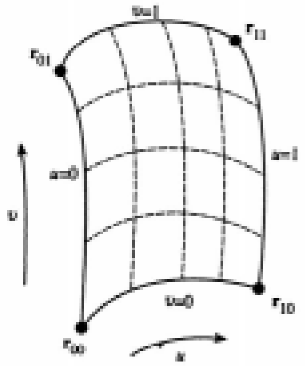
Chapter 6. Design of surfaces

6.1 Parameterization of surface patches

The ideas from the design of curves can be extended for surface patches. A rectangular surface patch bounded by *u=0, u=1, v=0* and *v=1* can be assumed with corner position vectors $\vec{r}_{00}, \vec{r}_{10}, \vec{r}_{01}$ and $\vec{r}_{11}$.



Rectangular surface patch

A set of position vectors $\vec{r}_{ij}$ ($i=0,1,\cdots,N,\ j=0,1,\cdots,M$) may be used as control points; more conveniently, corner derivatives values are specified instead.

6.2 Bezier patches

A Bezier patch defined by (N+1)$\times$(M+1) control points $\vec{r}_{ij}$ can be written as

$$\vec{r}(u,v) = \sum_{i=0}^{N}\sum_{j=o}^{M} b_i^N(u) b_j^M(v)\vec{r}_{ij}$$

where $b^N{}_i$, $b^M{}_j$ are the Nth and Mth Bernstein polynomials. It will be convenient to use matrix notations and let

$$\vec{u}_N = (1 \quad u \quad u^2 \quad ----- \quad u^N)^T, \vec{v}_M = (1 \quad v \quad v^2 \quad ----- \quad v^M)^T, \vec{P} = [\vec{r}_{ij}]$$

then
$$\vec{b}^N(u) = [b_0^N(u) \quad b_1^N(u) \quad ---- \quad b_N^N(u)]^T = C_N \vec{u}_N$$

where C_N is a coefficient matrix (N+1)$\times$(N+1) for given N.
and

$$\vec{r}(u,v) = (\vec{b}^N(u))^T \cdot \vec{P} \cdot \vec{b}^M(v)$$
$$= (\vec{u}_N)^T C_N^T \vec{P} C_M \vec{v}_M$$

Example <u>Bicubic Bezier patch</u>

In the case N=M=3, $\vec{b}^3(u) = \left[(1-u)^3 \quad 3u(1-u)^2 \quad 3u^2(1-u) \quad u^3\right]$

$$C_3 = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \vec{P} = \begin{bmatrix} \vec{r}_{00} & \vec{r}_{01} & \vec{r}_{02} & \vec{r}_{03} \\ \vec{r}_{10} & - & - & - \\ \vec{r}_{20} & - & - & - \\ \vec{r}_{30} & - & - & \vec{r}_{33} \end{bmatrix}$$

and $\vec{r}(u,v) = (\vec{u}_3)^T C_3^T \vec{P} C_3 \vec{v}_3,$ $\begin{matrix} \vec{u}_3 = (1 \quad u \quad u^2 \quad u^3) \\ \vec{v}_3 = (1 \quad v \quad v^2 \quad v^3) \end{matrix}$

Hence $\vec{r}(0,0) = \vec{r}_{00}, \vec{r}(0,1) = \vec{r}_{03}, \vec{r}(1,0) = \vec{r}_{30}, \vec{r}(1,1) = \vec{r}_{33}$

Also

$$\vec{r}_u(u,v) = (0 \quad 1 \quad 2u \quad 3u^2) C_3^T \vec{P} C_3 \vec{v}_3$$
$$\vec{r}_v(u,v) = (\vec{u}_3)^T C_3^T \vec{P} C_3 (0 \quad 1 \quad 2v \quad 3v^2)^T$$
$$\vec{r}_{uv}(u,v) = (0 \quad 1 \quad 2u \quad 3u^2) C_3^T \vec{P} C_3 \cdot (0 \quad 1 \quad 2v \quad 3v^2)^T$$

It is then possible to relate corner derivatives to $\vec{r}_{ij}$:

$$T = \begin{bmatrix} \vec{r}(0,0) & \vec{r}(0,1) & \vec{r}_v(0,0) & \vec{r}_v(0,1) \\ \vec{r}(1,0) & \vec{r}(1,1) & \vec{r}_v(1,0) & \vec{r}_v(1,1) \\ \vec{r}_u(0,0) & \vec{r}_u(0,1) & \vec{r}_{uv}(0,0) & \vec{r}_{uv}(0,1) \\ \vec{r}_u(1,0) & \vec{r}_u(1,1) & \vec{r}_{uv}(1,0) & \vec{r}_{uv}(1,1) \end{bmatrix} = Q\vec{P}Q^T$$

where $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix}$

If $\vec{r}, \vec{r}_u, \vec{r}_v, \vec{r}_{uv}$ at the corners are given, then $\vec{P}$ can be specified.

6.3 Rational surface patches

If we use homogeneous coordinates $\vec{R} = \omega(\vec{r},1)^T$ and follow the procedure in the previous section, then

$$\vec{R}(u,v) = (\vec{u}_N)^T C_N^T \vec{\Re} C_M \vec{v}_M, where \vec{\Re} = [\vec{R}_{ij}]$$

Hence $$\begin{bmatrix} \omega(u,v)\vec{r}(u,v) \\ \omega(u,v) \end{bmatrix} = (\vec{u}_N)^T C_N^T \begin{bmatrix} \omega_{ij}\vec{r}_{ij} \\ \omega_{ij} \end{bmatrix} C_M \vec{v}_M$$

Thus
$$\omega(u,v) = (\vec{u}_N)^T C_N^T [\omega_{ij}] C_M \vec{v}_M$$

and $$\vec{r}(u,v) = \frac{(\vec{u}_N)^T C_N^T [\omega_{ij}\vec{r}_{ij}] C_M \vec{v}_M}{\omega(u,v)}$$

6.4 Composite surface patches

In general, it is very difficult to join two surface patches along an edge having position, gradient and even curvature continuity. Most surface patches cannot be matched to have the same gradient. Depending on the application requirements of the geometry, (i.e. plane elasticity or plate bending), composite surface with the required continuity is called conforming, otherwise non-conforming.