Chapter 3 Differential Geometry of Curves

3.1. Parameterization and arc length

A curve in space $\Re^3$ can be represented as the intersection of two cartesian equations

$$g_1(x,y,z) = 0 \text{ and } g_2(x,y,z) = 0$$

or more conveniently, as a vector depending on a single parameter

$$\vec{r}(t) = (x(t), y(t), z(t))^T, \qquad t \in \Re$$

The same curve may have different parameterizations. The most natural parameter on a curve is its length s measured from an arbitrary point $\vec{r}(t_0)$

As $$ds = (dx^2 + dy^2 + dz^2)^{½}$$

$$\therefore \qquad S(t) = \int_{t_0}^{t} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}} dt = \int_{t_0}^{t} |\dot{\vec{r}}| \, dt$$

The speed of an particle moving along the curve is

$$\dot{s} = \frac{dS}{dt} = |\dot{\vec{r}}|$$

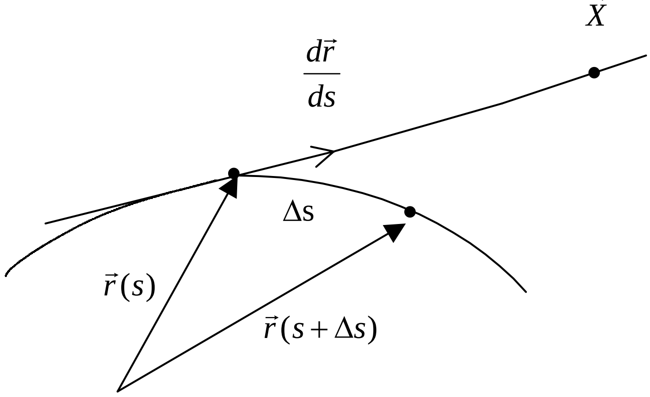
Since S is related to t, we may regard $\vec{r}(t) = \vec{r}(t(s))$ and the curve is paramaterized by S. In this case

$$\hat{t} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \dot{\vec{r}} \frac{1}{|\dot{\vec{r}}|}$$

is a unit vector – the <u>unit tangent</u> vector.
Equation of tangent is

$$\vec{X} = \vec{r} + \alpha\hat{t}, \quad (\alpha \in \Re)$$

for any $\vec{X}$ on the tangent line.



3.2 Frenet Frame

Since $\hat{t}$ is a unit vector, $\hat{t}\cdot\hat{t}$ =1, hence $\hat{t}\cdot\dot{\hat{t}}$ =0.

i.e. $\dot{\hat{t}}$ lies in the plane through $\vec{r}\perp$ to the tangent at $\vec{r}$.

Any point $\vec{X}$ in this <u>Normal plane</u> is governed by

$$\hat{t}\cdot(\vec{X}-\vec{r})=0$$

We will call the unit direction $\hat{n}$ <u>Principal normal</u> where

$$\hat{n}=\frac{\dot{\hat{t}}}{\left|\dot{\hat{t}}\right|}$$

Also, $d\hat{t}/ds$ measured the rate of rotation of $\hat{t}$ along the curve, let

$$\kappa=\left|\frac{d\hat{t}}{ds}\right|=\left|\frac{d\hat{t}}{dt}\cdot\frac{dt}{ds}\right|=\frac{\left|\dot{\hat{t}}\right|}{\dot{s}}$$

κ is called the <u>curvature</u> of the curve. The plane defined by $\hat{t}$ and $\hat{n}$ at $\vec{r}$ is called the <u>osculating plane</u> , any point $\vec{x}$ on it will satisfy

$$(\vec{X}-\vec{r})\cdot(\hat{t}\times\hat{n})=0$$

Finally, we pick a second normal direction, the <u>binormal</u> as

$$\hat{b}=\hat{t}\times\hat{n} \qquad\qquad \text{(unit vector)}$$

The change in $\hat{b}$ along the curve will measure the <u>torsion</u> τ(twisting). Now

$$\frac{d\hat{b}}{ds}=\frac{d}{ds}(\hat{t}\times\hat{n})=\frac{d\hat{t}}{ds}\times\hat{n}+\hat{t}\times\frac{d\hat{n}}{ds}=\hat{t}\times\frac{d\hat{n}}{ds}$$

therefore $\frac{d\hat{b}}{ds}\perp\hat{t}$. Also from $\hat{b}\cdot\hat{b}=1, \frac{\mathrm{d}\hat{\mathrm{b}}}{\mathrm{ds}}\perp\hat{\mathrm{b}}$

$\therefore$ the torsion can be defined as $\frac{d\hat{b}}{ds}=-\tau\hat{n}$, i.e.

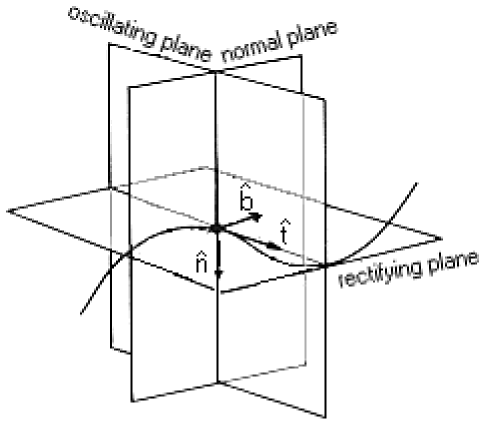
$|\tau|=\left|\frac{d\hat{b}}{ds}\right|$ and lie along $\hat{n}$ with a negative sign to preserve the right-hand screw direction as s increases. The plane determined by $\hat{t}$ and $\hat{b}$ is called the <u>rectifying plane</u>. Now we have the Frenet frame $(\hat{t},\hat{n},\hat{b})$

and $$\frac{d\hat{n}}{ds}=\frac{d\hat{b}}{ds}\times t+\hat{b}\times\frac{d\hat{t}}{ds}=-\tau(\hat{n}\times\hat{t})+\kappa(\hat{b}\times\hat{n})$$

$$=\tau\hat{\mathrm{b}}-\kappa\hat{\mathrm{t}}.$$

Combining the results together, we have the Frenet-Serret formula

$$\begin{bmatrix} \dot{\hat{t}} \\ \dot{\hat{n}} \\ \dot{\hat{b}} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa\dot{s} & 0 \\ -\kappa\dot{s} & 0 & \tau\dot{s} \\ 0 & -\tau\dot{s} & 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix}$$



Theorem If τ(t) =0 for all t, then the curve is planar and lies in the osculating plane at $\vec{r}(t_\circ)$. (Assignment)

Example: let $\vec{r}(t)$ represented the motion of a particle in space, then

velocity $\vec{v} = \frac{d\vec{r}}{dt} = \dot{s}\hat{t}$ ($\dot{s} = \frac{ds}{dt}$ is the speed)

acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\dot{s}\hat{t})}{dt} = \ddot{s}\hat{t} + \dot{s}\frac{d\hat{t}}{dt}$

$= \ddot{s}\hat{t} + \kappa(\dot{s})^2\hat{n}$

(in Physics, we often use $v = \dot{s}, a = \frac{dv}{dt} = \ddot{s}$)

3.3 Computational formulae

Let $\vec{r}(t)$ be a regular curve ($|\dot{\vec{r}}| \neq 0$), then

$$\boxed{\hat{t} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}}$$

Now, we follow the example above, and using

$$\ddot{\vec{r}} = a\hat{t} + \kappa v^2 \hat{n}$$

then $\quad \dot{\vec{r}} \times \ddot{\vec{r}} = v\hat{t} \times (a\hat{t} + \kappa v^2 \hat{n}) = \kappa v^3 \hat{b}$.

Hence

$$\boxed{\hat{b} = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{|\dot{\vec{r}} \times \ddot{\vec{r}}|}} \qquad \text{and} \qquad \boxed{\kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}}$$

Now, $\dddot{\vec{r}} = \dot{a}\hat{t} + a\dot{\hat{t}} + (2\kappa va + \dot{\kappa} v^2)\hat{n} + \kappa v^2 \dot{\hat{n}}$

$$= \dot{\mathrm{a}}\hat{\mathrm{t}} + \mathrm{a}\kappa \mathrm{v}\hat{\mathrm{n}} + (2\kappa \mathrm{va} + \dot{\kappa}\mathrm{v}^2)\hat{n} - \kappa^2 v^3 \hat{t} + \tau \kappa v^3 \hat{b}$$

and $\quad (\dot{\vec{r}} \times \ddot{\vec{r}}) \cdot \dddot{\vec{r}} = \kappa^2 v^6 \tau$

therefore $\quad \boxed{\tau = \frac{(\dot{\vec{r}} \times \ddot{\vec{r}}) \cdot \dddot{\vec{r}}}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}}$

Finally, $\quad \hat{n} = \hat{b} \times \hat{t}$, and $\rho = 1/\kappa$ is the <u>radius of curvature</u>.

Given a Frenet frame $(\hat{t}_\circ, \hat{n}_\circ, \hat{b}_\circ)$ at $\vec{\mathrm{r}}(0)$, we can also approximate $\vec{r}(t)$ near $\vec{r}(0)$ by using

$$\vec{r}(t) = \vec{r}(0) + \dot{\vec{r}}(0)t + \frac{\ddot{\vec{r}}(0)}{2}t^2 + \frac{\dddot{\vec{r}}(0)}{6}t^3 + \ldots$$

where $\quad \dot{\vec{r}}(0) = v_\circ \hat{t}_\circ, \quad \ddot{\vec{\mathrm{r}}}(0) = \mathrm{a}_\circ \hat{t}_\circ + \kappa_\circ v_\circ^2 \hat{n}_\circ \quad$ ...etc

to obtain $\vec{r}(t) = \vec{r}(0) + [v_\circ t + 0(t^2)]\hat{t}_\circ + [\frac{\kappa_\circ v_\circ^2}{2}t^2 + 0(t^3)]\hat{n}_\circ + [\frac{\tau_\circ \kappa_\circ v_\circ^3}{6}t^3 + 0(t^4)]\hat{b}_\circ$.

A straight line will be given by $\quad \vec{r} = \vec{r}(0) + s\hat{t}_\circ$.