**Fractal Geometry**

Reasons for studying this geometry:

(i). An extension of classical geometry such as Euclidean geometry, projective geometry. In classical geometries, the geometrical objects are smooth. In fractal geometry, the objects are ‘rough’. For instance, a fractal line is nowhere differentiable;

(ii). In dynamical systems, most chaotic regions are fractals.

**What is a fractal?**

Fractals have self-similar properties. If a portion of a fractal object is enlarged, the magnified portion always resembles the original figure.

E.g. coastal line of northern Europe, small cloud, fern etc., Cantor middle-thirds set.

How can we understand the geometric structure of fractals?

⇒ Definition of Dimension
Measurement of coastal line of H.K. island

Scale 1:100,000

Scales of measurement

How to measure a line segment of 1 m

<table>
<thead>
<tr>
<th>Scale</th>
<th>No. of measure</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}m$</td>
<td>2</td>
<td>$\frac{1}{2} \times 2 = 1m$</td>
</tr>
<tr>
<td>$\frac{1}{4}m$</td>
<td>4</td>
<td>$\frac{1}{4} \times 4 = 1m$</td>
</tr>
<tr>
<td>$r$</td>
<td>$N$</td>
<td>$rN = 1m$ or $r'^N = 1$</td>
</tr>
</tbody>
</table>
**How to measure square of area 1 m\(^2\)?**

<table>
<thead>
<tr>
<th>Scale</th>
<th>No. of measure</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{2}) m</td>
<td>4</td>
<td>(\left(\frac{1}{2}\right)^2 \times 4 = 1 m^2)</td>
</tr>
<tr>
<td>(\frac{1}{4}) m</td>
<td>16</td>
<td>(\left(\frac{1}{4}\right)^2 \times 16 = 1 m^2)</td>
</tr>
<tr>
<td>r</td>
<td>N</td>
<td>(r^2 N = 1 m^2) or (r^2 N = 1)</td>
</tr>
</tbody>
</table>

**Definition of Fractal Dimension**

Suppose that
- \(r\) is the scale of measure,
- \(N\) is the number of measure,
- \(L\) is the total length.

Then, the fractal dimension \(D\) is defined as

\[
r^D N = l
\]

as \(r \to 0\). Therefore,

\[
D = -\lim_{r \to 0} \left( \frac{\log N}{\log r} \right).
\]
Simple fractal objects
Example (a)
(Koch snowflake)

Fig. 14.8. The first four stages in the construction of Koch snowflake.

Fig. 14.9 Magnification of the Koch curve.
### Dimension

<table>
<thead>
<tr>
<th>Scale</th>
<th>No. of measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1/3</td>
<td>3 × 4</td>
</tr>
<tr>
<td>1/9</td>
<td>3 × 4²</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>1/3^n</td>
<td>3 × 4^n</td>
</tr>
</tbody>
</table>

\[ D = \frac{\log 3 \times 4^n}{\log 3^n} \]

\[ = \frac{\log 3 + n \log 4}{n \log 3} = \frac{1}{n} + \frac{\log 4}{\log 3} \]

As \( n \to \infty \), \( D = \frac{\log 4}{\log 3} \approx 1.26 \)

**Example (b) (Sierpinski Triangle)**
Dimension

<table>
<thead>
<tr>
<th>Scale</th>
<th>No. of measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>3</td>
</tr>
<tr>
<td>1/4</td>
<td>9</td>
</tr>
<tr>
<td>1/8</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2^n</td>
<td>3^n</td>
</tr>
</tbody>
</table>

\[ D = \frac{\log 3^n}{\log 2^n} = \frac{\log 3}{\log 2} \]

= 1.58…

**Iterated Function Systems (IFS)**

- A method to create many fractals.
- Create real life images such as fern.
- Applied to image compression.

**Definition**

Let \( 0 < \beta < 1 \). Let \( p_1, \ldots, p_n \) be points in the plane. Let \( A_i(p) = \beta(p-p_i)+p_i \) for each \( i = 1, \ldots, n \). The collection of functions \( \{A_1, \ldots, A_n\} \) is called an iterated function system.

If we choose a particular element, say \( A_1 \), the repeated iteration of \( A_1 \) to any point \( p \) in the plane converges to \( p_1 \).
To produce a fractal, we choose an arbitrary initial point in the plane and compute its orbit under random iteration of the $A_i$. This orbit converges with probability 1 to a specific subset of the plane.

**Definition**

Suppose $\{A_1, \ldots, A_n\}$ is an iterated function system. The set of points to which an arbitrary orbit in the plane converges is the attractor for the system.

**Example**

Let $A_1$ & $A_2$ in an IFS be defined as

\[
A_0 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x \\ y \end{pmatrix}
\]

\[
A_1 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x-1 \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Show that any orbit of the IFS tends to the Cantor middle-thirds sets.

**Solution:**

We note that the contraction factor here is $\frac{1}{3}$, and the fixed points are located at 0 & 1 along the x-axis.

Let $P_n$ be the $n^{th}$ point on the orbit with

\[
P_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix},
\]
Since \( y_n = \frac{y_{n-1}}{3} = \frac{y_0}{3^n} \), the orbit of any point tends toward the x-axis.

Computation of x-coordinates of points is as follows:

A sequence of iterations can be described by means of a sequence of 0’s & 1’s given by \((S_1, S_2, S_3, \ldots)\) where each \(S_j\) is either 0 or 1.

\[
S_j = k \quad (k = 0, 1)
\]

\(A_k\) is chosen at the \(j^{th}\) iteration.

Then \(X_n = \frac{x_0}{3^n} + \left( \frac{2S_1}{3^n} + \frac{2S_2}{3^{n-1}} + \frac{2S_3}{3^{n-2}} + \ldots \right)\)

where \(X_0\) is the initial x-coordinate.

As \(n \rightarrow 0\), the first term vanishes, implying that \(X_n\) is independent of \(X_0\). The remaining terms tend to an infinite series which is of the form

\[
\sum_{i=1}^{\infty} \frac{t_i}{3^i}
\]

where \(t_i\) is either 0 or 2. This series corresponds to points in the Cantor set.

**Example**

What is the IFS for the Sierpinski triangle?

**Solution:**

Construction of Sierpinski triangle
The IFS consists of 3 affine mappings:

1\textsuperscript{st} mapping: \( A \rightarrow A_1 \)
\[
A_1 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x - 0 \\ y - 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.
\]

2\textsuperscript{nd} mapping: \( A \rightarrow A_2 \)
\[
A_2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x \end{pmatrix}. \quad (\text{The origin is the fixed point.})
\]

3\textsuperscript{rd} mapping: \( A \rightarrow A_3 \)
\[
A_3 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x - 1 \\ y - 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.
\]

**Example**

What is the IFS of the box fractal?

Solution: Construction of the box fractal is shown below:
The unit square at the left is mapped to the five smaller squares. Therefore, the IFS consists of five affine mappings:

\[
A_1 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x - 0 \\ y - 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix},
\]

\[
A_2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix},
\]

\[
A_3 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x - 1/2 \\ y - 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix},
\]

\[
A_4 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x \\ y \end{pmatrix},
\]

\[
A_5 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x - 1 \\ y - 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}.
\]

We may include rotation to IFS as follows:

\[
A \begin{pmatrix} x \\ y \end{pmatrix} = \beta \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},
\]

\[P = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\] is the fixed point. Any other point is first contracted by a factor of \(\beta\) toward \(P\) and then rotated by angle \(\theta\) about \(P\).

**Example**

Find the attractor of the following IFS.

\[
P_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

\[\theta = \pi/4 \quad \text{and} \quad \beta = \frac{1}{2}.
\]
Solution: The attractor is as follows:

IFS code for the generation of Fern

It consists of 4 affine mappings.

If \[ A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}, \]

Then, the code is

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.16</td>
<td>0</td>
<td>0</td>
<td>0.01</td>
</tr>
<tr>
<td>0.85</td>
<td>0.04</td>
<td>-0.04</td>
<td>0.85</td>
<td>0</td>
<td>1.6</td>
<td>0.85</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.26</td>
<td>0.23</td>
<td>0.22</td>
<td>0</td>
<td>1.6</td>
<td>0.07</td>
</tr>
<tr>
<td>-0.15</td>
<td>0.28</td>
<td>0.26</td>
<td>0.24</td>
<td>0</td>
<td>0.44</td>
<td>0.07</td>
</tr>
</tbody>
</table>