Chapter 1 Transformation

1.1 What is Geometry? – Studies of figures

Felix Klein in his famous Erlangen Program (1872) defined geometry as a discipline concerned with properties of figures which remain unchanged under certain groups of transformations.

Look at the set of transformations forming a group (i.e. identity element, inverse element & composition)

<u>Example 1</u> Euclidean Geometry is the study of the properties of 3D figures invariant under rigid body motions (transformations) and Similarity transformations.

Let $\vec{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{position}$ and write $\vec{p}' = R\vec{p} + \vec{b}$, $R_{3\times 3}$ rotation matrix,

$\vec{b}_{3x1}$ transition

Any vector $\Delta\vec{p}$ will be transformed as $\Delta\vec{p}' = R \cdot \Delta\vec{p}$.

If R is a rigid body motion, it will preserve distance, i.e.

$$(\Delta\vec{p}')^T \cdot \Delta\vec{p}' = (\Delta\vec{p})^T R^T R \Delta\vec{p} = (\Delta\vec{p})^T \cdot \Delta\vec{p} \text{ true for all } \vec{p}.$$

Therefore $R^T R = I_3$, i.e. R is <u>orthonormal</u>.

As rigid body motion will not change the sense of the axes (left hand or right hand), so also det(R)= 1.

It will be convenient to write the transformation as

$$P = \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \text{ and } P' = BP = \begin{bmatrix} R & \vec{b} \\ 0 & 1 \end{bmatrix} P$$

The transformation B is known as an <u>isometry</u> because it preserves distance.

Special forms of B in 2D include

1.1.1 Pure transformation: $B=B_t$, $R=I_3$, $\vec{b} = (b_1, b_2, 0)^T$

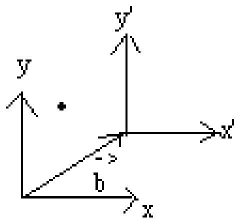
1.1.2 Pure rotation about z-axis: $B = B_\theta = \begin{bmatrix} R_\theta & 0 \\ 0 & 1 \end{bmatrix}$ with

$$\vec{b} = 0, R = R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{bmatrix}$$

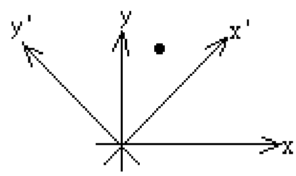
Note: under these transformations, a position vector $\vec{p} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ becomes

$\vec{p}' = \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix}$ referring to the basic coordinate system $(x, y, z)^T$, i.e. the point $\vec{p}$ is moving to $\vec{p}'$ measured in the same coordinate system.

Sometimes it would be convenient to regard the physical location of the point remains fixed. If it is given by $\vec{p}$ in the coordinate system $(x, y, z)^T$, it will be given by $\vec{p}'$ in the coordinate system $(x', y', z')^T$ $\vec{p}' = B_t^{-1} \cdot \vec{p} = \begin{bmatrix} I_3 & -\vec{b} \\ 0 & 1 \end{bmatrix} \cdot \vec{p}$



or $\vec{p}' = B_\theta^{-1} \cdot \vec{p} = B_\theta^T \cdot \vec{p} = \begin{bmatrix} R_\theta^T & 0 \\ 0 & 1 \end{bmatrix} \cdot \vec{p}$



in these two cases, regarded as coordinate transformations.

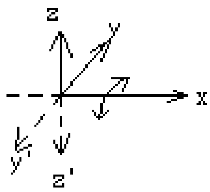
1.1.3 A rotation with center at $\vec{b} = (b_1, b_2, 0)^T$: $B = R_{\theta,b}$

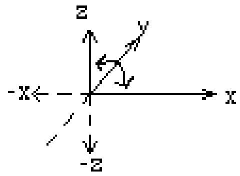
$$B = R_{\theta,b} = B_t B_\theta B_t^{-1} = \begin{bmatrix} I_3 & \vec{b} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_\theta & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} I_3 & -\vec{b} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_\theta & \begin{matrix} b_1(1-\cos\theta) + b_2 \sin\theta \\ -b_1 \sin\theta + b_2(1-\cos\theta) \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix}$$

($R_{\theta,b}^{-1}$ is also a rotation with center at $\vec{b}$, Cederberg p.93)

1.1.4 Rotation about x-axis (y-axis) by π is a reflection in the y-z plane (z-x plane)

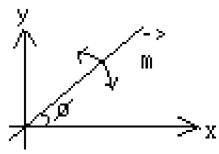
$$B_{\pi x} = \begin{bmatrix} 1 & & & \\ & -1 & & 0 \\ & & -1 & \\ 0 & & & 1 \end{bmatrix}, \; B_{\pi y} = \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & -1 & \\ & 0 & & 1 \end{bmatrix}$$





1.1.5 Reflection about an axis $\vec{m} = (m_1, m_2, 0) = (\cos\phi, \sin\phi, 0)$

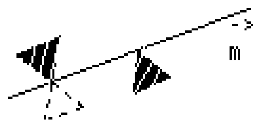
$$B_{\pi m} = B_{\phi} B_{\pi x} B_{\phi}^{-1} = \begin{bmatrix} \cos 2\phi & \sin 2\phi & & \\ \sin 2\phi & -\cos 2\phi & & 0 \\ & & -1 & \\ & 0 & & 1 \end{bmatrix}$$



(i.e. reflection changes the sense of the axes in 2D, but $\det(B_{\pi m})=1$ in 3D)

1.1.6 A glide reflection with axis $\vec{m}$ is the product of a reflection about $\vec{m}$ and a translation $\vec{b} = \lambda \vec{m}$ along $\vec{m}$

$$B_{gm} = B_g B_{\pi m} = \begin{bmatrix} & \lambda m_1 \\ I_3 & \lambda m_2 \\ & 0 \\ 0 & 1 \end{bmatrix} \cdot B_{\pi m}$$



Finally, Similarity transformation in x-y plane is represented by

$$B_s = \begin{bmatrix} \mu R & \vec{b} \\ 0 & 1 \end{bmatrix}$$ where $\mu \neq 0$, distance will be scaled by μ.

Only angles and ratio of distances remain invariant.

1.1.7 A central similarity (homothety, or dilatation) with center at $\vec{b}$ is given by

$$B = D_b = \begin{bmatrix} \mu I_b & \vec{b}(1-\mu) \\ 0 & 1 \end{bmatrix} \text{ with } D_b \begin{pmatrix} \vec{b} \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{b} \\ 1 \end{pmatrix}$$

1.1.8 A sprial similarity is given by: $\mu = e^{a\theta}$ and $R = R_\theta$

$$B = H_\theta = \begin{bmatrix} e^{a\theta} R_\theta & 0 \\ 0 & 1 \end{bmatrix}$$

1.1.9 Example of ratio of distances: Pythagoras Theorem.

For any right-angle Δ, $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$

1.2 Projections

We are concerned with transformations that produce 2D images of 3D objects onto an image plane.

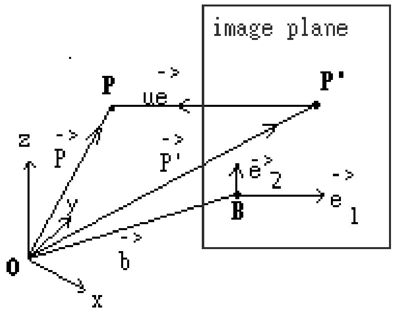
1.2.1 Parallel Projection

In this case, every point P is projected parallel to a vector $\vec{e}$ to become P' on the image plane.

$$\vec{p}' = \vec{p} - u\,\vec{e}$$

$$= \vec{b} + u_1\,\vec{e}_1 + u_2\,\vec{e}_2$$

where $\vec{b}$ is the origin of the coordinate system B-$(\vec{e}_1, \vec{e}_2)$ in the image plane. $(\vec{e}_1, \vec{e}_2, \vec{e})$ need not be orthogonal.



Therefore $u = \dfrac{(\vec{p} - \vec{b}) \cdot (\vec{e}_1 \times \vec{e}_2)}{\vec{e} \cdot (\vec{e}_1 \times \vec{e}_2)}$

Let $\vec{n} = (\vec{e}_1 \times \vec{e}_2)$, the image plane is $\vec{n} \cdot (\vec{p}' - \vec{b}) = 0$

and $\vec{p}' = \vec{p} - \dfrac{(\vec{p} - \vec{b}) \cdot \vec{n}}{\vec{e} \cdot \vec{n}} \cdot \vec{e} = \vec{p} - \left(\dfrac{\vec{p} \cdot \vec{n}}{\vec{e} \cdot \vec{n}}\right)\vec{e} + \dfrac{\vec{b} \cdot \vec{n}}{\vec{e} \cdot \vec{n}}\vec{e}$

i.e. $\begin{pmatrix} \vec{p}' \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} I_3 - \dfrac{\vec{e} \cdot n^T}{\vec{e} \cdot \vec{n}} & & d\,\vec{e} \\ 0 \quad 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix}$ where $d = \dfrac{\vec{b} \cdot \vec{n}}{\vec{e} \cdot \vec{n}}$

relating P and its image P'. Also on the image plane

$$u_1 = \frac{(\vec{p} - \vec{b}) \cdot (\vec{e}_2 \times \vec{e})}{\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e})} = \frac{(\vec{p} - \vec{b}) \cdot (\vec{e}_2 \times \vec{e})}{\vec{e} \cdot \vec{n}},$$

$$u_2 = \frac{(\vec{p} - \vec{b}) \cdot (\vec{e}_1 \times \vec{e})}{\vec{e}_2 \cdot (\vec{e}_1 \times \vec{e})} = \frac{(\vec{p} - \vec{b}) \cdot (\vec{e} \times \vec{e}_1)}{\vec{e} \cdot \vec{n}}.$$

as the coordinates of P' relative to B$(\vec{e}_1, \vec{e}_2)$.

(Orthographic projection)

In case the projection is perpendicular to the image plane:

$\vec{e} = \vec{n} = \vec{e}_1 \times \vec{e}_2$, then

$$u_1 = (\vec{p} - \vec{b}) \cdot \vec{e}_1, \qquad u_2 = (\vec{p} - \vec{b}) \cdot \vec{e}_2, \qquad u = (\vec{p} - \vec{b}) \cdot \vec{n}$$

i.e. $$\begin{pmatrix} \vec{U} \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u \\ 1 \end{pmatrix} = \begin{bmatrix} e_1^T & & & 0 \\ e_2^T & & & 0 \\ n^T & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} & I_3 & & -\vec{b} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix} = \begin{bmatrix} E^T & -E^T \vec{b} \\ 0_3^T & 1 \end{bmatrix} \cdot \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix}$$

where $E_{3\times 3}^T = \begin{pmatrix} e_1^T \\ e_2^T \\ n^T \end{pmatrix}$ is a rotation with $E^T E = I_3$.

u is not a required datum in the image plane.

1.2.2 Axonometric projection

while orthographic projections along principal axes of an object are common to obtain multi-view mechanical drawings, a special form of orthographic projection is used to give simulated 3D images: - the axonometric projection which scales the axes in fixed proportions.

Define the rotation matrix E as the product of two consecutive rotations such that

$$\begin{pmatrix} \vec{U} \\ 1 \end{pmatrix} = \begin{bmatrix} R_{\theta x} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_{\beta y} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix}$$

where

$$R_{\theta x} \cdot R_{\beta y} = \begin{bmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\beta & & \sin\beta \\ & 1 & \\ -\sin\beta & & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ \sin\beta\sin\theta & \cos\theta & -\cos\beta\sin\theta \\ -\sin\beta\cos\theta & \sin\theta & \cos\beta\cos\theta \end{bmatrix}$$

The axes $\vec{p} = (1,0,0)^T, (0,1,0)^T, (0,0,1)^T$ will be transformed into

$\vec{U}_1 = (\cos\beta \;\; \sin\beta\sin\theta \;\; -\sin\beta\cos\theta)^T$, $\qquad \vec{U}_2 = (0 \;\; \cos\theta \;\; \sin\theta)^T$ and

$\vec{U}_3 = (\sin\beta \;\; -\cos\beta\sin\theta \;\; -\cos\beta\cos\theta)^T$

To develop an isometric image on the image plane, set

$\left|\vec{U}_1\right| = \left|\vec{U}_2\right| = \left|\vec{U}_3\right|$ ignoring the u component (the third component), i.e.

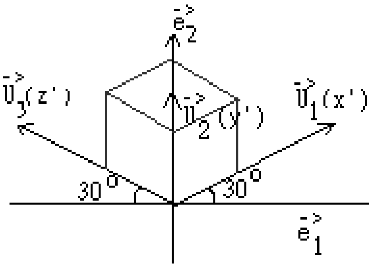
$\cos^2\beta + \sin^2\beta\sin^2\theta = \cos^2\theta = \sin^2\beta + \cos^2\beta\sin^2\theta.$

We may select a value of θ and compute β from above.

For example, let $\sin^2\theta = \frac{1}{3}$, then $\sin^2\beta = \frac{1}{2}$.

The angle α that the x' -axis ($\vec{U}_1$) made with e_1-axis is

$$\tan\alpha = \frac{U_{1y}}{U_{1x}} = \frac{\sin\beta\sin\theta}{\cos\beta} \text{ or } \alpha = 30^\circ$$



Other requirement such as $\frac{1}{2}\left|\vec{U}_1\right| = \left|\vec{U}_2\right| = \left|\vec{U}_3\right|$ can also be constructed. They are used for hand drafting purposes.

1.2.3 Central projection (perspective transformation)

In the case, the image P' of the object point P is in the line from a specific view point P_v, (the center of projection)

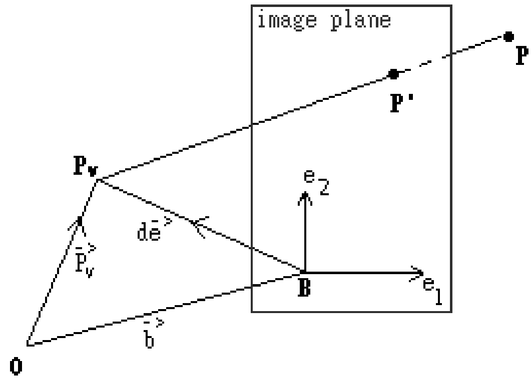
we have

$$\vec{p}' = \vec{p}_v + u(\vec{p} - \vec{p}_v)$$

$$= \vec{b} + u_1\vec{e}_1 + u_2\vec{e}_2$$

Let $\vec{n} = \vec{e}_1 \times \vec{e}_2$ to describe the image plane $\vec{n}\cdot(\vec{p}' - \vec{b}) = 0$

Therefore



$$u = \frac{(\vec{b} - \vec{p}_v)\cdot(\vec{e}_1 \times \vec{e}_2)}{(\vec{p} - \vec{p}_v)\cdot(\vec{e}_1 \times \vec{e}_2)} = \frac{(\vec{b} - \vec{p}_v)\cdot\vec{n}}{(\vec{p} - \vec{p}_v)\cdot\vec{n}}$$

$$\vec{p}' = \vec{p}_v + u(\vec{p} - \vec{p}_v) = \frac{[(\vec{b} - \vec{p}_v)\cdot\vec{n}]\,\vec{p} + (\vec{p}\cdot\vec{n})\,\vec{p}_v - (\vec{b}\cdot\vec{n})\,\vec{p}_v}{(\vec{p} - \vec{p}_v)\cdot\vec{n}}$$

Although this is a rational form, we can still use homogenous coordinates to write $\omega' = (\vec{n}\cdot\vec{p} - \vec{p}_v\cdot\vec{n})\omega$ and

$$\begin{pmatrix} \omega'\vec{p}' \\ \omega' \end{pmatrix} = \begin{pmatrix} x'\omega' \\ y'\omega' \\ z'\omega' \\ \omega' \end{pmatrix} = \begin{bmatrix} \vec{p}_v\,n^T - [(\vec{p}_v - \vec{b})\cdot\vec{n}]I_3 & -(\vec{b}\cdot\vec{n})\,\vec{p}_v \\ n^T & -\vec{p}_v\cdot\vec{n} \end{bmatrix} \cdot \begin{bmatrix} x\omega \\ y\omega \\ z\omega \\ \omega \end{bmatrix}$$

Also

$$u_1 = \frac{(\vec{p}_v - \vec{b})\cdot[\vec{e}_2 \times (\vec{p} - \vec{p}_v)]}{\vec{e}_1\cdot[\vec{e}_2 \times (\vec{p} - \vec{p}_v)]} = \frac{(\vec{p} - \vec{p}_v)\cdot[\vec{e}_2 \times (\vec{b} - \vec{p}_v)]}{(\vec{p} - \vec{p}_v)\cdot\vec{n}}$$

$$u_2 = \frac{(\vec{p}_v - \vec{b})\cdot[\vec{e}_1 \times (\vec{p} - \vec{p}_v)]}{\vec{e}_2\cdot[\vec{e}_1 \times (\vec{p} - \vec{p}_v)]} = \frac{(\vec{p} - \vec{p}_v).[\vec{e}_1 \times (\vec{p}_v - \vec{b})]}{(\vec{p} - \vec{p}_v)\cdot\vec{n}}$$

These can be further simplified if the origin B in the image plane is chosen such that $\vec{e}$ joining the view point to this origin is normal to this image plane, then $\vec{p}_v = \vec{b} + d\,\vec{e} = \vec{b} + d\,\vec{n}$ and

$$u_1 = \frac{d(\vec{p} - \vec{b}) \cdot \vec{e}_1}{d - (\vec{p} - \vec{b}) \cdot \vec{n}}$$

$$u_2 = \frac{d(\vec{p} - \vec{b}) \cdot \vec{e}_2}{d - (\vec{p} - \vec{b}) \cdot \vec{n}}$$

$$u = \frac{d}{d - (\vec{p} - \vec{b}) \cdot \vec{n}}$$

Writing $\Omega' = (d - (\vec{p} - \vec{b}) \cdot \vec{n})\Omega$ and

$$\Omega' \begin{pmatrix} \vec{U} \\ 1 \end{pmatrix} = \Omega' \begin{pmatrix} u_1 \\ u_2 \\ u \\ 1 \end{pmatrix} = \begin{bmatrix} d & & & 0 \\ & d & & 0 \\ & & 0 & d \\ 0 & 0 & -1 & d \end{bmatrix} \cdot \begin{bmatrix} & e_1^T & & \\ & e_2^T & & 0 \\ & n^T & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} I_3 & -\vec{b} \\ 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \Omega$$

$$= \begin{bmatrix} de_1^T & -de_1^T\,\vec{b} \\ de_2^T & -de_2^T\,\vec{b} \\ 0 & d \\ -n^T & d + n^T\,\vec{b} \end{bmatrix} \cdot \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix} \Omega$$

u is not a required datum on the image plane. In case $\vec{e}_1 = \vec{x}$, $\vec{e}_2 = \vec{y}$ and B is the origin, then

$$u_1 = \frac{dx}{d - z}, \quad u_2 = \frac{dy}{d - z}$$

1.3 Projective views

1.3.1 Vanishing points and vanishing planes

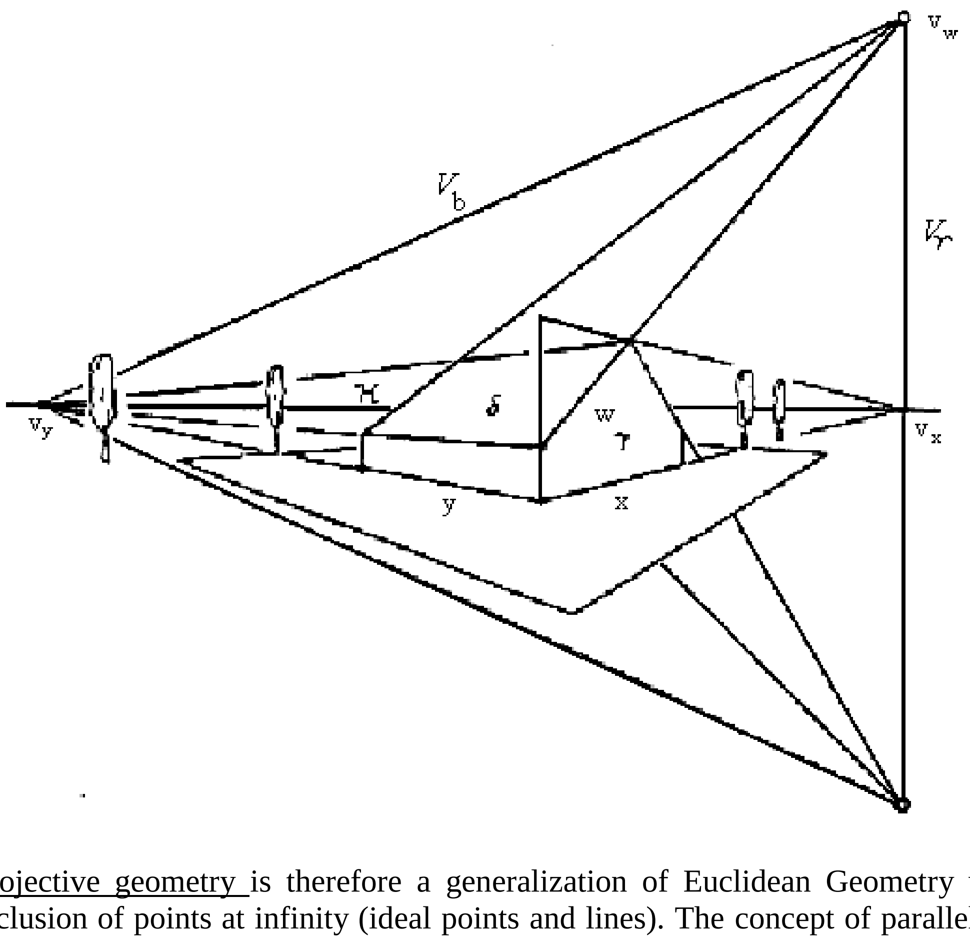
Points at infinity may appear on the image plane as vanishing points. Consider the point at infinity of a line L in 3D in the direction $\vec{u}_L$. A general point on L can be represented by $\vec{p} = \vec{a} + \lambda \vec{u}_L$. The image of P in the central projection as $\lambda \to \infty$ is given by substituting $\vec{p}$ into the projection formulae to obtain

$$u_1 = \frac{-d\,\vec{u}_L \cdot \vec{e}_1}{\vec{u}_L \cdot \vec{n}}$$

$$u_2 = \frac{-d\,\vec{u}_L \cdot \vec{e}_2}{\vec{u}_L \cdot \vec{n}}$$

(u_1, u_2) is the vanishing point in the image plane so long as $\vec{u}_L \cdot \vec{n} \neq 0$

In case $\vec{u}_L \cdot \vec{n} = 0$, the vanishing point for $\vec{u}_L$ is at infinity.



Projective geometry is therefore a generalization of Euclidean Geometry with the inclusion of points at infinity (ideal points and lines). The concept of parallelism and similarity no longer apply. Only concept of incidence remains.

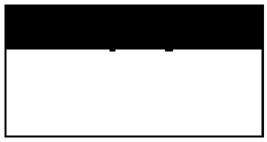
In 2D projective geometry

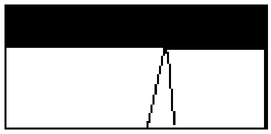
One ordinary point and one ideal points determine a line through the ordinary point in the direction of the ideal point.

Two ideal points determine the ideal line where all ideal points lie, (e.g. the horizon).

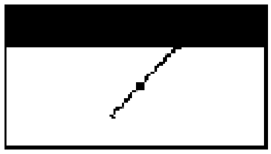
Two lines which are parallel determine an ideal point.

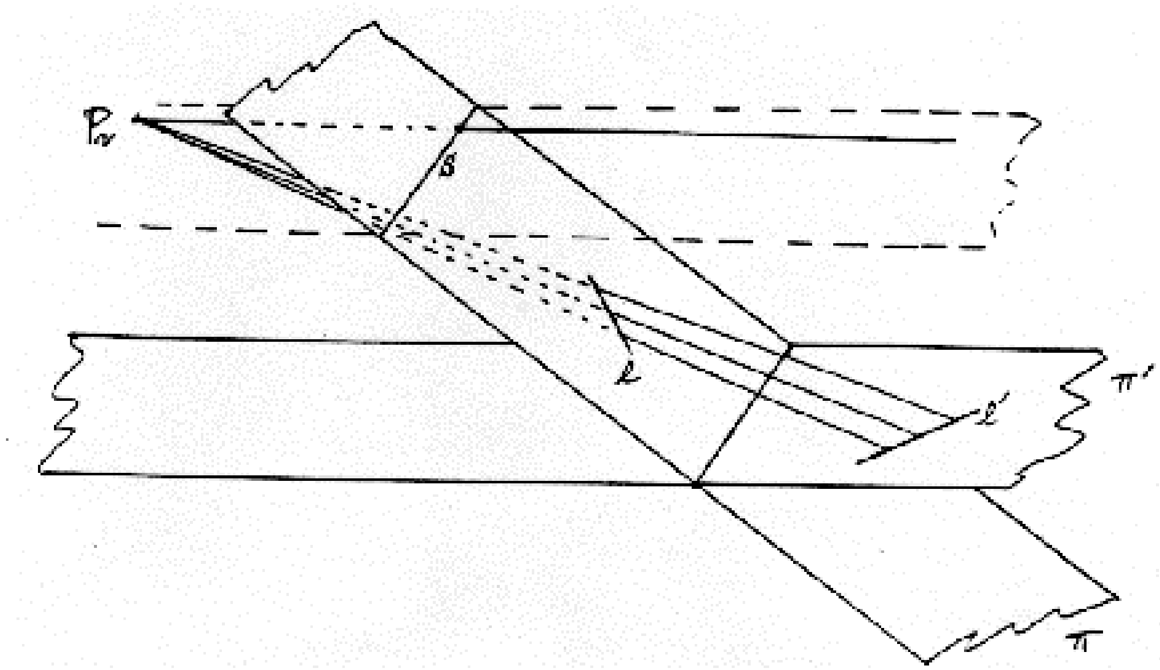
An ordinary line and the ideal line determine the ideal point.



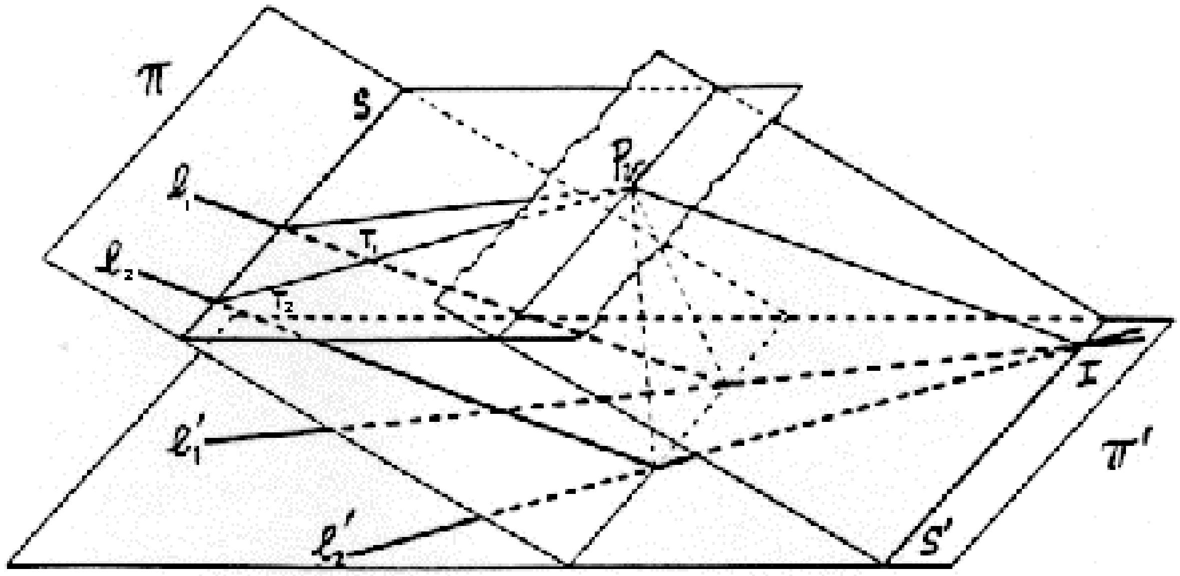


From a central projection, in 3D, a line l in plane π is transformed into a line l' in plane π', (except the line s which lies on the plane through P_v parallel to π'), and vice versa, (except the line s', which lies on the plane through P_v parallel to π).

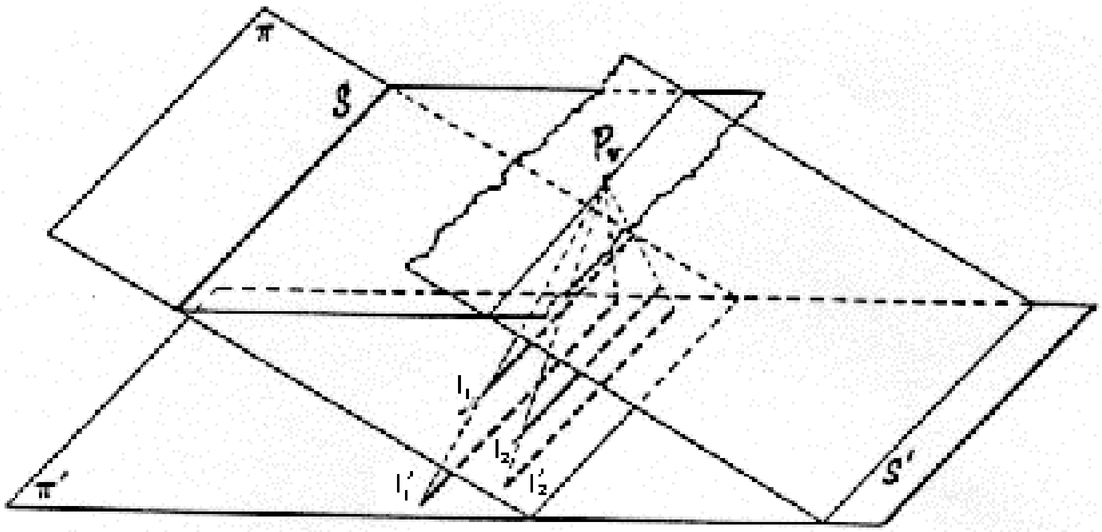




If l_1 and l_2 are parallel lines in π, they are transformed into lines l_1' and l_2' on π' meeting at a point I on S' .

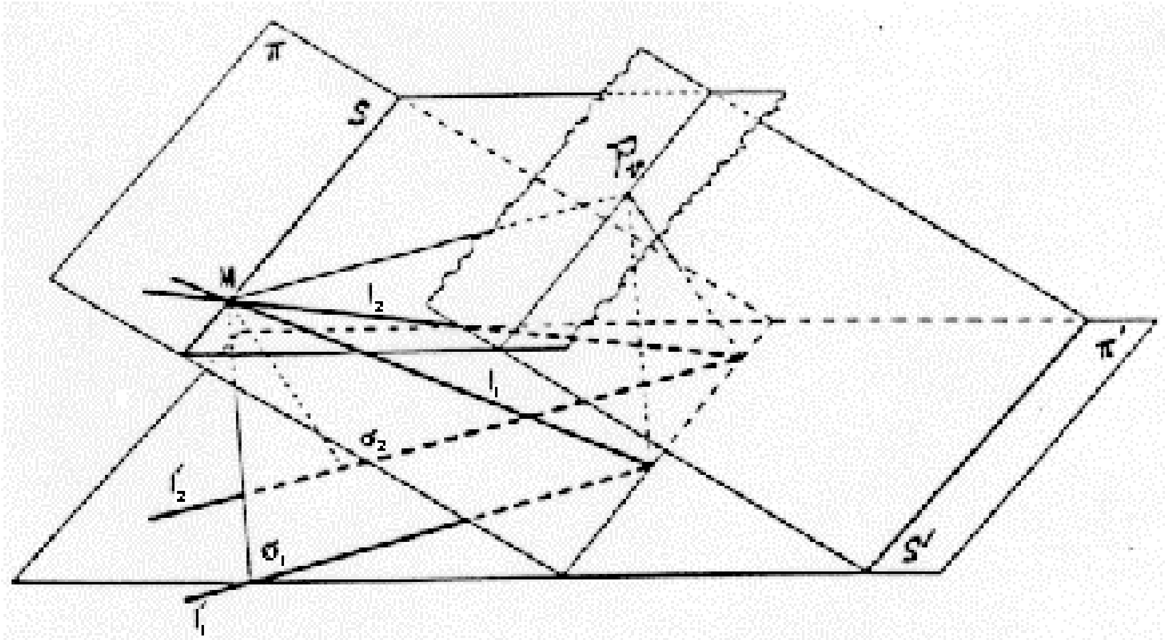


Proof: this follows because the planes T_1, T_2 determined by the point P_v and the lines l_1 and l_2 respectively, intersect in a line P_vI parallel to π and meeting π' in a point I on the special line S' .
Exceptions to this rule are lines parallel to S; these lines are mapped onto lines parallel to the intersection of π and π' (or parallel to S')



Note that I is the image of the ideal point for l_1 and l_2 on π' (i.e. I is the vanishing point).

Conversely, let l_1 and l_2 in the plane π intersect in a point M on the special line S. Then their images under a central projection are two parallel lines l_1' and l_2' in

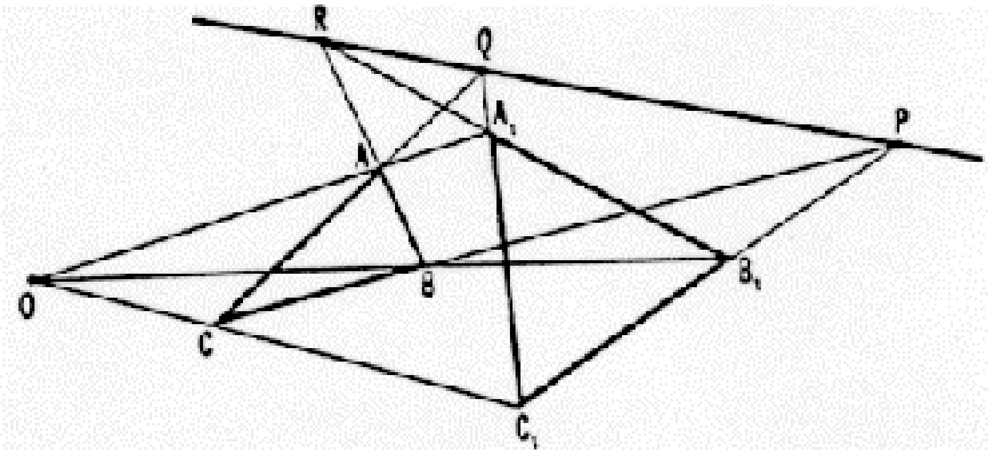


π' .

Proof: the planes σ_1 and σ_2 determined by the point P_v and the lines l_1 and l_2 respectively, intersect in the line MP_v parallel to π' . It follows that the projection l_1' and l_2' are parallel in π' .

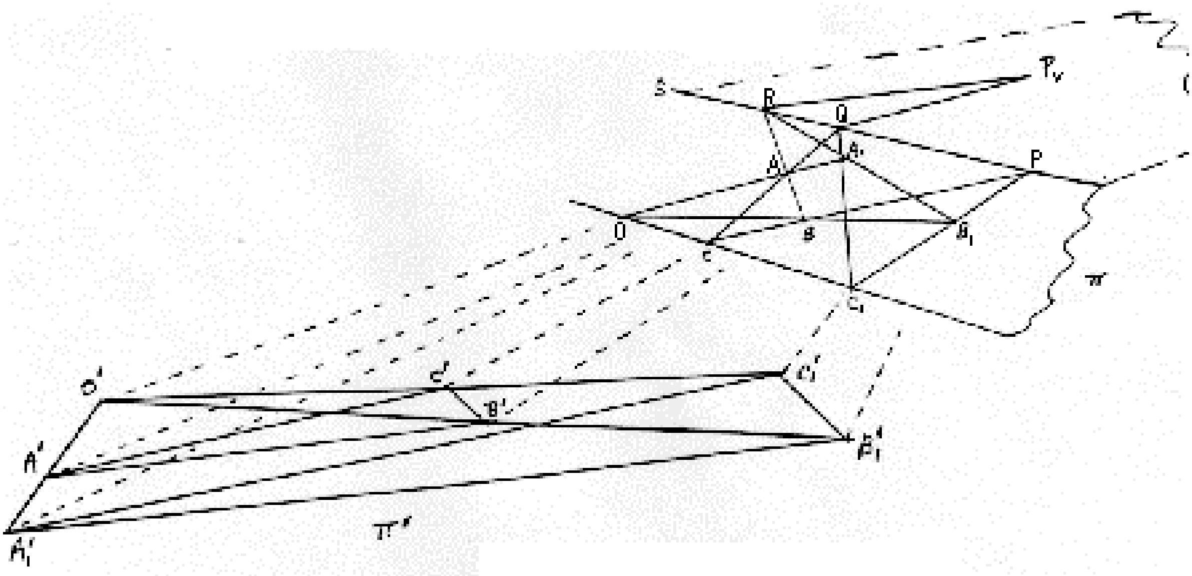
We may use these properties to prove a fundamental result in Renaissance geometry.

<u>Desargues' Theorem</u>(狄氏定理1639): If two ΔABC and $\Delta A_1B_1C_1$ are located in a plane so that lines AA_1, BB_1 and CC_1 are concurrent, then the points of intersection of lines AB and A_1B_1, AC and A_1C_1, BC and B_1C_1 are collinear, and vice versa.



Note that the theorem expresses a property of lines and points in a plane not necessarily tied to a pair of triangles. For example OCC_1 is collinear for ΔPBB_1 and ΔQAA_1; or B is the center of concurrency for ΔPRB_1 and CAO.

Proof of Desargues' Theorem:
Project the plane π of the triangles onto a plane π' such that QR is the special line s



of π.

we have $A'B' // RP_v // A_1^{'}B_1^{'}$ and $A'C' // QP_v // A_1^{'}C_1^{'}$
Also O' will be concurrent with $A'A_1^{'}$, $C'C_1^{'}$ and $B'B_1^{'}$
It follows $O'B'/O'B_1^{'} = O'A'/O'A_1^{'} = O'C'/O'C_1^{'}$, hence
$B'C'$ and $B_1^{'}C_1^{'}$ are parallel, therefore
the point P of intersection of BC and B_1C_1 lies on the special line S of π, i.e. P, Q, R are collinear.

Conversely, if P, Q, R are collinear, then lines of the corresponding vertices of the ΔPB_1B and ΔQA_1A are concurrent at R. By the first part of the assertion, the points C_1, C, and O of intersection of the corresponding sides are collinear.

Q.E.D.