Chapter 5. Differential Geometry of Surfaces

5.1 Surface in parametric form

In 3D, a surface can be represented by

(1). Explicit form $z = f(x,y)$

(2). Implicit form $f(x,y,z) = 0$

(3). Vector form $\vec{r}(x,y) = (x,y,f(x,y))^T$, or more general $\vec{r}(u,v)=(x(u,v),y(u,v),z(u,v))^T$ depending on two parameters.

Example 1. The sphere of radius a has the geographical form

$$\vec{r}(\theta,\phi)=(a\cos\theta\cos\phi, a\cos\theta\sin\phi, a\sin\theta)^T \qquad 0 \le \theta \le \pi,\quad 0 \le \phi \le 2\pi$$

Example 2. The cylinder built on the curve $\vec{r}(t) = (x(t),y(t))^T$, $a \le t \le b$ in the xy-plane has the form

$\vec{r}(u,v)=(x(u),y(u),v)^T$, $\quad a \le u \le b$, $-\infty < v < \infty$

Example 3. Surface of revolution by rotating a curve $\vec{r}(t) = (p(t),0,q(t))^T$ $(a \le t \le b)$ about the z-axis

$$\vec{r}(u,v) = R_z\vec{r}(t) = \begin{bmatrix} \cos v & -\sin v & \\ \sin v & \cos v & \\ & & 1 \end{bmatrix}\begin{pmatrix} p(u) \\ 0 \\ q(u) \end{pmatrix} = (p(u)\cos v, p(u)\sin v, q(u))^T$$

$$a \le u \le b,\ 0 \le v \le 2\pi$$

Specical cases are: a torus with $\vec{r}(t) = (R + a\cos t, 0, a\sin t)^T$, $\quad 0 \le t \le 2\pi$

A cone with $\vec{r}(t) = (t, 0, mt)^T$, $\quad -\infty \le t \le \infty$

<u>Tangent vectors</u> on the surface are $\vec{r}_u(u,v)$ and $\vec{r}_v(u,v)$. Hence a unit <u>normal</u> $\hat{n}$ at r(u,v) is given by

$$\hat{n} = \pm\frac{\vec{r}_u(u,v)\times\vec{r}_v(u,v)}{|\vec{r}_u(u,v)\times\vec{r}_v(u,v)|}$$

assuming $\vec{r}_u \times \vec{r}_v \neq 0$ at (u,v), ( non-singular point).

If the surface is given implicitly $f(x,y,z) = 0$, then

$$\vec{n} = \pm\frac{\nabla f}{|\nabla f|}$$

## 5.2 Metric properties

Distance on the surface is measured by

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = d\vec{r} = \vec{r}_u du + \vec{r}_v dv = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = A d\vec{u}$$

$$ds^2 = d\vec{r} \cdot d\vec{r} = (d\vec{r})^T d\vec{r} = d\vec{u}^T A^T A d\vec{u} = d\vec{u}^T B d\vec{u}$$

where $B = A^T A = \begin{bmatrix} \vec{r}_u \cdot \vec{r}_u & \vec{r}_u \cdot \vec{r}_v \\ \vec{r}_v \cdot \vec{r}_u & \vec{r}_v \cdot \vec{r}_v \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ in standard notation

This is the $1^{st}$ fundamental form of the surface:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

The unit tangent $\hat{t}$ along the curve $\vec{r}$ (t) = $\vec{r}$ (u(t),v(t)) is

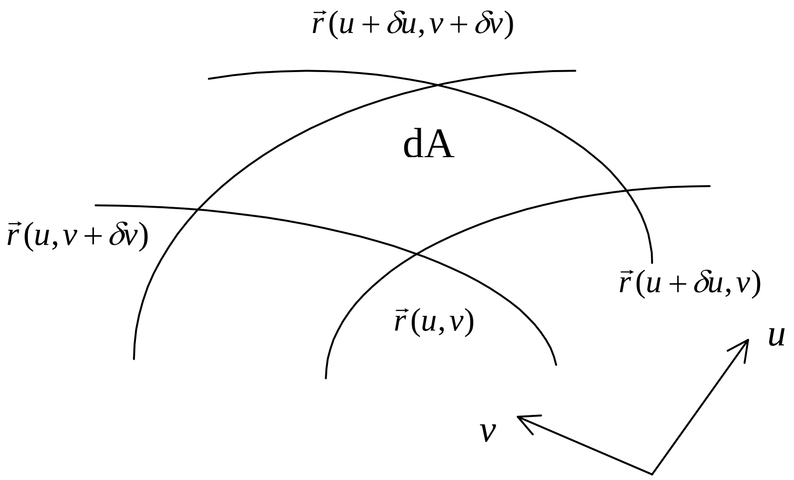
$$\hat{t} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = \frac{A\dot{\vec{u}}}{(\dot{\vec{u}}^T B\dot{\vec{u}})^{1/2}}$$

The length of the segment of the curve $\vec{r}$ (t) from t = $t_0$ to t = $t_1$ is

$$s = \int_{t_2}^{t_1} |\dot{\vec{r}}| \, dt = \int_{t_2}^{t_1} (\dot{\vec{u}}^T B\dot{\vec{u}})^{1/2} dt$$

If two curves $\vec{r}_i(t) = \vec{r}(u_i(t), v_i(t))$ intersect at an angle θ on the surface, then

$$\cos\theta = \hat{t}_1 \cdot \hat{t}_2 = \frac{\dot{\vec{u}}_1 A^T A\dot{\vec{u}}_2}{(\dot{\vec{u}}_1 B\dot{\vec{u}}_1)^{1/2}(\dot{\vec{u}}_2 B\dot{\vec{u}}_2)^{1/2}} = \frac{\dot{\vec{u}}_1 B\dot{\vec{u}}_2}{dS_1 \cdot dS_2}$$



An elementry area dA will be given by

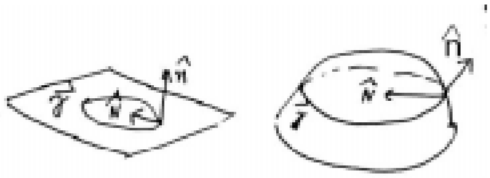
$$dA = \left| [\vec{r}(u+du,v) - \vec{r}(u,v)] \times [\vec{r}(u,v+dv) - \vec{r}(u,v)] \right|$$

$$\approx \left| \vec{r}_u du \times \vec{r}_v dv \right|$$

$$= \left| \vec{r}_u \times \vec{r}_v \right| dudv$$

We have $\left| \vec{r}_u \times \vec{r}_v \right|^2 = |\vec{r}_u|^2 \cdot |\vec{r}_v|^2 - (\vec{r}_u \cdot \vec{r}_v)^2 = EG - F^2 = \det(B) = |B|$

Thus $A_R = \iint_R |B|^{\frac{1}{2}} du \cdot dv$ for any Region R on the surface.



5.3 Curvatures

Recalled that for a general space curve $\vec{\gamma}(t)$ ,

$\dot{\vec{\gamma}}(t) = \dot{s}\hat{t}$ and $\ddot{\vec{\gamma}}(t) = \ddot{s}\hat{t} + \dot{s}^2 k\hat{N}$

where $\hat{N}$ is the principal normal to the curve, not to confuse with the normal $\hat{n}$ to the surface. If $\vec{\gamma}(t)$lies on the surface, then $\vec{\gamma}(t) = \vec{r}(u(t), v(t))$, so that

$$\dot{\vec{r}} = \vec{r}_u \dot{u} + \vec{r}_v \dot{v}$$

Differentiating again

$$\ddot{\vec{r}} = \vec{r}_{uu}\dot{u}^2 + 2\vec{r}_{uv}\dot{u}\dot{v} + \vec{r}_{vv}\dot{v}^2 + \vec{r}_u \ddot{u} + \vec{r}_v \ddot{v}$$

Notice that the surface normal $\hat{n}$ is perpendicular to $\hat{t}$ , $\vec{r}_u$ and $\vec{r}_v$

$$\therefore \ddot{\vec{r}} \cdot \hat{n} = \dot{S}^2 k\hat{N} \cdot \hat{n} = \hat{n} \cdot \vec{r}_{uu}\dot{u}^2 + 2\hat{n} \cdot \vec{r}_{uv}\dot{u}\dot{v} + \hat{n} \cdot \vec{r}_{vv}\dot{v}^2$$

$$= \dot{\vec{u}}^T \begin{bmatrix} \hat{n} \cdot \vec{r}_{uu} & \hat{n} \cdot \vec{r}_{uv} \\ \hat{n} \cdot \vec{r}_{uv} & \hat{n} \cdot \vec{r}_{vv} \end{bmatrix} \dot{\vec{u}} = \dot{\vec{u}}^T D \dot{\vec{u}}$$

The right-hand side is the second fundamental form of the surface,

$D = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$ is standard notation.

The normal curvature $\kappa_n$ of the curve $\vec{\gamma}(t)$ in the surface is defined to be

$$\kappa_n = \frac{\ddot{\vec{r}} \cdot \hat{n}}{\dot{S}^2} = \frac{\dot{\vec{u}}^T D \dot{\vec{u}}}{\dot{\vec{u}}^T B \dot{\vec{u}}}$$

Note that $\kappa_n = \kappa \hat{N} \cdot \hat{n} = \kappa \cos\theta$, i.e. it is the component of $\kappa$ in the direction of $\hat{n}$. The other component of $\kappa$ in the tangent plane is known as the geodesic curvature $\kappa_g$, because of orthogonality, it has the magnitude

$$\kappa_g{}^2 = \kappa^2 - \kappa_n{}^2 = \kappa^2 \sin^2\theta$$

A surface curve $\vec{\gamma}(t)$, for which $\kappa_g = 0$ at every point is called a geodesic, (as straight as possible on the surface).

Consider row $\kappa_n$ as a function of the direction $\dot{\vec{u}} = \left(\frac{du}{dt}, \frac{dv}{dt}\right)^T$,

Let $\lambda = \frac{dv}{du}$ then

$$\kappa_n = \kappa_n(\lambda) = \frac{L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}$$

As λ changes direction in the surface, $\kappa_n$ will achieve a maximum and a minimum value unless L: M: N = E: F: G, in that case $\kappa_n$ is independent of λ (such locations are called umbilic points)

Setting $\frac{d\kappa_n}{d\lambda} = 0$, the principal directions λ and the corresponding principal curvatures k are governed by

$$(FN - GM)\lambda^2 + (EN - GL)\lambda + (EM - FL) = 0$$

and

$$(EG - F^2)k^2 - (LG + NE - 2FM)k - (LN - M^2) = 0$$

The solutions satisfy

$$E + F(\lambda_1 + \lambda_2) + G\lambda_1\lambda_2 = 0$$

$$\tfrac{1}{2}(k_1 + k_2) = \frac{LG - 2FM + NE}{2(EG - F^2)}$$

$$k_1 k_2 = \frac{LN - M^2}{EG - F^2} = \frac{\det(D)}{\det(B)}$$

$K = k_1 k_2$ is called the Gaussian Curvature.

$H = \tfrac{1}{2}(k_1 + k_2)$ is the mean Curvature (Germain Curvature).

If the two directions $\lambda_1$, $\lambda_2$ correspond to

$$d\vec{r}_1 = d\vec{r}(\lambda_1) = \vec{r}_u du_1 + \vec{r}_v dv_1$$

$$d\vec{r}_2 = d\vec{r}(\lambda_2) = \vec{r}_u du_2 + \vec{r}_v dv_2$$

then $\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2 = E\frac{du_1}{dt}\frac{du_2}{dt} + F\left(\frac{du_1}{dt}\frac{dv_2}{dt} + \frac{du_2}{dt}\frac{dv_1}{dt}\right) + G\frac{dv_1}{dt}\frac{dv_2}{dt} = 0$

therefore the two principal directions are orthogonal.

Some geometrical meaning of the curvatures are the following.

1) A surface is called minimal if H = 0 everywhere. A minimal surface with boundary *l* has the smallest surface area among all surfaces with boundary *l*.
2) If the principal directions are taken as the parametric curves, then $F \equiv 0 \equiv M$ and

$$k_1 = \frac{L}{E}, k_2 = \frac{N}{G}$$

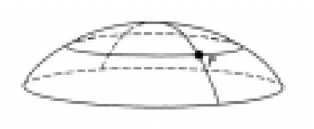
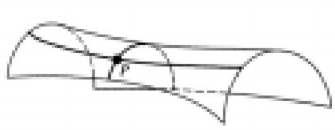
curvature in any other direction $\lambda$ is then given by

$$K(\lambda) = \frac{L + N\lambda^2}{E + G\lambda^2} = k_1\frac{E}{E + G\lambda^2} + k_2\frac{G\lambda^2}{E + G\lambda^2}$$

$$= k_1\cos^2\psi + k_2\sin^2\psi$$

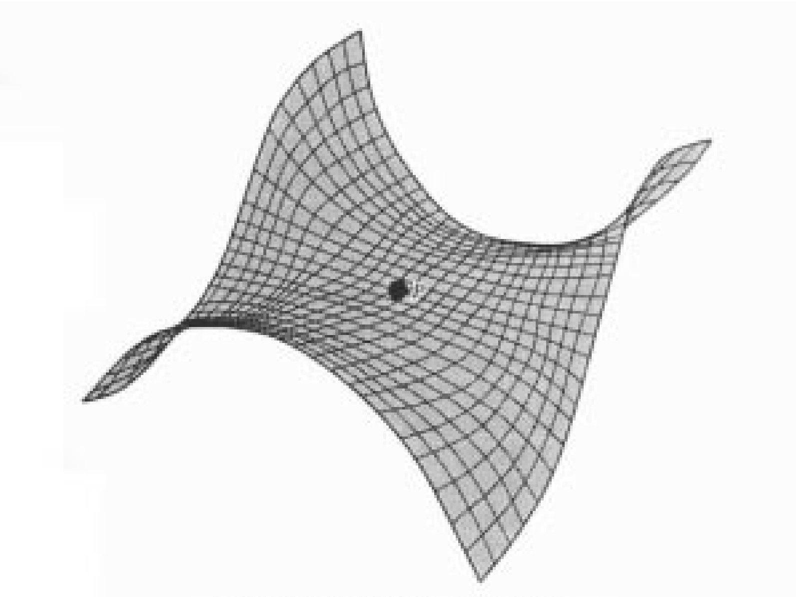
where $\psi$ is the angle between $\vec{r}_u$ and $\dot{\vec{r}} = \vec{r}_u\dot{u} + \vec{r}_v\dot{v}$, $\left(\lambda = \frac{\dot{v}}{\dot{u}}\right)$ which is known as the Euler's formula.

3) If K>0 at a point P on the surface, then P is an elliptic point. As $k_1$, $k_2$ have the same sign, so all the surface is bending the same way in all directions.


4) If K<0 at a point P, then it is an hyperbolic point. Tangent directions at P can bend away or towards the tangent plane.

5) If K=0 then either
(a) One principal curvature only is zero. The point P is a parabolic point, and one of the principal direction is straight near P.
(b) Both principal curvature are zero. The point is a special type of umbilic point and is planar.
Note: Isolated planar points can exist on surfaces which is far from planar. E.g. the monkey saddle surface $z = x(x+\sqrt{3}y)(x-\sqrt{3}y)$ at P(0,0,0)



The monkey saddle point

6) Consider a point P on the surface $z = \psi(x,y)$ By a change of coordinates, choose the origin at P, and *x, y* axes along the principal directions at P, also z-axis in the direction of the surface normal at P, then the surface has equation $z = f(x,y)$ local to (0,0,0) with $f(0,0)=0$

$$\frac{\partial f(0,0)}{\partial x} = \frac{\partial f(0,0)}{\partial y} = 0 \qquad (\because \text{x-y is the tangent plane})$$

and
$$z = f(x,y) = \tfrac{1}{2}\left\{\frac{\partial^2 f(0,0)}{\partial x^2}x^2 + 2\frac{\partial^2 f(0,0)}{\partial x \partial y}xy + \frac{\partial^2 f(0,0)}{\partial y^2}y^2\right\} + O(x^3, y^3)$$

Now, taking $\vec{r} = (x, y, f(x, y))$

$$\vec{r}_x = \left(1,0,\frac{\partial f}{\partial x}\right),\ \vec{r}_y = \left(0,1,\frac{\partial f}{\partial y}\right)$$

At $P$ = (0,0,0), $\vec{r}_x(p) = (1,0,0)$, $\vec{r}_y(p) = (0,1,0)$ and $\hat{n} = (0,0,1)$

$$E = \vec{r}_x(p)\cdot\vec{r}_x(p) = 1,\ F = \vec{r}_x(p)\cdot\vec{r}_y(p) = 0,\ G = \vec{r}_y(p)\cdot\vec{r}_y(p) = 1$$

the principal directions are orthogonal. Also

$$\vec{r}_{xx}(p) = \left(0,0,\frac{\partial^2 f}{\partial x^2}\right)_p,\ \vec{r}_{xy}(p) = \left(0,0,\frac{\partial^2 f}{\partial x \partial y}\right)_p,\ \vec{r}_{yy}(p) = \left(0,0,\frac{\partial^2 f}{\partial y^2}\right)_p$$

Hence
$$L = \hat{n}\cdot\vec{r}_{xx}(p) = \frac{\partial^2 f(0,0)}{\partial x^2}$$
$$M = \hat{n}\cdot\vec{r}_{xy}(p) = \frac{\partial^2 f(0,0)}{\partial x \partial y}$$
$$N = \hat{n}\cdot\vec{r}_{yy}(p) = \frac{\partial^2 f(0,0)}{\partial y^2}$$

$$\begin{aligned}\therefore z &= \tfrac{1}{2}\left\{Lx^2 + 2Mxy + Ny^2\right\} + O\left(x^3, y^3\right)\\ &= \tfrac{1}{2}\left\{k_1 x^2 + k_2 y^2\right\} + O\left(x^3, y^3\right)\end{aligned}$$

These conics justify the terminology used. The simplest surface on which all three cases of Gaussian curvature $\begin{pmatrix} > \\ K = 0 \\ < \end{pmatrix}$ occur is the torus.

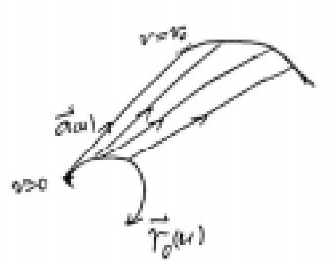
## 5.4 Special cases

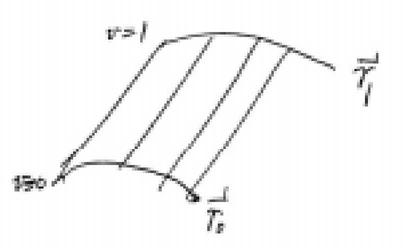
### 5.4.1 Developable surfaces

Consider a surface in $\mathbf{R}^3$ which is constructed by a moving straight line, this so called <u>ruled surface</u> has the form

$$\vec{r}(u,v) = \vec{r}_0(u) + v\vec{a}(u)$$

where $\vec{r}_0(u)$ is the position vector of a point on a given line, $\vec{a}(u)$ is the direction of the moving line (generators)



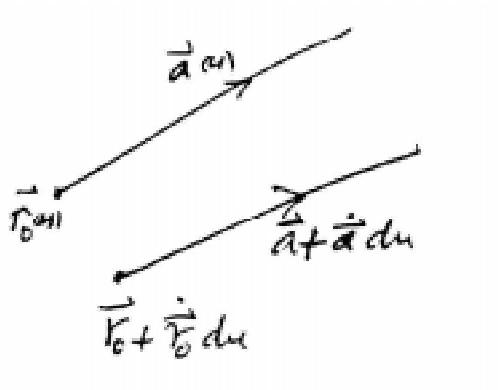




or if straight lines are used to join two given curves $\vec{r}_0(u),\ \vec{r}_1(u)$ then

$$\vec{r}(u,v) = (1-v)\vec{r}_0(u) + v\vec{r}_1(u)$$

Examples are cylinders and cones.
Now we look for conditions so that a ruled surface can be unrolled into a flat plane without distortion, (i.e. distances are preserved). If a ruled surface is developable, then all the generators eventually lie on a plane, therefore they are either parallel or intersect one another.
Now the intersection of two generators $\vec{a}$ and $\vec{a} + \dot{\vec{a}}du$ is governed by

$$\left(\vec{r}_0 + \dot{\vec{r}}du - \vec{r}_0\right)\cdot\left[\vec{a} \times \left(\vec{a} + \dot{\vec{a}}du\right)\right] = \dot{\vec{r}}_0 \cdot \left(\vec{a} \times \dot{\vec{a}}\right) = 0$$



In case all the generators are parallel, the above condition is also satisfied. Therefore it is the condition for a ruled surface becomes developable. As

$$\vec{r}_u = \dot{\vec{r}}_0 + v\dot{\vec{a}}, \quad \vec{r}_v = \vec{a}, \quad \hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{\left|\vec{r}_u \times \vec{r}_v\right|}$$

$$\vec{r}_{uu} = \ddot{\vec{r}}_0 + v\ddot{\vec{a}}, \quad \vec{r}_{uv} = \dot{\vec{a}}, \quad \vec{r}_{vv} = 0$$

therefore, $N = \hat{n} \cdot \vec{r}_{vv} = 0$

$$M = \hat{n} \cdot \vec{r}_{uv} = \hat{n} \cdot \dot{\vec{a}} = \frac{1}{\left|\vec{r}_u \times \vec{r}_v\right|}\left(\dot{\vec{r}}_0 + v\dot{\vec{a}}\right) \times \vec{a} \cdot \dot{\vec{a}} = 0$$

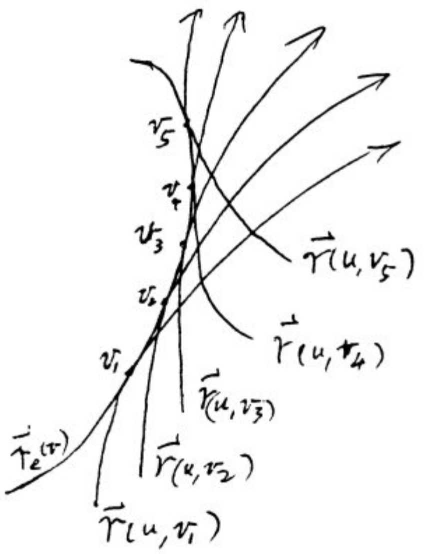
$$\therefore K = \frac{\det(D)}{\det(B)} = \frac{LN - M^2}{\det(B)} = 0 \qquad \text{for developable surfaces.}$$

In case the ruled surface is governed by two curves $\vec{r}_0(u)$, $\vec{r}_1(u)$, the condition becomes $(\vec{r}_1 - \vec{r}_0) \cdot (\dot{\vec{r}}_0 \times \dot{\vec{r}}_1) = 0$, this fact is used in the tangent plane method of generating developable surface passing through two curves.

5.4.2 Envelope of space curves

Regarding $\vec{r} = \vec{r}(u,v)$ as a family of curves $\vec{r} = \vec{r}_{(v)}(u)$ depending on a parameter v. There may exist a curve $\vec{r} = \vec{r}_e(v)$ which is tangential to every curve $\vec{r}_{(v)}(u)$ at the parametric value $v$. Such a curve, if it exists, is called the envelope of the family of curves.
In terms of the original parameters $\vec{r}(u,v)$ it implies $\vec{r}_u$ is parallel to $\vec{r}_v$ and the surface normal is not defined at these points:

$$\vec{r}_u \times \vec{r}_v = 0$$




In case of the developable surface, the generators will have an envelope if they are not parallel. An envelope satisfies

$$\vec{r}_u \times \vec{r}_v = \left(\dot{\vec{r}}_0 + v\dot{\vec{a}}\right) \times \vec{a} = 0$$

so long as the generators are not parallel, $\dot{\vec{a}} \times \vec{a} \neq 0$ hence the location of common tangent occurs at

$$v = \frac{\left(\vec{a} \times \dot{\vec{a}}\right) \cdot \left(\dot{\vec{r}}_0 \times \vec{a}\right)}{\left|\vec{a} \times \dot{\vec{a}}\right|^2}$$

and the equation of the envelope of the generators is

$$\vec{r}_e = \vec{r}_0(u) + \frac{(\vec{a} \times \dot{\vec{a}}) \cdot (\dot{\vec{r}}_0 \times \vec{a})}{\left|\vec{a} \times \dot{\vec{a}}\right|^2} \vec{a}(u)$$