Chapter 4 Design of Curves

4.1 Interpolation and approximation

There are two ways in which a set of control points (knots) $\{P_0, P_1, \cdots P_N\}$ can be used to describe a curve. The first is called <u>interpolation</u> in which the curve passes through each control point. The second is called <u>approximation</u> in which the curve does not necessarily pass through any of the control points, but usually passes close to all of them. The latter is useful in design, where we would like to change the shape of the curve quickly by alternating some pre-defined parameters. With this in mind, we would like to define a curve by

$$\vec{r}(u) = \sum_{i=0}^{N} \vec{r}_i f_i(u), \quad u \in [0,1],$$

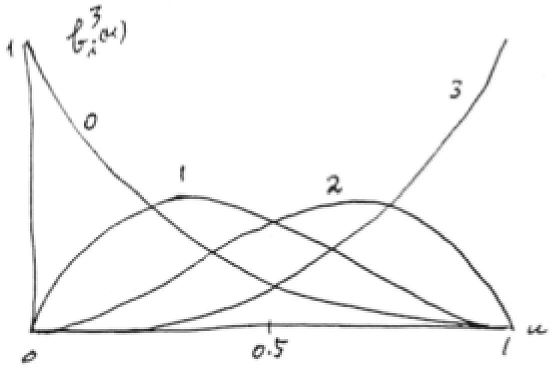
where $\vec{r}_i$ are given vectors and f_i are called <u>blending functions</u>. The curve is parameterized to lie between u=0 and u=1. Both interpolation and approximation can be written in this form, familiar examples for interpolation would be Larange interpolation, Hermite cubic interpolation, cubic splines etc.

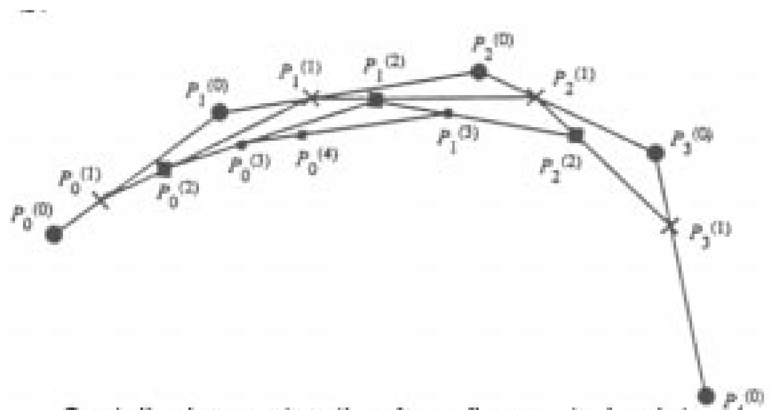
4.2 Bezier curves (1970)

The blending function of the Bezier curves of order N are the Bernstein polynomials

$$b_i^N(u) = \binom{N}{i} u^i (1-u)^{N-i}$$

so that $\vec{r}(u) = \sum_{i=0}^{N} \vec{r}_i b_i^N(u), \qquad 0 \le u \le 1.$





Casteljau's construction from five control points

There is a simply recursive construction of the Bezier curves by <u>Casteljau's algorithm</u>. Given $\vec{r}_i = \vec{r}_i^{(0)}$, a sequence of points $\vec{r}_i^{(j)}$ $(j = 0,1,\ldots N)$ are defined recursively if

$$\vec{r}_i^{(j+1)}(u) = (1-u)\vec{r}_i^{(j)} + u\vec{r}_{i+1}^{(j)}, \quad u \in [0,1]$$

The point $\vec{r}_0^{(N)}(u)$ lies on the Bezier curve of order N. As u varies from 0 to 1, the locus of the point $\vec{r}_0^{(N)}(u)$ produces the complete curve.

For example, in the quardratic case

$$\begin{aligned}\vec{r}(u) &= \vec{r}_0^{(2)}(u) = (1-u)\vec{r}_0^{(1)} + u\vec{r}_1^{(1)} \\ &= (1-u)[(1-u)\vec{r}_0^{(0)} + u\vec{r}_1^{(0)}] + u[(1-u)\vec{r}_1^{(0)} + u\vec{r}_2^{(0)}] \\ &= (1-u)^2\vec{r}_0^{(0)} + 2u(1-u)\vec{r}_1^{(0)} + u^2\vec{r}_2^{(0)} \\ &= b_0^2(u)\vec{r}_0 + b_1^2(u)\vec{r}_1 + b_2^2(u)\vec{r}_2\end{aligned}$$

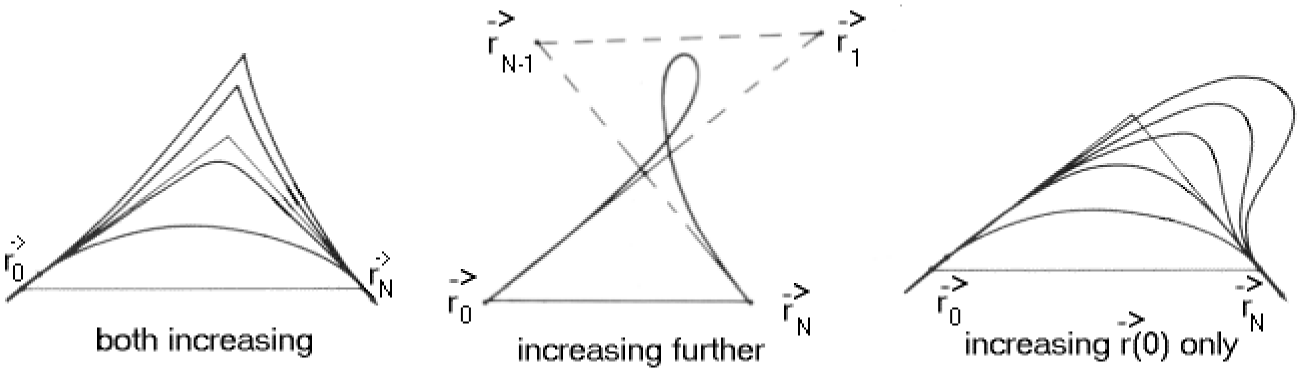
We note also $\vec{r}(0) = \vec{r}_0$, $\vec{r}(1) = \vec{r}_N$, $\dot{\vec{r}}(0) = N(\vec{r}_1 - \vec{r}_0)$, $\dot{\vec{r}}(1) = N(\vec{r}_N - \vec{r}_{N-1})$

Therefore we have the first properties of the Bezier curves:

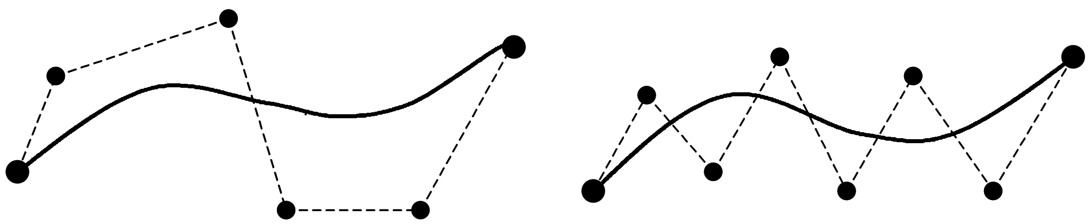
(1) The curve passes through the end control points and is tangential at these points to the first and last line segments defined by the control points.

(2) The Bezier curve has the convex hull property, i.e. it must lie inside the convex hull of the control points $\vec{r}_i$ as shown by the Casteljau' s construction algorithm.

(3) The effect of increasing $\dot{\vec{r}}(0)$ and $\dot{\vec{r}}(1)$, (or moving the points $\vec{r}_1$ and $\vec{r}_{N-1}$ further out), may be seen as follows.



(4) Disadvantage: The Bezier curve is a blend of all the control points, local features may be smoothed out, (diminishing variation property). As the curve does not vary in an erratic manner, it may be an advantage in practice.



4.3 B-splines

B-splines are produced by polynomial blending functions of the form

$$B_0(v) = \begin{cases} 1 & 0 \le v \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$B_j(v) = \frac{1}{j+1}[vB_{j-1}(v) + (j+1-v)B_{j-1}(v-1)]$$

The span over which $B_j(v)$ is non-zero is given by $0<v<j+1$, $(j>0)$.

Therefore B_j are functions having <u>finite support</u> of length of j+1.

Example:

$$B_1(v) = \frac{1}{2}[vB_0(v) + (2-v)B_0(v-1)]$$

$$= \begin{cases} \frac{1}{2}v & 0 \le v \le 1 \\ \frac{1}{2}(2-v) & 1 \le v \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$B_2(v) = \frac{1}{3}[vB_1(v) + (3-v)B_1(v-1)]$$

$$= \begin{cases} \frac{1}{6}v^2 & 0 \le v \le 1 \\ \frac{1}{6}(-2v^2 + 6v - 3) & 1 \le v \le 2 \\ \frac{1}{6}(v^2 - 6v + 9) & 2 \le v \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$B_3(v) = \frac{1}{4}[vB_2(v) + (4-v)B_2(v-1)]$$

$$= \begin{cases} \frac{1}{24}[2+(v-2)]^3 & 0 \le v \le 1 \\ \frac{1}{24}[4-6(v-2)^2 - 3(v-2)^3] & 1 \le v \le 2 \\ \frac{1}{24}[4-6(v-2)^2 + 3(v-2)^3] & 2 \le v \le 3 \\ \frac{1}{24}[2-(v-2)]^3 & 3 \le v \le 4 \\ 0 & \text{otherwise} \end{cases}$$

We can centre the spline on the origin by considering

$$B_j(x + \frac{j+1}{2}), \quad -\frac{j+1}{2} \le x \le \frac{j+1}{2}$$

Consider the curve represented by the jth order blending function in the form

$$\vec{r}(u) = \sum_{i=0}^{N} (j+1)B_j(Nu - i + \frac{j+1}{2})\vec{r}_i, \quad 0 \le u \le 1$$

where the N+1 knots $\vec{r}_i$ are uniformly spaced at u=0, $1/N, 2/N, \ldots, N-1/N, 1$.

In particular, the cubic B-spline (j=3) is frequently used:

$$\vec{r}(u) = \sum_{i=0}^{N} 4B_3(Nu - i + 2)\vec{r}_i = \sum_{i=0}^{N} B(Nu - i)\vec{r}_i$$

where B(x)=4B$_3$(x+2) is the usual form of the cubic blending function.

For any non-integer value of Nu, only four of the terms in the summation are non-zero, consequently each span of the curve $(k-1)/N \le u \le k/N$ is determined by at most four consecutive vertices $\vec{r}_{k-2}, \vec{r}_{k-1}, \vec{r}_k, \vec{r}_{k+1}$, i.e.

$$\vec{r}(u) = B(Nu - k + 2)\vec{r}_{k-2} + B(Nu - k + 1)\vec{r}_{k-1} + B(Nu - k)\vec{r}_k + B(Nu - k - 1)\vec{r}_k$$

for $-1 \le$ Nu-k ≤ 0

But

$$\sum_{m=-1}^{2} B(\mu + m) = \frac{1}{6}\{[2 + \mu - 1]^3 + [4 - 6\mu^2 - 3\mu^3] + [4 - 6(\mu + 1)^2 + 3(\mu + 1)^3] + [2 - (\mu + 2)]^3\}$$

$$= 1$$

Therefore

(1) $\vec{r}(u)$ is a convex combination of $\vec{r}_{k-2}, \vec{r}_{k-1}, \vec{r}_k, \vec{r}_{k+1}$ and the whole span must lie inside the convex hull.

(2) Consider the case Nu=k, (k≠0 or N), then

$$\vec{r}(u) = B(2)\vec{r}_{k-2} + B(1)\vec{r}_{k-1} + B(0)\vec{r}_k + B(-1)\vec{r}_{k+1}$$

$$= (\vec{r}_{k-1} + 4\vec{r}_k + \vec{r}_{k+1})/6$$

$$= \frac{2}{3}\vec{r}_k + \frac{1}{3}(\frac{\vec{r}_{k-1} + \vec{r}_{k+1}}{2})$$

i.e. the curve passes through the point which is one-third of the way joining $\vec{r}_k$ to the mid-point of the line joining $\vec{r}_{k-1}$ to $\vec{r}_{k+1}$.

(3) Consider the case Nu=(k-1)+$\frac{1}{2}$, (k≠0 or N), then

$$\vec{r}(u) = B\left(\frac{3}{2}\right)\vec{r}_{k-2} + B\left(\frac{1}{2}\right)\vec{r}_{k-1} + B\left(-\frac{1}{2}\right)\vec{r}_k + B\left(-\frac{3}{2}\right)\vec{r}_{k+1}$$

Since $B\left(\frac{3}{2}\right) = B\left(-\frac{3}{2}\right) = 0.02083333$ and $B\left(\frac{1}{2}\right) = B\left(-\frac{1}{2}\right) = 0.47916667$

therefore $\vec{r} \approx \frac{1}{2}(\vec{r}_{k-1} + \vec{r}_k)$

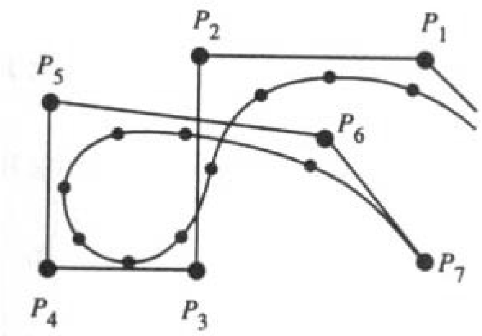
(4) Now $\vec{r}(0) = B(0)\vec{r}_0 + B(-1)\vec{r}_1 = \frac{2}{3}\vec{r}_0 + \frac{1}{6}\vec{r}_1 \neq \vec{r}_0$

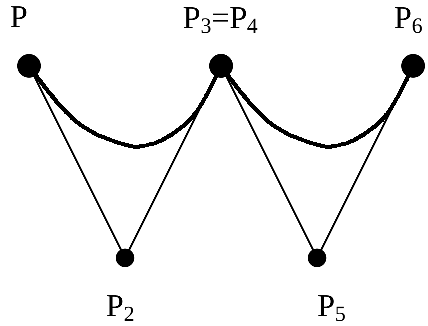
$$\vec{r}(1) = B(1)\vec{r}_{N-1} + B(0)\vec{r}_N = \frac{1}{6}\vec{r}_{N-1} + \frac{2}{3}\vec{r}_N \neq \vec{r}_N$$

Hence the spline does not pass through these knots. To force the spline to pass the end knots, we may extend the knots by two phantom control points $\vec{r}_{-1}$ and $\vec{r}_{N+1}$ at the ends by choosing $\vec{r}_{-1} = 2\vec{r}_o - \vec{r}_1$, $\vec{r}_{N+1} = 2\vec{r}_N - \vec{r}_{N-1}$ and

$$\vec{r}(u) = \sum_{i=-1}^{N+1} B(Nu - i)\vec{r}_i$$

(5) The effect of a double knot is to pull the curve towards that point. Closed curves may be generated by choosing $\vec{r}_N = \vec{r}_o$.





4.4 Rational Parametric Curves

We may generalized the polynomial curves above into rational curves by using homogeneous coordinates in R^4, $\vec{r} \to (\vec{r},1)^T$ and define $\vec{R} = \omega(\vec{r},1)^T$

4.4.1 The N^{th} degree rational curve is given by

$$\vec{R}(u) = \sum_{i=0}^{N} b_i^N(u)\vec{R}_i \qquad \text{where } \vec{R}(u) = \omega(u)[\vec{r}(u),1]^T$$

$$\text{hence} \quad \vec{r}(u) = \frac{\sum_{i=0}^{N} b_i^N(u)\vec{r}_i\omega_i}{\sum_{i=0}^{N} b_i^N(u)\omega_i}, \qquad 0 \le u \le 1$$

4.2 Rational B-splines

In this case $\vec{R}(u) = \sum_{i=0}^{N}(j+1)B_j(Nu - i + \frac{j+1}{2})\vec{R}_i$

$$\text{hence } \vec{r}(u) = \frac{\sum_{i=0}^{N}\omega_i B_j(Nu - i + \frac{j+1}{2})\vec{r}_i}{\sum_{i=0}^{N}\omega_i B_j(Nu - i + \frac{j+1}{2})}$$

An extension of this definition is to change the uniform knots $u_k=k/N$ into a non-uniform set $\{u_0, u_1, u_2, \cdots, u_{N+J+1}\}$.

The non-uniform B-splines $B_{i,j}(u)$ is now defined by

$$B_{i,0}(u) = \begin{cases} 1 & u_i \le u < u_{i+1} \qquad i = 0,1,\ldots,N+J \\ 0 & \text{otherwise} \end{cases}$$

$$B_{i,j}(u)=\frac{u-u_i}{u_{i+j}-u_i}B_{i,j-1}(u)+\frac{u_{i+j+1}-u}{u_{i+j+1}-u_{i+1}}B_{i+1,j-1}(u) \qquad ,i=0,1,\cdots,N+J-j$$

The only restriction on the knots is that it is non-decreasing.
The non-uniform rational B-spline of order J is then given by

$$\vec{r}^{(J)}(u)=\frac{\sum_{k=0}^{N}\omega_k B_{k,J}(u)\vec{r}_k}{\sum_{k=0}^{N}\omega_k B_{k,J}(u)}, \quad u_0\le u\le u_N$$

NURBS is one of the most popular design curve in modern CAD.
But as example, we will consider the
<u>Rational quadratic curves</u>-conic sections
In this case

$$\vec{R}(u)=(1-u)^2\vec{R}_0+2u(1-u)\vec{R}_1+u^2\vec{R}_2, \quad \vec{R}=\omega(\vec{r},1)^T$$

with $\omega(u)=(1-u)^2\omega_0+2u(1-u)\omega_1+u^2\omega_2$

Therefore $\vec{C}(u)=\vec{r}(u)=\dfrac{(1-u)^2\omega_0\vec{r}_0+2u(1-u)\omega_1\vec{r}_1+u^2\omega_2\vec{r}_2}{\omega(u)}$

lying in the convex hull of $\vec{r}_0,\vec{r}_1,\vec{r}_2$ is a plane curve. Also $\vec{C}(u)$ is a general quadratic; with suitable choices of the vectors $\vec{r}_i$ and weights ω_i, it can represent any conic segments exactly.

First of all, if $\omega_i \rightarrow k\omega_i$ ($k\neq 0$), $\vec{C}(u)$ remains the same.

Secondly, let $u=\dfrac{t}{p(1-t)+t}, 1-u=\dfrac{p(1-t)}{p(1-t)+t}$

then $\vec{C}(t)=\dfrac{p^2(1-t)^2\omega_0\vec{r}_0+2pt(1-t)\omega_1\vec{r}_1+t^2\omega_2\vec{r}_2}{p^2(1-t)^2\omega_0+2pt(1-t)\omega_1+t^2\omega_2}$

therefore the shape is not changed if $\omega_i \rightarrow p^{2-i}\omega_i=\hat{\omega}_i$.

In particular, choose $p=\sqrt{\omega_2/\omega_0}$ then $\hat{\omega}_0=\hat{\omega}_2$. Hence all $\vec{C}(u)$ with $\omega_0, \omega_2\neq 0$ can be transformed into $\hat{\omega}_0=\hat{\omega}_2=1$.

For this particular choice of ω_i, we have

$$\vec{C}(u)=\frac{(1-u)^2\vec{r}_0+2u(1-u)\omega_1\vec{r}_1+u^2\vec{r}_2}{(1-u)^2+2u(1-u)\omega_1+u^2}$$

The singularities, corresponding to the points at infinity of $\vec{C}(u)$, are determined by the real roots of the denominator

$$\omega(u)=(1-u)^2+2u(1-u)\omega_1+u^2=0$$

They are given by $u=\dfrac{1-\omega_1\pm\sqrt{\omega_1^2-1}}{2-2\omega_1}$

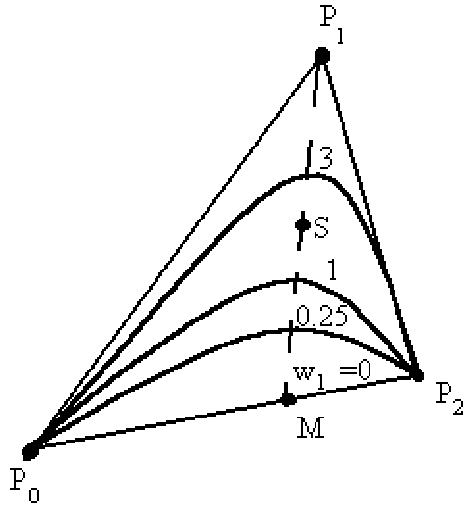
The curve reduces to
(1) a straight line if space $\omega_1=0$
(2) an elliptic segment if $0<\omega_1<1$
(3) a parabolic segment if $\omega_1=1$
(4) a hyperbolic segment if $\omega_1>1$

Geometrically, the curved are governed by the parameter

$$s = \frac{\omega_1}{1+\omega_1} \quad \text{as}$$

$$S = \vec{C}\left(\frac{1}{2}\right) = \frac{\vec{r}_0\, 2\omega_1 \vec{r}_0 + \vec{r}_2}{2(1+\omega_1)} = \frac{1}{1+\omega_1}\cdot\frac{(\vec{r}_0+\vec{r}_2)}{2} + \frac{\omega_1}{1+\omega_1}\vec{r}_1$$

$$= (1-s)M + sP_1$$



The curve $\vec{C}(u)$ passes through $\vec{r}_0 (= P_0)$ and $\vec{r}_2 (= P_2)$ with the slope from

$$\dot{\vec{R}} = \begin{pmatrix} \dot{\omega}\vec{r} + \omega\dot{\vec{r}} \\ \dot{\omega} \end{pmatrix}$$

hence $\quad \dot{\vec{r}}(0) = \dfrac{2\omega_1}{\omega_0}(\vec{r}_1 - \vec{r}_0)$

$\dot{\vec{r}}(1) = \dfrac{2\omega_1}{\omega_2}(\vec{r}_2 - \vec{r}_1)$, $\quad$ the tangents meets at $\vec{r}_1 (= P_1)$.

4.5 Composite Curves

Consider the problem of joining the curve segment $\vec{r}^{(1)}(u), (0 \le u \le 1)$ with the curve segment $\vec{r}^{(2)}(v), (0 \le v \le 1)$. Usually we would like to ensure continuity of the curve, its slope and even the curvature.

Therefore, we have the conditions

$$\vec{r}^{(1)}(1) = \vec{r}^{(2)}(0)$$

$$\frac{1}{\varepsilon_1}\dot{\vec{r}}^{(1)}(1) = \frac{1}{\varepsilon_2}\dot{\vec{r}}^{(2)}(0) = \hat{t}$$

where $\hat{t}$ is the common unit tangent and ε_i are parameters.

Since $\quad \kappa\hat{b} = \dfrac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\left|\dot{\vec{r}}\right|^3}$

For curvature continuity, it gives

$$\frac{\dot{\vec{r}}^{(2)}(0) \times \ddot{\vec{r}}^{(2)}(0)}{\left|\dot{\vec{r}}^{(2)}(0)\right|^3} = \frac{\dot{\vec{r}}^{(1)}(1) \times \ddot{\vec{r}}^{(1)}(1)}{\left|\dot{\vec{r}}^{(1)}(1)\right|^3}$$

i.e. $\quad \hat{t} \times \ddot{\vec{r}}^{(2)}(0) = \left(\dfrac{\varepsilon_2}{\varepsilon_1}\right)^2 \hat{t} \times \ddot{\vec{r}}^{(1)}(1)$

This equation is satisfied by

$$\ddot{\vec{r}}^{(2)}(0) = \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \ddot{\vec{r}}^{(1)}(1) + \lambda \dot{\vec{r}}^{(1)}(1)$$

where λ is an arbitrary scalar.

Example: <u>Composite Bezier cubic curve</u>

$$\vec{r}(u) = (1-u)^3 \vec{r}_0 + 3u(1-u)^2 \vec{r}_1 + 3u^2(1-u)\vec{r}_2 + u^3 \vec{r}_3$$

then since

$$\vec{r}(0) = \vec{r}_0, \quad \vec{r}(1) = \vec{r}_3$$

$$\dot{\vec{r}}(0) = 3(\vec{r}_1 - \vec{r}_0), \quad \dot{\vec{r}}(1) = 3(\vec{r}_3 - \vec{r}_2)$$

$$\ddot{\vec{r}}(0) = 6(\vec{r}_0 - 2\vec{r}_1 + \vec{r}_2)$$

$$\ddot{\vec{r}}(1) = 6(\vec{r}_1 - 2\vec{r}_2 + \vec{r}_3)$$

Continuity of curve implies $r_3^{(1)} = r_0^{(2)}$

Continuity of slope implies $\frac{3}{\varepsilon_1}(\vec{r}_3^{(1)} - \vec{r}_2^{(1)}) = \frac{3}{\varepsilon_2}(\vec{r}_1^{(2)} - \vec{r}_0^{(2)}) = \hat{t}$

$\therefore$ the three points $r_2^{(1)}, r_3^{(1)} = r_0^{(2)}, r_1^{(2)}$ must be collinear.

Now $6(\vec{r}_0^{(2)} - 2\vec{r}_1^{(2)} + \vec{r}_2^{(2)}) = 6\theta^2(\vec{r}_1^{(1)} - 2\vec{r}_2^{(1)} + \vec{r}_3^{(1)}) + 3\lambda(\vec{r}_3^{(1)} - \vec{r}_2^{(1)})$

i.e. $\vec{r}_2^{(2)} = \vec{r}_3^{(1)} - \theta^2(\vec{r}_2^{(1)} - \vec{r}_1^{(1)}) + (\theta^2 + 2\theta + \lambda/2)(\vec{r}_3^{(1)} - \vec{r}_2^{(1)})$

hence $\vec{r}_2^{(2)}$ is also determined from $\vec{r}_1^{(1)}, \vec{r}_2^{(1)},$ and $\vec{r}_3^{(1)}$.

The only free vertex in $\vec{r}^{(2)}$ is $\vec{r}_3^{(2)}$. Also from the above requirements show that $\vec{r}_1^{(1)}, \vec{r}_2^{(1)}, \vec{r}_3^{(1)} = \vec{r}_0^{(2)}, \vec{r}_1^{(2)}, \vec{r}_2^{(2)}$ are all coplanar. But the curve can still be 3D as $\vec{r}_0^{(1)}$ and $\vec{r}_3^{(2)}$ can be defined freely.