Chapter 2 Figures and Shapes

2.1 Polyhedron in n-dimension

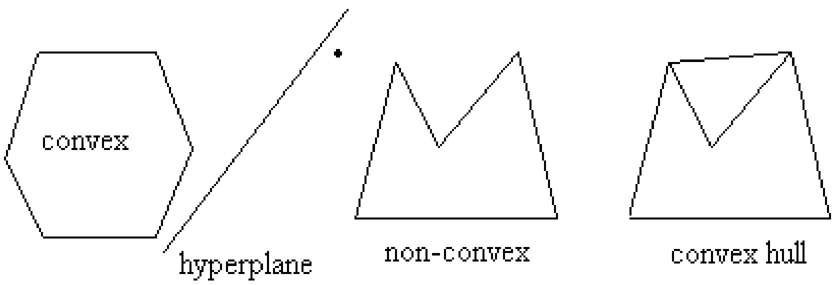
In linear programming we know about the simplex method which is so named because the feasible region can be decomposed into simplexes.

A zero-dimensional simplex is a point, an 1D simplex is a straight line segment, a 2D simplex is a triangle, a 3D simplex is a tetrahedron.

In general, a n-dimensional simplex has n+1 vertices not all contained in a (n-1)-dimensional hyperplane. Therefore simplex is the simplest building block in the space it belongs.

An n-dimensional polyhedron can be constructed from simplexes with only possible common face as their intersections. Such a definition is a bit vague and such a figure need not be connected. Connected polyhedron is the prototype of a closed manifold. We may use vertices, edges and faces (hyperplanes) to define a polyhedron. A polyhedron is convex if all convex linear combinations of the vertices $\vec{V_i}$ are inside itself, i.e. $\sum_{all\_i} \alpha_i \vec{V_i}$ is contained inside for all $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$.

If a polyhedron is not convex, then the smallest convex set which contains it is called the convex hull of the polyhedron.



Separating hyperplane Theorem

For any given point outside a convex set, there exists a hyperplane with this given point on one side of it and the entire convex set on the other.

Proof: Because the given point will be outside one of the supporting hyperplanes of the convex set.

2.2 Platonic Solids

Known to Plato (about 500 B.C.) and explained in the Elements (Book XIII) of Euclid (about 300 B.C.), these solids are governed by the rules that the faces are the regular polygons of a single species and the corners (vertices) are all alike. Begin from the equilateral Δ, the simplest regular polygon. To make a solid, at least three must join at any corner, so can four or five. But when six join at a corner they lie flat to make a regular hexagon.

Starting from a corner and by adding more polygons of the same kind according to the rules, we have

Four Δ to make the tetrahedron,

Eight Δ to make the octahedron,

Twenty Δ to make the icosahedron.

(e.g. six Δ is a deltahedron violating the second rule).

Turn now to the square face, three can join at a corner but four joining lie flat.

Six □ to make the cube.

Three regular pentagons can join at a corner, but more will not fit together.

Twelve  to make the dodecahedron.

Three regular hexagons joining at a corner lie flat ; three regular polygons with more sides cannot join at a corner.
Therefore there are five and only five solids fitting the rules. The Platonic solids are convex; there is a circumscribing sphere passing through all the corners and there is an inscribing sphere touching all the faces, (e.g. having spherical symmetry).

Duality the interchanging relations faces ↔ corners, edges ↔ edges subdivide the solids into three sets:

Solids	faces F	corners V	edges E	duality
tetrahedron	4	4	6	self-dual
cube	6	8	12	dual
octahedron	8	6	12	
dodecahedron	12	20	30	dual
icosahedron	20	12	30	

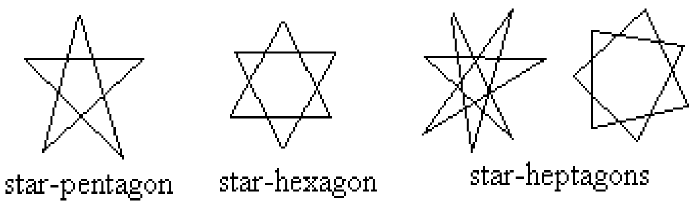
e.g. the cube has 6 four-sided faces meeting by threes in 8 corners,
octahedron has 8 three-sided faces meeting by fours in 6 corners.
Therefore the octahedron can be constructed from a cube by making a point at the middle of each face of a cube and connecting those points by lines.

Euler's formula (~1750) is true for any simply connected convex polyhedron (of genus=0): $V-E+F=2$.
For multi-connected figures, we have the Euler-Poincare' formula:

$$V-E+F=2-2g$$

Where g=0 for a sphere
g=1 for a torus (sphere with one handle)
g=2 for a solid figure eight (sphere with two handles)
……

If instead of regular polygons as faces, regular star polygons were used,



then four more regular solids (the Kepler-Poinsot solids) ~1600 ~1850 will be obtained. They satisfy the regular rules, but are not convex solids.

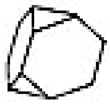
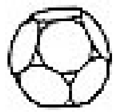
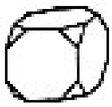
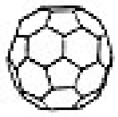
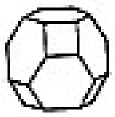
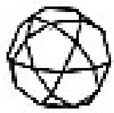
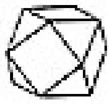
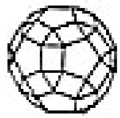
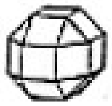
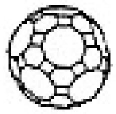
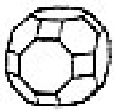
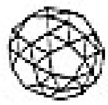
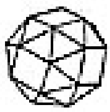
Small stellated dodecahedron Great dodecahedron
Great stellated dodecahedron Great icosahedron

They can be constructed from the dodecahedron and icosahedron by stellating, i.e. by extending the faces of the solid until they intersect again to form new shapes.
The Plato and Kepler-Poinsot solids are collectively known as the nine regular solids.

2.3 Archimedean Solids

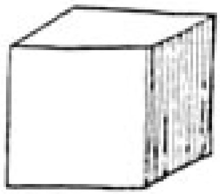
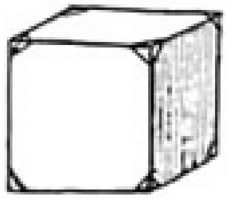
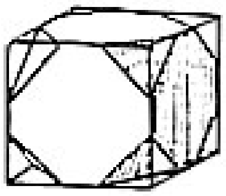
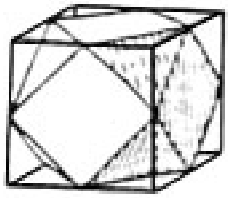
If the face rule of a regular solid is relaxed by allowing the faces to be made of several sorts of regular polygons but the corners are still required to be all alike, then we come to the semi-regular solids. There are thirteen (types) of them, all known to Archimedes (~250 B.C.) They share with the Platonic solids the property of convexity and can be inscribed in a sphere that touches all its corners. Since they have more corners than the Platonic solids, they can provide closer approximations to a sphere, (e.g. the modern football is made from a truncated icosahedron). They can be obtained from the Platonic solids by truncation.

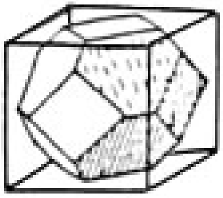
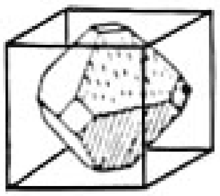
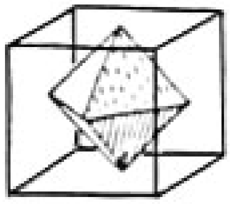
The archimedean or semiregular polyhedra:

Name	Diagram	Name	Diagram
truncated tetrahedron		truncated dodecahedron	
truncated cube		truncated icosahedron	
truncated octahedron		icosidodecahedron	
cuboctahedron		small rhombicosidodecahedron	
small rhombicuboctahedron		great rhombicosidodecahedron	
great rhombicuboctahedron		snub dodecahedron	
snub cube			

name	c	e	f3	f4	f5	f6	f8	f10
truncated tetrahedron	12	18	4	-	-	4	-	-
truncated cube	24	36	8	-	-	-	6	-
truncated octahedron	24	36	-	6	-	8	-	-
cuboctahedron	12	24	8	6	-	-	-	-
small rhombicuboctahedron	24	48	8	18	-	-	-	-
great rhombicuboctahedron	48	72	-	12	-	8	6	-
snub cube	24	60	32	6	-	-	-	-
truncated dodecahedron	60	90	20	-	-	-	-	12
truncated icosahedron	60	90	-	-	12	20	-	-
icosidodecahedron	30	60	20	-	12	-	-	-
small rhombicosidodecahedron	60	120	20	30	12	-	-	-
great rhombicosidodecahedron	120	180	-	30	-	20	-	12
snub dodecahedron	60	150	80	-	12	-	-	-

A sequence of truncation of a cube:

Cube	transition of a truncated cube		cuboctahedron
 (a)	 (b)	 (c)	 (d)

truncated octahedron	transition to an octahedron	
 (e)	 (f)	 (g)

2.4 Barycentric coordinates (introduced by Mobius 1827)

We mentioned that points inside a simplex can be represented by convex combination of the vertices. More specific

<u>1D simplex</u>: A straight line passing through two vertices $\vec{V}_1, \vec{V}_2$ is given by

$$\vec{P}(t) = \alpha_1 \vec{V}_1 + \alpha_2 \vec{V}_2 , \qquad \alpha_1 + \alpha_2 = 1$$

$$= (1-t)\vec{V}_1 + t\vec{V}_2 , \quad t \in \Re$$

(t<0) |← t →|← 1-t →| (t>1)

$\vec{V}_1$ $\vec{P}$ $\vec{V}_2$

(α_1 , α_2) is the barycentric coordinate of $\vec{P}$ with respect to $\vec{V}_1$, $\vec{V}_2$.

We have $ratio(\vec{V}_1, \vec{P}, \vec{V}_2) = \dfrac{length(\vec{P}-\vec{V}_1)}{length(\vec{V}_2-\vec{P})} = \dfrac{length[t(\vec{V}_2-\vec{V}_1)]}{length[(1-t)(\vec{V}_2-\vec{V}_1)]} = \dfrac{t}{1-t}$

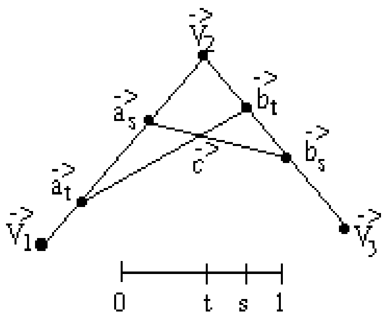
For example, given three vertices $\vec{V}_1, \vec{V}_2, \vec{V}_3$ and

$$\vec{a}_t = (1-t)\vec{V}_1 + t\vec{V}_2$$

$$\vec{a}_s = (1-s)\vec{V}_1 + s\vec{V}_2$$

$$\vec{b}_t = (1-t)\vec{V}_2 + t\vec{V}_3$$

$$\vec{b}_s = (1-s)\vec{V}_2 + s\vec{V}_3$$



let $\vec{c}$ be the intersection of the straight lines $\vec{a_t b_t}$ and $\vec{a_s b_s}$, then ratio $(\vec{a}_t, \vec{c}, \vec{b}_t) = \frac{s}{1-s}$ and ratio $(\vec{a}_s, \vec{c}, \vec{b}_s) = \frac{t}{1-t}$.

Proof: $\vec{c}$ satisfies the two equations

$$\vec{c} = (1-s)\,\vec{a}_t + s\,\vec{b}_t \quad \text{and} \quad \vec{c} = (1-t)\,\vec{a}_s + t\,\vec{b}_s$$

Q.E.D.

This result is a CAD version of the famous Menelaus Theorem

$$ratio(\vec{b}_s, \vec{b}_t, \vec{V}_2) \cdot ratio(\vec{V}_2, \vec{a}_t, \vec{a}_s) \cdot ratio(\vec{a}_s, \vec{c}, \vec{b}_s) = -1$$

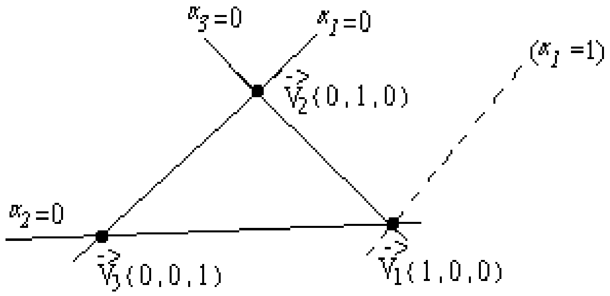
($b_t c a_t$ is a line segment meeting the sides of the $\Delta b_s V_2 a_s$)

2D simplex Given $\vec{V}_1, \vec{V}_2, \vec{V}_3$ in $\Re^2$ and write

$$\vec{P} = \alpha_1 \vec{V}_1 + \alpha_2 \vec{V}_2 + \alpha_3 \vec{V}_3 \text{ with } \alpha_1 + \alpha_2 + \alpha_3 = 1.$$

The coefficients (α_1, α_2, α_3) are the barycentric coordinates of $\vec{P}$ in $\Re^2$ with respect to $\vec{V}_i$. They are sometimes known as the area coordinates as

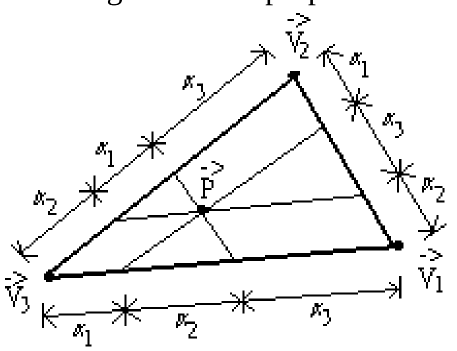
$$\alpha_1 = \frac{area(P, V_2, V_3)}{area(V_1, V_2, V_3)}$$

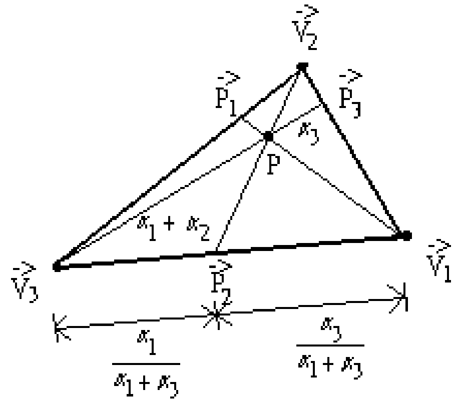


$$\alpha_2 = \frac{area(V_1, P, V_3)}{area(V_1, V_2, V_3)}$$

$$\alpha_3 = \frac{area(V_1, V_2, P)}{area(V_1, V_2, V_3)}$$

More geometrical properties are illustrated below (in ratios):





An immediate consequence is known as the Ceva's Theorem

$$ratio(\vec{V}_1, \vec{P}_3, \vec{V}_2) \cdot ratio(\vec{V}_2, \vec{P}_1, \vec{V}_3) \cdot ratio(\vec{V}_3, \vec{P}_2, \vec{V}_1) = 1$$

2.5 Curvatures

Recall that the curvature of a plane curve is defined as

$$\text{Curvature} = \frac{\text{angle of embrace}}{\text{length of arc}} = \frac{\theta}{AB}$$

Gauss introduced an auxiliary circle, of unit radius, onto which points on the curve may be mapped by the rule of parallel normals, then

$$\text{Curvature} = \frac{\text{arc length of image}}{\text{length of arc}} = \frac{ab}{AB}$$

If the curve has different curvature at different points, then we may take limit on the right hand side as the arc length tends to zero, obtaining curvature at a point.

<u>Gauss Map</u> (1825, 1827)
To measure the curvature of an area A_s on a surface, we may first map the boundary of A_s by the rule of parallel normals onto a unit sphere and define

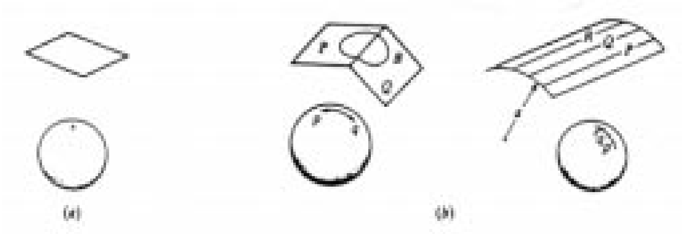
$$\text{Gaussian curvature } \kappa = \frac{a_s}{A_s}$$

Where a_s is the mapped area on the unit sphere and equals to the <u>solid angle</u> 2 substended at the centre of the sphere.

Notice that a sphere of unit radius has total surface area 4π, hence the total solid angle surrounding a point in three-dimensional space is 4π.

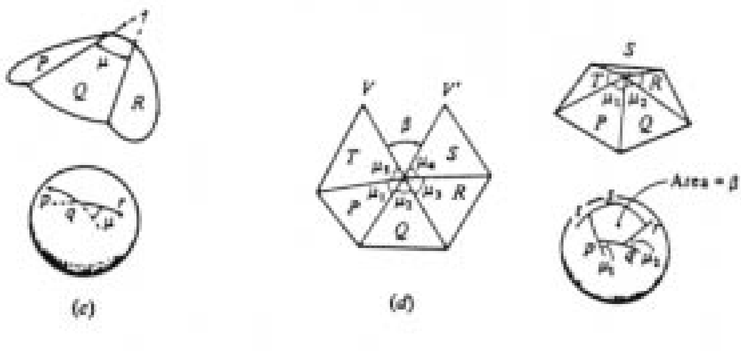
2.5.1 Curvature of piecewise flat surfaces

(a) Plane surface. Since the normals at all points are parallel, the entire plane maps onto a single point of the unit sphere, therefore $\theta = 0$ and $\kappa = 0$.



(b) Roof with one fold. Planes P and Q will be mapped onto p and q. The fold has no unique normal and will map onto the arc pq of a great circle lying in a diametral plane perpendicular to the fold. A contour such as B maps to zero area on the unit sphere, hence κ =0. Similarly, for a cylinder with infinitely many parallel folds.

(c) Roof with two folds intersecting at an angle μ. Planes P, Q and R will be mapped onto p,q and r. The arcs pq and qr follow great circles perpendicular to the folds, hence intersect each other at the same angle μ. The curvature of any enclosed area is still zero.

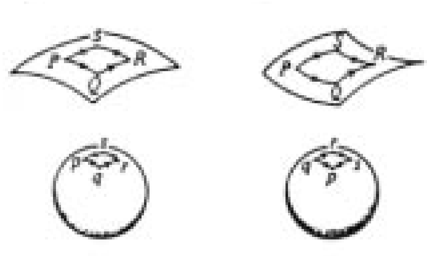


(d) Roof with a convex vertex. The number of flat surfaces is immaterial, but for a convex vertex to occur, $\sum_i \mu_i < 2\pi$. There is a positive <u>angular defect</u> $\beta = 2\pi - \sum_i \mu_i$ when the roof is flattened onto a plane.

In this case, the Guass map encloses an area proportional to β

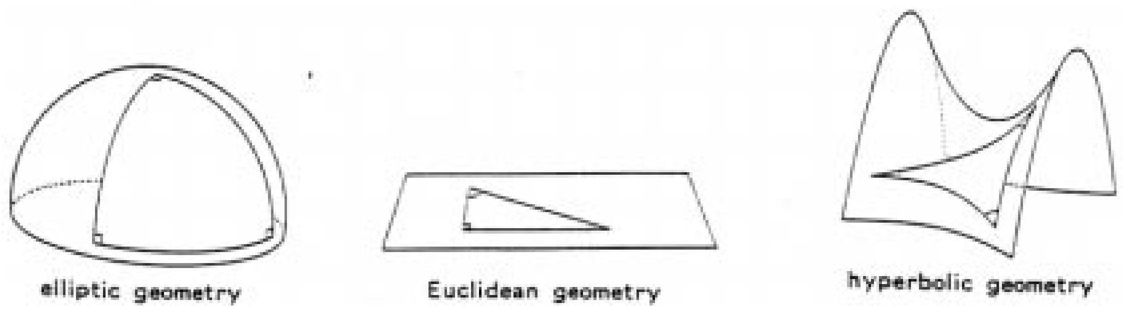
and $\kappa = \dfrac{\beta}{A_s}$ $(\beta = \theta)$

It should be noted that the contours B and b transverse in the same direction, which is the case when a convex surface is mapped onto the unit sphere. For a saddle surface, the reverse will happen and κ will be negative.



2.5.2 Parallel postulate, Angular excess and defect

The differences in the Guass maps from different geometries are intrinsic. It all starts from the parallel postulate.



(a) In the Euclidean plane, acceptance of the parallel postulate implies that the sum of the interior angles of a Δ is π.

Exer: The angle sum of a plane n-gon is $(n-2)\pi$.

(b) In spherical geometry, (consider geometry on a spherical surface), 'lines' are great circles. They are of finite length and any two lines meet at two points. In fact there are no parallel lines and two lines determine an area (a lune):

$$\text{Area } A_\alpha = \text{area of sphere} \times \frac{\alpha}{2\pi} = 2\alpha \qquad \text{(radius =1)}$$

$$A_\Delta = \text{area of spherical } \Delta = \alpha_1 + \alpha_2 + \alpha_3 - \pi$$
$$= \text{sum of interior angles} - \pi$$
$$= \text{Angular excess.}$$

Hence the angular excess is always positive on a sphere.

Exer. The area of a spherical n-gon is equal to its angular excess, namely, the sum of its interior angles less the corresponding sum for a plane figure; i.e. $A_n = \sum_i \alpha_i - (n-2)\pi$

Return now to the roof with a pointed vertex. Its Gauss map encloses an area $a_s = pqrst$ with exterior angles μ_i, hence

$$a_s = \sum_i (\pi - \mu_i) - (n-2)\pi$$
$$= n\pi - \sum_i \mu_i - (n-2)\pi$$
$$= 2\pi - \sum_i \mu_i = \beta$$

i.e. the angular defect β of the flattened vertex equals the mapped area and in turn equals the angular excess of the spherical n-gon.

(c) The hyperbolic plane was constructed by Lobatchevsky (1829) and Bolyai (1832) independently where more than one 'line' can be drawn passing a point and parallel to a given 'line'. There are also lines which are ultra-parallel (not parallel but never meet). In this case the sum of the interior angles of a hyperbolic Δ is always less than π, i.e. the angular excess is negative with A $= \pi$ - sum of interior angles.

2.5.3 Curvature of closed surfaces

For a complete sphere of radius R, its total curvature

$$\kappa = \frac{a}{A} = \frac{4\pi}{4\pi R^2} = \frac{1}{R^2}$$

For a polyhedron, its total curvature is the sum of curvatures of all its vertices, hence

$$\kappa = \frac{\text{total angular excess}}{\text{total surface area}} = \frac{\text{total angular defect}}{\text{total surface area}} = \frac{\mathrm{T}}{\mathrm{A}}$$

For instance, at any vertex of a cube there are three angles of $\pi/2$ so the angular defect is $2\pi - 3\times\pi/2$. The total angular defect of the cube is $\mathrm{T} = 8\times\pi/2 = 4\pi$.

Exer. Determine the total angular defect for each of the five platonic solids.

Descartes's Formula:

For any polyhedron, the total angular defect is related to the Euler characteristic by

$$\mathrm{T} = 2\pi\chi = 2\pi(\mathrm{V} - \mathrm{E} + \mathrm{F})$$

proof:

$$\begin{aligned} \mathrm{T} &= \sum_{vertices} \text{angular defect} \\ &= \sum_{vertices} (2\pi - \text{sum of face angles at the vertex}) \\ &= 2\pi V - \sum_{faces} (\text{sum of interior angles of the face}) \\ &= 2\pi V - \sum_{faces} (n_f - 2)\pi \end{aligned}$$

(n_f is the number of edges on the faces, i.e. the faces is n_f-gon)

$$\begin{aligned} &= 2\pi V - \sum_{faces} n_f \pi + \sum_{faces} 2\pi \\ &= 2\pi V - 2\pi E + 2\pi F \\ &= 2\pi(V - E + F) = 2\pi\chi \end{aligned}$$

Q.E.D.

We have also obtained the Gauss-Bonnet Theorem for any closed polygonalised (or smooth) surface:

$$\kappa A = T = 2\pi\chi \ .$$

Examples:

(1) S^2 (Sphere of genus = 0)

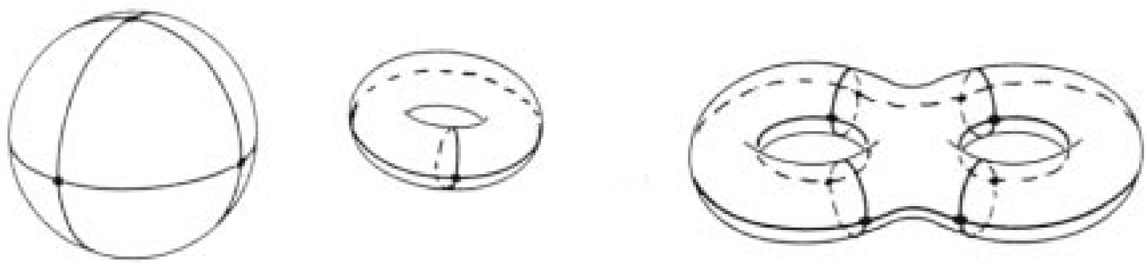
V = 6, E = 12, F = 8 to give χ = 2 .

(2) T^2 (torus of genus = 1)

V = 1, E = 2, F = 1 to give χ = 0 .

(3) $T^2 \# T^2$ (double toridal surface of genus =2)

V = 8, E = 16, F = 6 to give χ = -2 .



The aspect of a surface's nature which is unaffected by deformation is called the topolgy of the surface (invariant under differentiable transformations). The Euler number and total Gaussian curvature are topology properties.