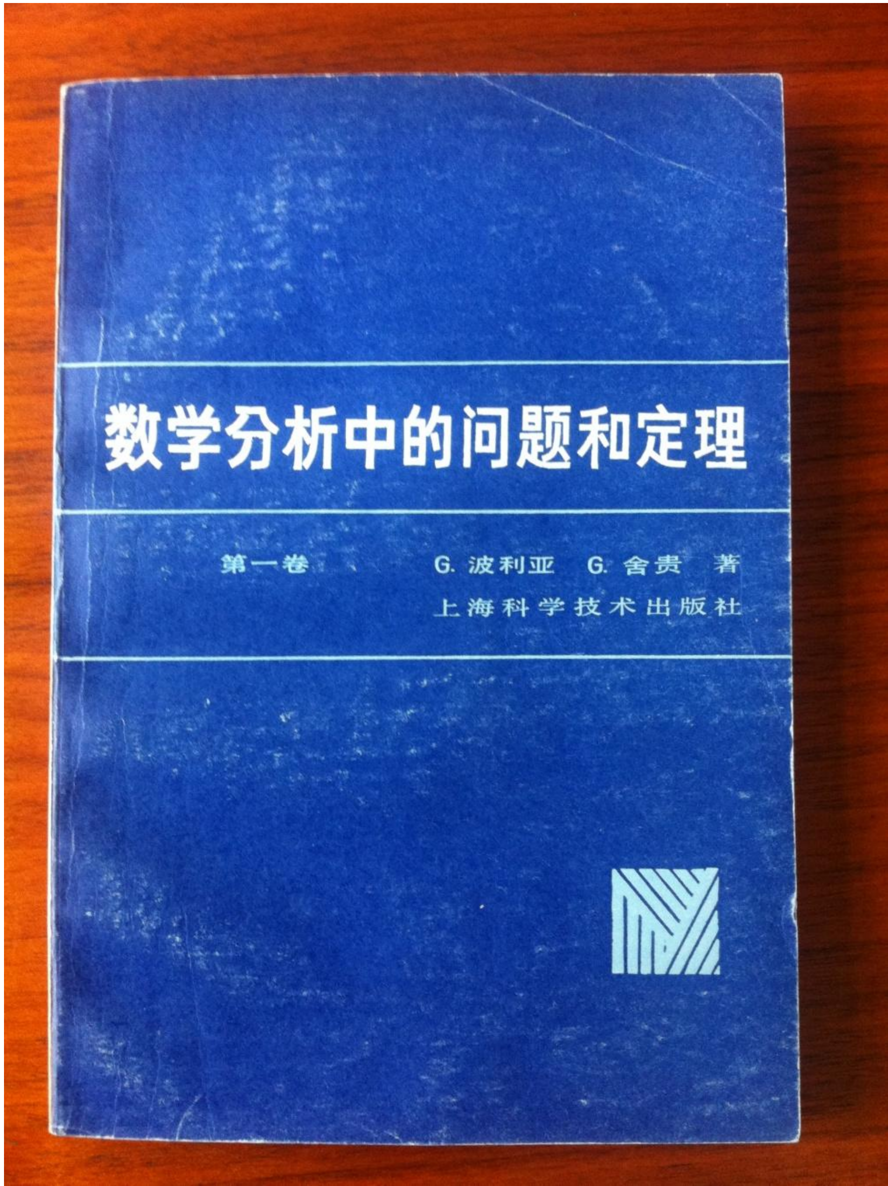
- 如果$N \longrightarrow \infty$则$g(x) \longrightarrow \infty$.
- 若$N=\infty$, 则$g(1)= \infty$.

总结：一步之遥总会到达，却需等无穷长时间.

Polya的尴尬事

- 导致他引入常返的概念.
- … he and his fianc èe (would) also set out for a stroll in the woods, and then suddenly I met them there. And then I met them the same morning repeatedly. I don't remember how many times, but certainly much too often and I felt embarrassed. It looked as if I was snooping around which was, I assure you, not the case. I met them by accident - but how likely was it that it happened by accident and not on purpose?

George Polya



常返

- X_n Polya 随机游动，Y_n 对方情侣，$Z_n = X_n - Y_n$ 仍是随机游动，
- $X_n = Y_n \iff Z_n = 0$
- 相遇很多次 $\iff$ 随机游动回到原点很多次，$\iff$ 随机游动一定回到原点
- 问题: 从原点出发的随机游动是否一定回到原点? 需要研究的第一个问题.
- 常返：$\rho_{xx} = P_x(\tau_x < \infty) = 1.$
- 判别准则：：$\rho_{xx} = 1. \iff$ 返回无穷多次的概率为1. $\iff G(x,x) = \infty.$

- 一维情形，从原点出发一步之后到了1或-1，由对称性，可只考虑一半，即从1出发的随机游动会不会回到原点

- $f(x) = 1-\ x/N$， $N\longrightarrow\infty$, $f(x)\longrightarrow 0$
- 一维随机游动是常返

- 二维随机游动是常返的而三维随机游动是非常返.

- Kakutani的评语：A drunk man will find his way home, but a drunk bird may get lost forever.

角谷静夫，(1911-2004)

W. Feller的一个问题

- 选定一个图G=(V,E), 你能判断图上随机游动是否常返?

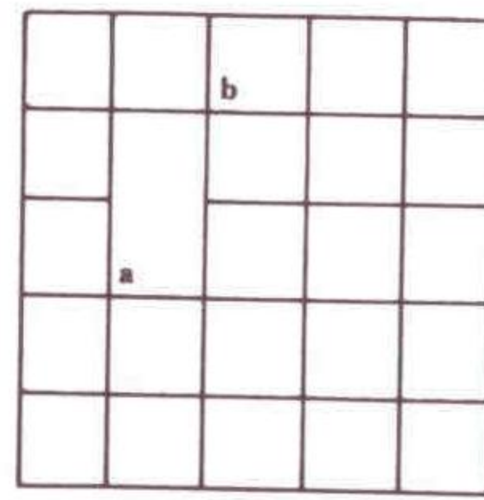
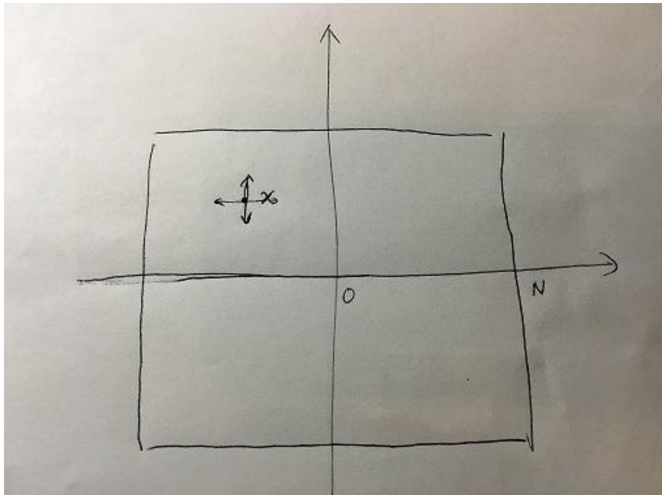


FIGURE 4.2

- 再来认真看二维情形
- N = 方框的边界
- 记$f(x) = P_x(\sigma = \tau_0) = P(\tau_0 < \tau_N)$
- $f(0)=1$, $f(N) = 0$,
- $f(x) = \frac{1}{4}f(x+e_1) + \frac{1}{4}f(x+e_2) + \frac{1}{4}f(x+e_3) + \frac{1}{4}f(x+e_4)$



电网络

- N不太大，手工可解
- N很大，用计算机也能解

- 但方程组有物理意义
- 假设每条边的电阻为1,
- 原点电势=1，边界N电势=0
- f(x) = 位置x点的电势

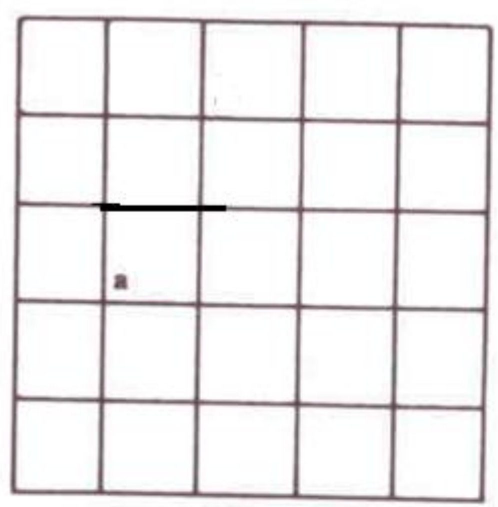
电网络

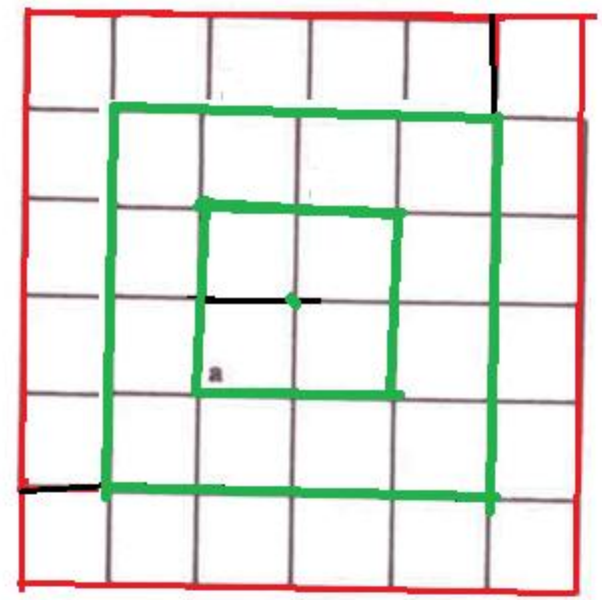
- 一维情形:
- $R(x \longleftrightarrow y)$ = x与y之间的电阻
- $f(x) = 1 - \frac{x}{N} = 1 - \frac{R(0 \leftrightarrow x)}{R(0 \leftrightarrow N)}$
- 当$N \longrightarrow \infty$时， $R(0 \longleftrightarrow N)$趋于无穷， 而 $R(0 \longleftrightarrow x)$并不增大

- 二维情形也有类似结论
- 需要证明： $R(0 \longleftrightarrow N)$ 随着$N \longrightarrow \infty$而趋于无穷

电网络

- R(0 ⟷N) = 0与N之间的电阻
- $> \frac{1}{4} + \frac{1}{12} + \ldots. + \frac{1}{8N-4}$





随机游动与电网络

- 定理：

随机游动常返当且仅当$R(0 \leftrightarrow \infty)$ 是无穷

推论I=练习I：

三维格点图上$R(0 \leftrightarrow \infty)$ 是有限的

- 没有导线=导线电阻无穷大
- 有限图上$R(0\leftrightarrow\infty) = \infty$

Feller的问题

命题： 若S是G的子图，则$R_S(0 \leftrightarrow \infty) \geqq R_G(0 \leftrightarrow \infty)$

推论2：如果图G上的随机游动是常返的，S是G的子图，则图S上的随机游动是常返的

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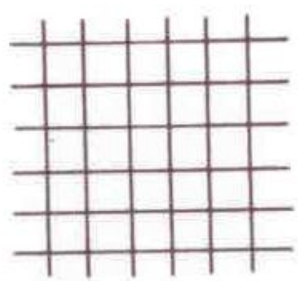

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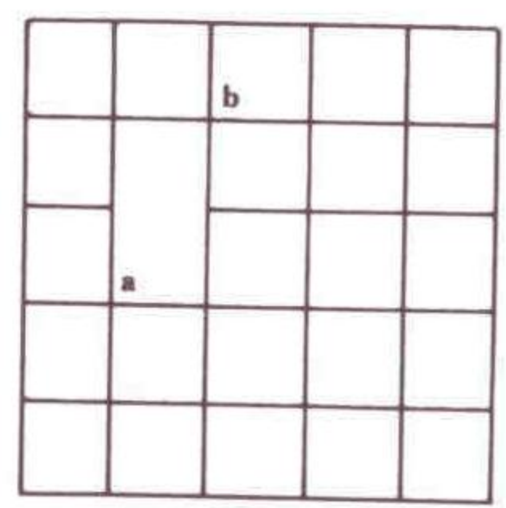

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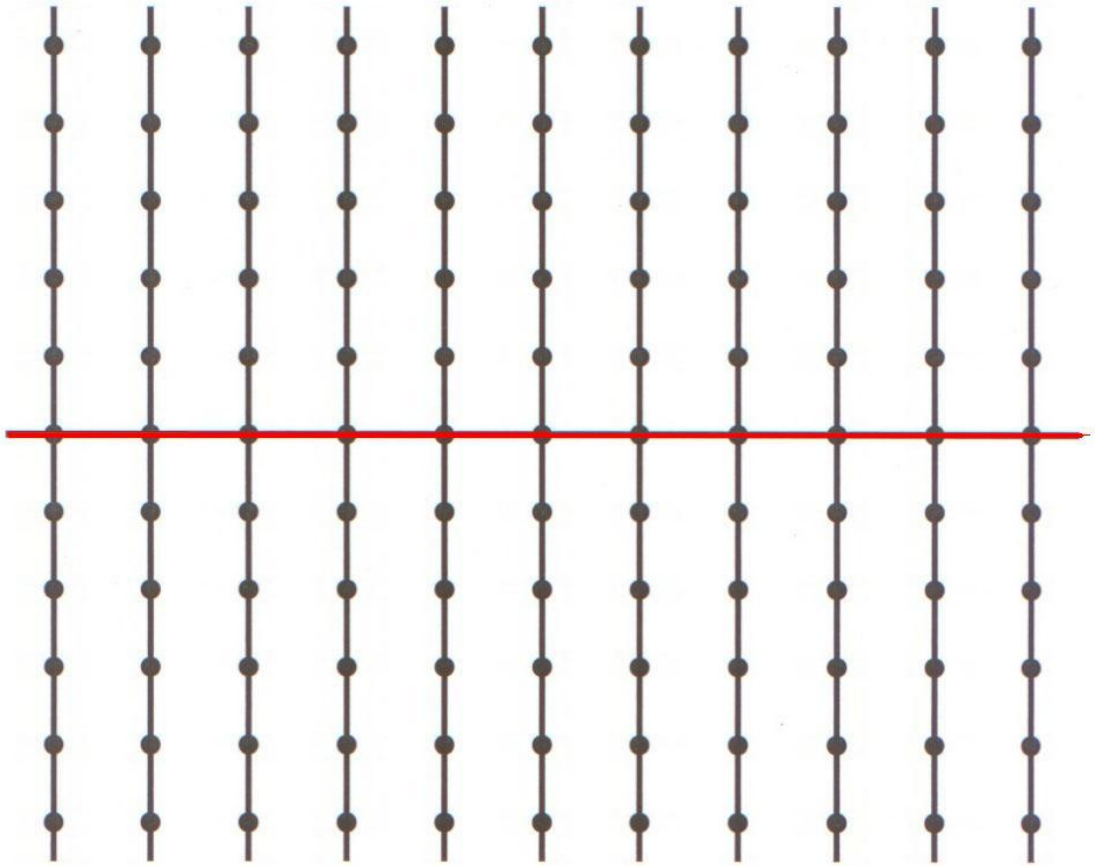
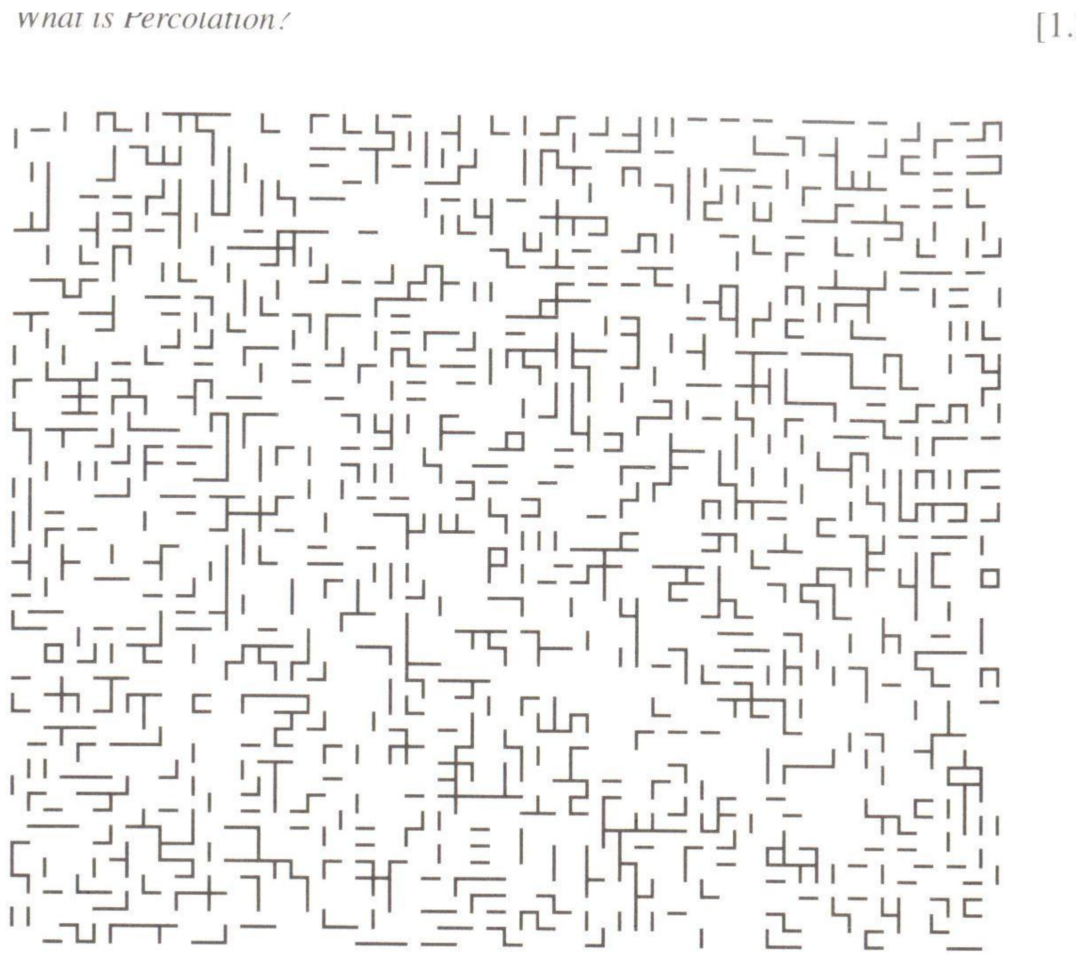


渗流模型



(a) $p = 0.25$

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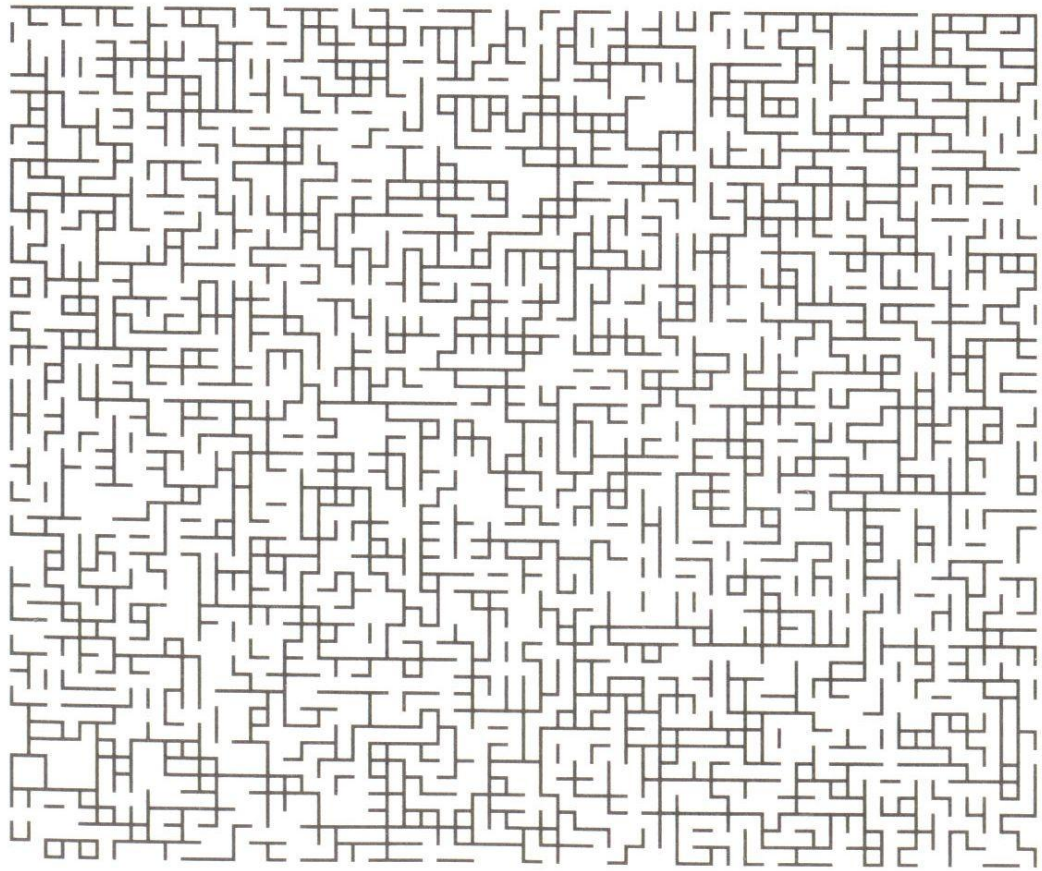


(b) $p = 0.49$

- 泛泛而论，二维图是常返的，而三维不常返

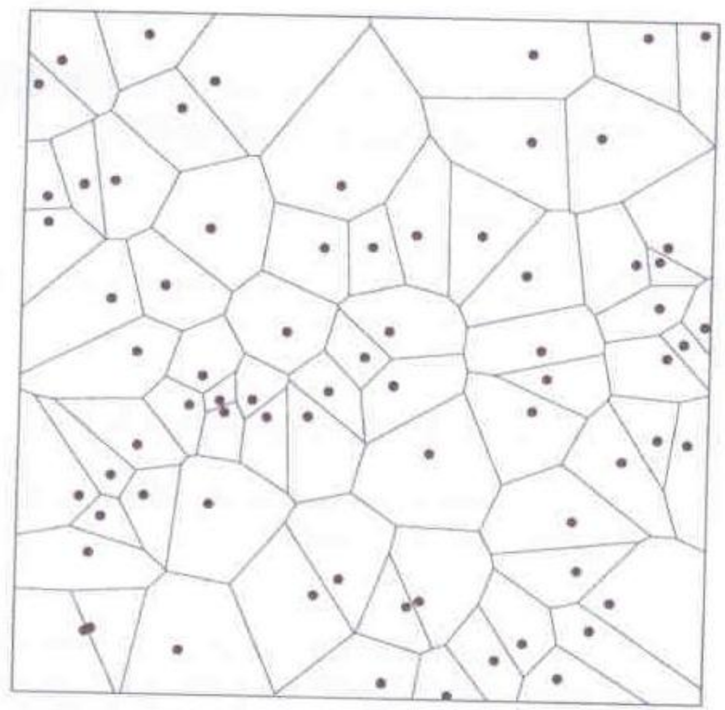


Fig. 1. A sample of Voronoi tessellation. Dotted points represent generating points. Polygon for each point is determined by the perpendicular bisectors between the point and its neighbouring points.

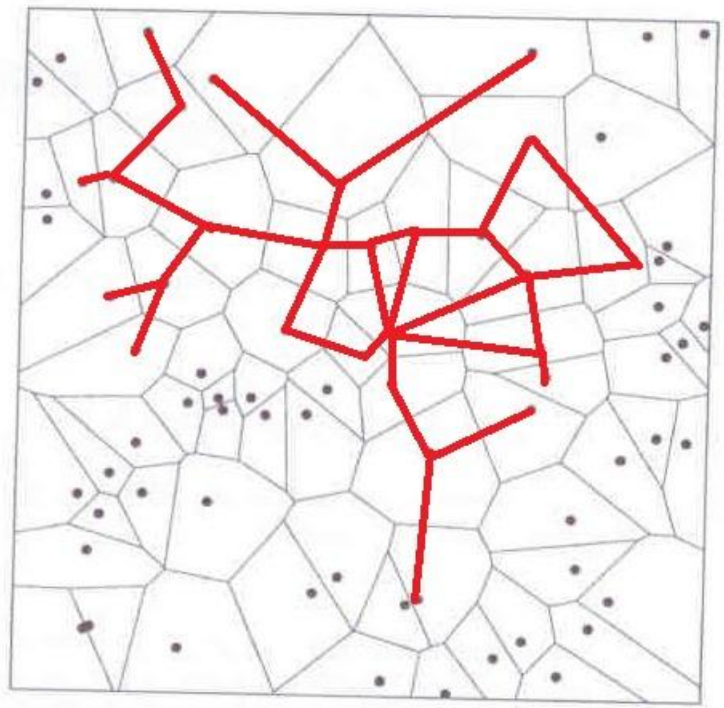


Fig. 1. A sample of Voronoi tessellation. Dotted points represent generating points. Polygon for each point is determined by the perpendicular bisectors between the point and its neighbouring points.

- 真正有难度的是三维格点的子图

楔子 wedge

$\{(x,y,z)\ ;\ |z| \leqq g(x)\}$

是常返的当且仅当$\Sigma\, 1/(x\, g(x)) = \infty$.

Scherk's graph

三维格点的渗流模型

定理(Grimmett, Kesten & Zhang).

The infinite open cluster of the Bernoulli bond percolation of Z^3 is transient.

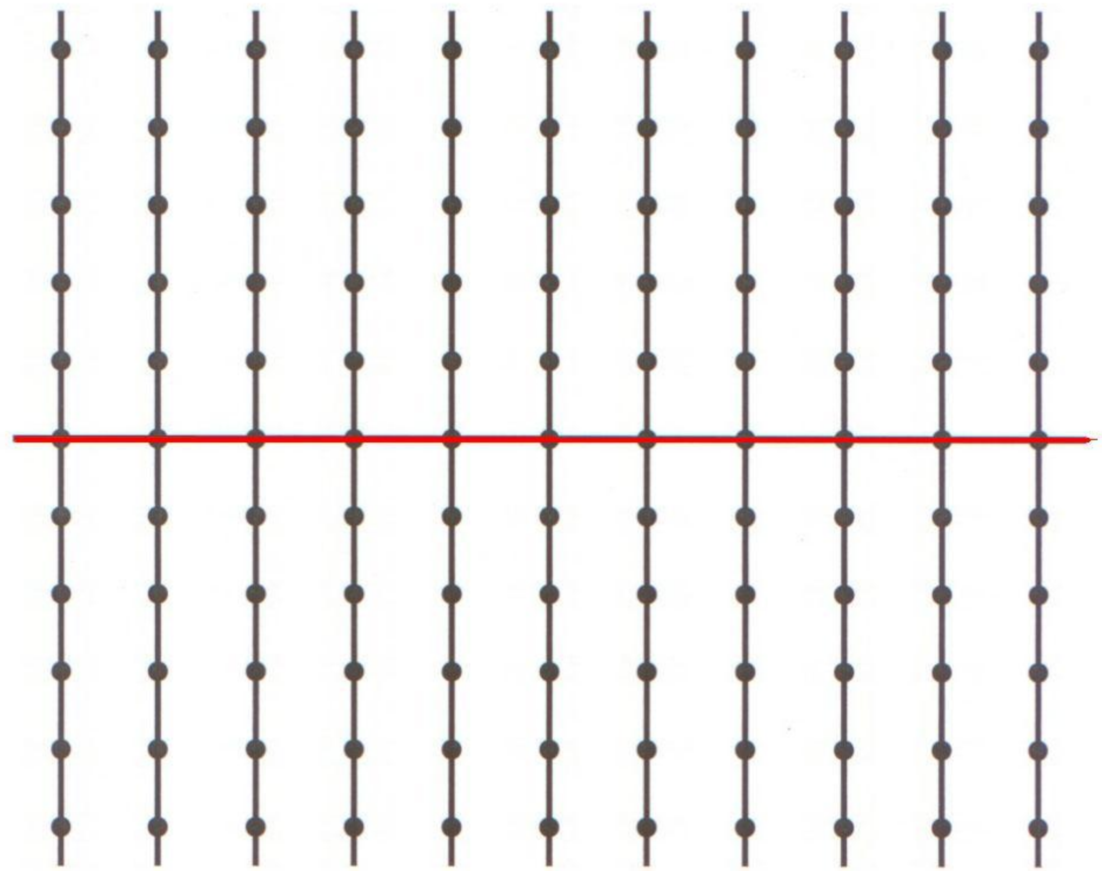
Scherk’s graph

相遇问题

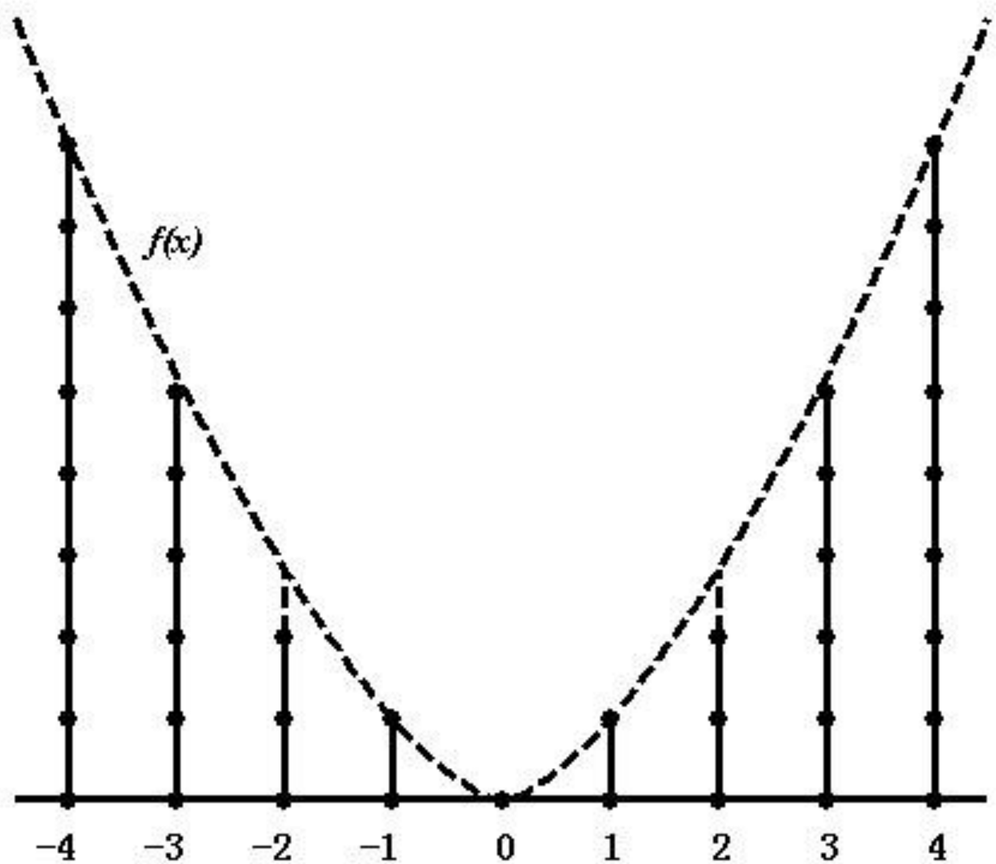
- 常返源自相遇
- 图越小越容易常返
- 图越小越容易相遇么？
- 常返而不相遇的例子（Liggett，1974）

相遇问题

- 2004年



相遇问题没有单调性！



相遇问题

- $f(x) = x^{\alpha}$， $\alpha \le 1/5$, 必定相遇. (Chen, Wei & Zhang, 2008).

- $f(x) = x^{\alpha}$, 若$\alpha < 1$, 则必定相遇；　若$\alpha > 1$, 则不一定相遇. (Barlow, Peres & Sousi, 2010).

- 若$f(x) \le x \log x$, 则必定相遇.
- 若$f(x) \ge x(\log x)^2$, 则不一定相遇.

- 记$h(n) = \max\{1, f(i), -n \le i \le n\}$, 若 $\Sigma_n 1/h(n) = \infty$, 则必定相遇(Chen & Chen, 2011).

- Wedge. Z^{d+1}的子图
- $W = \{(x_1, x_2, \ldots., x_d, z), \ |x_i| \leq f_i(z), \ 1 \leq I \leq d\}$.
- Terry Lyons证明W上简单随机游动是常返的当且仅当
- $\Sigma_n [\Pi_{i=1}^d f_i (n)]^{-1} = \infty$.
- 这也是两个随机游动必定相遇的充分必要条件.（陈新兴）

- Z^2上的开簇. 每一条边或以概率p开, 以概率1-p闭, 所有开边组成一些连通子图, 其中最大的一个是无穷的, 考虑其上的简单随机游动. 它是常返的, 其上两个独立随机游动必定相遇. 证明基于Barlow的热核估计. (Chen & Chen 2010, Barlow, Peres & Sousi 2010).

- 随机环境中的随机游动(RWRE), Z^2的每一条边赋以随机权重$\mu_e \geq 1$， $\{\mu_e, e\in E\}$, i.i.d. Barlow & Deuschel 得到了这一类随机游动的热核估计, 利用他们的结果可以证明其上两个独立随机游动必定相遇. 这个结论可以进一步用来证明随机环境中的选举模型的不变分布一定是a δ_0 + (1-a) δ_1. (Shan & Chen, 2011).

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- 谢谢大家！