the proof of decomposition theorem

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We use $\mathcal{H}$ to mean the cohomology of complex of sheaves and $\mathbb{H}$ to mean the hypercohomology. dim means the dimension of complex varieties.

1 preparation results

1.1 semisimple of classical Hodge theory

See Hodge II section 4.2.

Let S be a smooth connected variety, Use $\mathcal{C}$ to denote the subcategory of all the families of structure of Hodge, whose object H satisfies the condition: there exist a Zariski open subset U such that there exist a projective and smooth morphism $f : X \to U$ and $H|_U$ is a direct factor of $R^i f_* \mathbb{Q}_X \otimes \mathbb{Q}(k)$ for integers i and k.

Then, for any proper and smooth morphism $f : X \to S$, using Chow lemma and resolution of singularities, we get a projective and smooth morphism $f' : X' \to f^{-1}(U) \to U$, and that $(R^i f_* \mathbb{Q}_X)|_U$ is a direct factor of $R^i f'_* \mathbb{Q}_{X'}$, thus lies in $\mathcal{C}$.

Theorem 1. *The objects in $\mathcal{C}$ are semisimple as local systems on S.*

Proof. We first claim that the category $\mathcal{C}$ satisfies the following proposition:

(i) $\mathcal{C}$ is stable under direct factor, direct sum, tensor product. the constant family $\mathbb{Q}(k)$ lies in $\mathcal{C}$.

(ii) all homogeneous Hodge structure are polarizable.

(iii) for all $H \in \mathcal{C}$, there exist $H_{\mathbb{Z}}$ such that $H = H_{\mathbb{Z}} \otimes \mathbb{Q}$.

(iv) for all $H \in \mathcal{C}$, the largest constant sub-local system is a constant sub-family of Hodge structure of H.

In order to verify this, we set $\mathcal{C}_1$ to be the direct factors of the families of Hodge structure in the form $R^i f_* \mathbb{Q}_X$ for projective and smooth morphism $f : X \to S$. Then it is obvious $\mathcal{C}_1$ is stable under direct factor. From the fact $R^k(f \times g)_* \mathbb{Q}_{X \times Y} = \oplus_{i+j=k} R^i f_* \mathbb{Q}_X \otimes R^j g_* \mathbb{Q}_Y$, we know $\mathcal{C}_1$ is stable under tensor product. Also since $R^i(f \sqcup g)_* \mathbb{Q}_{X \sqcup Y} = R^i f_* \mathbb{Q}_X \oplus R^i g_* \mathbb{Q}_Y$, we know $\mathcal{C}_1$ is stable under direct sum.

(ii) comes from the polarization of $(R^i f_* \mathbb{Q}_X)_s = \mathsf{H}^i(f^{-1}(s), \mathbb{Q})$ is polarized. (iii) comes from the fact that $R^i f_* \mathbb{Q}_X = R^i f_* \mathbb{Z}_X \otimes \mathbb{Q}$. (iv) is the result of Hodge II 4.1.2.

In fact, for $R^n f_* \mathbb{Q}_X$, the largest constant sub-local system is $(R^n f_* \mathbb{Q}_X)_s^{\pi_1(S,s)} = \mathsf{H}^0(S, R^n f_* \mathbb{Q}_X)$. On the other hand, we have the morphisms $\mathsf{H}^n(\bar{X}, \mathbb{Q}) \to \mathsf{H}^n(X, \mathbb{Q}) \twoheadrightarrow \mathsf{H}^0(S, R^n f_* \mathbb{Q}_X)$. The weight miracle shows that the composition is also surjective. Further more, we also have the morphism $\mathsf{H}^0(S, R^n f_* \mathbb{Q}_X) \hookrightarrow \mathsf{H}^n(f^{-1}(s), \mathbb{Q})$. This shows that $\mathsf{H}^0(S, R^n f_* \mathbb{Q}_X)$ is the image of two polarized Hodge structure and hence is a polarized Hodge structure.

So the category of the form $H\otimes\mathbb{Q}(k)$ for $k\in\mathbb{Z}$ satisfies these four propositions. As for the category $\mathcal{C}$, it suffices to note that if H is a family of Hodge structure and U is a Zariski open set of S, then the subobject, polarization and integer result on $H|_U$ can extent uniquely to H.

Next, we have a lemma, where G is the real algebraic group $\mathbb{C}^*$ acting on the Hodge structures by $z^p\overline{z}^q$ on H^{pq}.

Lemma. *If V is a sub-local system of rank 1 of $H_{\mathbb{C}}=H\otimes\mathbb{C}$ such that $V^{\otimes n}$ is trivial, then for any $t\in G$, we have tV is also a sub-local system of $H_{\mathbb{C}}$.*

Proof. It suffice to proof $(tV)^{\otimes n}\subseteq H_{\mathbb{C}}^{\otimes n}$ is so. But from the triviality of $V^{\otimes n}$, it is generated by a global section v. According to (iv), we obtain $(tV)^{\otimes n}=tV^{\otimes n}$ is generated by the global section tv and hence is locally constant. □

Now, we can prove the theorem. We induct on the dimension of H. without loss of generality, we can assume H is homogeneous. Set d the smallest dimension of the sub-local system of $H_{\mathbb{C}}$. Let W be the sum of all sub-local system of $H_{\mathbb{C}}$ of dimension d and hence is defined in $\mathbb{Q}$. Then W is a complex semisimple local system of dimension e, then $W_{\mathbb{Q}}$ is rational semisimple local system.

Set $H=H_{\mathbb{Z}}\otimes\mathbb{Q}, W_{\mathbb{Z}}=W\cap H_{\mathbb{Z}}$, then we have $\wedge^e W=\wedge^e W_{\mathbb{Z}}\otimes\mathbb{C}$, so $\pi_1(S)$ acts on it just by ± 1. Thus using the lemma, we obtain $t(\wedge^e W)$ is also local system. If V is one of the local system of $H_{\mathbb{C}}$ of dimension d and V' is the complementary in W, we have $\wedge^e W=\wedge^d V\otimes\wedge^{e-d}V'\subseteq\wedge^d H_{\mathbb{C}}\otimes\wedge^{e-d}H_{\mathbb{C}}$ is sub-local system. Apply the lemma, we have $\wedge^e tW=\wedge^d tV\otimes\wedge^{e-d}tV'\subseteq\wedge^d H_{\mathbb{C}}\otimes\wedge^{e-d}H_{\mathbb{C}}$ is also local system. Thus $\wedge^d tV\subset\wedge^d H_{\mathbb{C}}$ and $tV\subset H_{\mathbb{C}}$ is sub-local system. From the definition of W, we have $tV\subset W$ and hence W is stable under the action of G. Then $W_{\mathbb{Q}}$ is sub-Hodge structure of H and is orthogonal direct factor because of the polarization of H. Finally the result holds from the induction. □

1.2 a splitting criterion

Given a stratified space X with smallest stratum F of dimension s which is smooth and closed in X. Let U be the compliment of F and denote the inclusion map $F\xrightarrow{i}X\xleftarrow{j}U$.

For an element K in $P(X)$, we know that $K\in\mathsf{D}^{\leq -s}$ and $\operatorname{supp}\mathcal{H}^{-s}(K)\subseteq F$.

From the theorem of intermediate extension, we know that $j_{!*}j^*K=\tau_{\leq -s-1}j_*j^*K$. Since $K=\tau_{\leq -s}K$, we have a natural map $f:K\to\tau_{\leq -s}j_*j^*K$. From the distinguished triangle $i_!i^!K\to K\to j_*j^*K\xrightarrow{+1}$, we get a long exact sequence:

$$\mathcal{H}^{-s-1}(K)\xrightarrow{\alpha}\mathcal{H}^{-s-1}(j_*j^*K)\to\mathcal{H}^{-s}(i_!i^!K)\xrightarrow{\beta}\mathcal{H}^{-s}(K)\xrightarrow{\gamma}\mathcal{H}^{-s}(j_*j^*K)\tag{1}$$

Now we give a criterion for whether K is a direct sum of two intermediate extension complexes.

Theorem 2. *If K satisfies the following property:*

$$\dim\mathcal{H}^{-s}(i_!i^!K)_x=\dim\mathcal{H}^{-s}(i_*i^*K)_x,\forall x\in F\tag{2}$$

(caution that $\dim\mathcal{H}^{-s}(i_*i^*K)_x=\dim\mathcal{H}^{-s}(K)_x$)

Then the following are equivalent:

(i) $K=j_{!*}j^*K\oplus\mathcal{H}^{-s}(K)[s]$.

(ii) $\beta:\mathcal{H}^{-s}(i_!i^!K)\to\mathcal{H}^{-s}(K)$ *is an isomorphism.*

(iii) the morphism $f : K \to \tau_{\leq -s} j_* j^* K$ can be lifted to a morphism $\tilde{f} : K \to \tau_{\leq -s-1} j_* j^* K = j_{!*} j^* K$.

Proof. $(i) \implies (ii)$: From the property of $j_{!*}$, we know $i^! j_{!*} j^* K \in {}^{\mathsf{p}}\mathsf{D}_F^{\geq 1} = \mathsf{D}_F^{\geq -s+1}$. Since $i_!$ is exact we get $\mathcal{H}^{-s}(i_! i^! j_{!*} j^* K) = 0$.

$(ii) \iff (iii)$: apply $\mathsf{Hom}(K, -)$ to the following distinguished triangle:

$$\tau_{\leq -s-1} j_* j^* K \to \tau_{\leq -s} j_* j^* K \to \mathcal{H}^{-s}(j_* j^* K)[s] \xrightarrow{+1}$$

From the fact that both K and $\mathcal{H}^{-s}(j_* j^* K)[s]$ are in $P(X)$ since $(\mathcal{H}^{-s}(j_* j^* K))_U = 0$, we know $\mathsf{Hom}(K, \mathcal{H}^{-s}(K)[s][-1]) = 0$, and this gives an exact sequence:

$$0 \to \mathsf{Hom}(K, \tau_{\leq -s-1} j_* j^* K) \xrightarrow{t} \mathsf{Hom}(K, \tau_{\leq -s} j_* j^* K) \to \mathsf{Hom}(K, \mathcal{H}^{-s}(j_* j^* K)[s])$$

Thus $f \in \mathsf{im} t$ if and only if the composite of f and $\tau_{\leq -s} j_* j^* K \to \mathcal{H}^{-s}(j_* j^* K)[s]$ is zero. However, since $\mathcal{H}^{-s}(j_* j^* K)[s] \in \mathsf{D}^{\geq -s}$, we know $\mathsf{Hom}(K, \mathcal{H}^{-s}(j_* j^* K)[s]) = \mathsf{Hom}(\tau_{\geq -s} K, \mathcal{H}^{-s}(j_* j^* K)[s]) = \mathsf{Hom}(\mathcal{H}^{-s}(K), \mathcal{H}^{-s}(j_* j^* K))$ and the composition is precisely the morphism γ in (1), for both of them comes from the truncations of the adjoint map $K \to j_* j^* K$.

Now we know that $\gamma = 0$ if and only if β is surjective and thus according to (2), is equivalent to β is isomorphism.

$(ii)(iii) \implies (i)$: If the lift $\tilde{f}$ exist, we have an exact sequence in the category $P(X)$:

$$0 \to N \to K \to j_{!*} j^* K \to C \to 0$$

Because the map $\tilde{f}$ is isomorphism restricted at U, the supports of N and C are in F. But from the proposition of intermediate extension, we know $j_{!*} j^* K$ don't have non-zero quotient whose support is in F, which shows that C is equal to zero.

Now, because N is a perverse sheaf supports in F, it must be in the form $\mathcal{H}^{-s}(N)[s]$. Then the distinguished triangle

$$N \to K \to j_{!*} j^* K \xrightarrow{+1}$$

gives a long exact sequence:

$$0 \to \mathcal{H}^{-s-1}(K) \to \mathcal{H}^{-s-1}(j_* j^* K) \to \mathcal{H}^{-s}(N) \to \mathcal{H}^{-s}(K) \to 0$$

where the first map is equal to α in (1) and is surjective because of (ii). Hence we get $\mathcal{H}^{-s}(N) = \mathcal{H}^{-s}(K)$ and the boundary map in the middle is zero, which shows that the map in $\mathsf{Hom}(j_{!*} j^* K, N[1]) = \mathsf{Hom}(\tau_{\geq -s-1} j_{!*} j^* K, N[1]) = \mathsf{Hom}(\mathcal{H}^{-s-1}(j_* j^* K), \mathcal{H}^{-s}(N))$ is zero. This implies that the distinguished triangle above splits, which means $K = N \oplus j_{!*} j^* K = \mathcal{H}^{-s}(K)[s] \oplus j_{!*} j^* K$. This is (i). □

1.3 degenerate of spectral sequence

Given a triangulated category D with a t-structure, we use simply H to denote the cohomology with respect to this t-structure.

For an element $K \in \mathsf{D}$ and a map $\eta : K \to K[2]$, we use the same notion for the morphism of cohomology $\eta : \mathsf{H}^*(K) \to \mathsf{H}^{*+2}(K)$.

Theorem 3. *If $\eta^j : \mathsf{H}^{-j}(K) \to \mathsf{H}^j(K)$ are isomorphism for all integer j, then we have*

$$\bigoplus_j \mathsf{H}^j(K)[-j] \simeq K.$$

Proof. Let $P^{-j} = \mathsf{ker}(\eta^{j+1} : \mathsf{H}^{-j}(K) \to \mathsf{H}^{j+2}(K)), j \geq 0$. Then we have a splitting short exact sequence:

$$0 \to P^{-j} \to \mathsf{H}^{-j}(K) \to \mathsf{H}^{j+2}(K) \to 0$$

Thus $\mathsf{H}^{-j}(K) = P^{-j} \oplus \eta \mathsf{H}^{-j-2}(K) = \oplus_{i \geq 0} \eta^i P^{-j-2i}, j \geq 0$.

Now for any exact functor F, we have the spectral sequence $E_2^{p,q} = F^p \mathsf{H}^q(K) \Rightarrow F^{p+q}(K)$. We use induction to prove this spectral sequence degenerate at E_2-page.

Assume $E_r = E_2$, we will prove d_r in the E_r-page are all zero. We need only prove that $d_r(F^p(\eta^i P^{-q-2i})) = 0, i, q \geq 0$. Since η acts on K, we have a natural map $\eta : E_2^{p,q} \to E_2^{p,q+2}$. This gives a commutative diagram:

$$\begin{tikzcd}
F^p(\eta^i P^{-q-2i}) \arrow[r, "d_r"] \arrow[d, "\eta^{i+q+1}"] & F^{p+r}(\eta^i \mathsf{H}^{-q-2i-r+1}) \arrow[d, "\eta^{i+p+1}"] \\
F^p(\eta^{2i+q+1} P^{-q-2i}) \arrow[r, "d_r"] & F^{p+r} \mathsf{H}^{q+2i+3-r}
\end{tikzcd}$$

According to the condition, we know that the map on the right $\eta^{2i+p+1} : \mathsf{H}^{-q-2i-r+1} \to \mathsf{H}^{q+2i+3-r}$ is injective. Then the map on the left being equal to zero implies that the map on the top is equal to zero. This finishes our proof. $\square$

1.4 affine morphism

Theorem 4. *Let $f : X \to Y$ be an affine morphism, F be a constructible sheaf on X. If $\mathsf{dim\,supp}\, F \leq d$, then $\mathsf{dim\,supp}\, R^q f_*(F) \leq d - q$.*

Proof. See SGA 4 XIII. $\square$

Corollary 5. *Let $f : X \to Y$ be an affine morphism, them f_* is right t-exact and $f_!$ is left t-exact.*

Proof. For any complex of sheaves $K \in {}^{\mathfrak{p}}\mathsf{D}^{\leq 0}(X)$, we have $\mathsf{dim\,supp}\, \mathcal{H}^q(K) \leq -q$. Thus by the theorem above, we obtain $\mathsf{dim\,supp}\, R^p f_*(\mathcal{H}^q(K)) \leq -q - p$.

Using the spectral sequence:

$$R^p f_*(\mathcal{H}^q(K)) \Rightarrow R^{p+q} f_*(K)$$

We know that $\mathsf{dim\,supp}\, R^{p+q} f_*(K) \leq -(p+q)$, which shows that $f_*(K) \in {}^{\mathfrak{p}}\mathsf{D}^{\leq 0}(Y)$. This proves that f_* is right t-exact.

Using Verdier duality, we know $f_! = D f_* D$ and D maps ${}^{\mathfrak{p}}\mathsf{D}^{\geq 0}$ between ${}^{\mathfrak{p}}\mathsf{D}^{\leq 0}$. This proves the second assertion. $\square$

1.5 projection

We consider the projection $p : Y \times X \to Y$, where X is smooth of dimension d. Then we have $p^*[d] = p^![-d]$ is t-exact. The result is:

Theorem 6. *The functor $p^*[d] : P(Y) \to P(Y \times X)$ makes $P(Y)$ a fully faithful subcategory of $P(Y \times X)$.*

Proof. When K, L are two elements in $D^b(Y)$, we have the isomorphism:

$$p^* R\mathcal{H}om(K, L) \xrightarrow{\sim} R\mathcal{H}om(p^* K, p^* L)$$

If K, L are in $P(Y)$, we have $R\mathcal{H}om(K,L) \in \mathsf{D}^{\geq 0}(Y)$, thus applying ${}^0p_*\mathcal{H}^0$, we obtain:

$$ {}^0p_*p^*\mathcal{H}^0 R\mathcal{H}om(K,L) \xrightarrow{\sim} {}^0p_*\mathcal{H}^0 R\mathcal{H}om(p^*K[d], p^*L[d]) $$

Because ${}^0p_*p^* = \mathsf{id}$ for sheaves on Y, we can take the functor Γ, which shows that

$$ \mathsf{Hom}(K,L) = \mathsf{Hom}(p^*K[d], p^*L[d]) $$

This proves our assertion. $\square$

Now, we denote $p^*[d]$ and its left and right adjoint ${}^{\mathfrak{p}}\mathcal{H}^d p_!$ and ${}^{\mathfrak{p}}\mathcal{H}^{-d}p_*$ by u^*, $u_!$ and u_*, respectively. Therefore, from the property of fully faithful, we have $\mathsf{Hom}(A,B) = \mathsf{Hom}(u^*A, u^*B) = \mathsf{Hom}(A, u_*u^*B)$ for any $A, B \in P(Y)$, which shows $u_*u^*B = B$ for any $B \in P(Y)$.

If S is a simple element of $P(Y)$, namely, $S = j_{!*}\mathcal{L}[\mathsf{dim}\, U]$ for some smooth subset U of Y and some simple sheaf $\mathcal{L}$ on U. then $u^*S = j_{!*}p^*\mathcal{L}[\mathsf{dim}\, U + d]$ is also simple in $P(Y \times X)$. Because $P(Y)$ is Noetherian and Artian, the element of $P(Y)$ has finite length. Let K is an element in $P(Y)$, we have a filtration $0 = K_0 \subset K_1 \subset \cdots \subset K_n = K$ with simple quotients, then $0 = u^*K_0 \subset u^*K_1 \cdots \subset u^*K_n = u^*K$ is a filtration with simple quotients of the form u^*S. This implies the sub-object of u^*K are all comes from $u^*P(Y)$.

Then, given an element K in $P(Y \times X)$, we have a morphism $u^*u_*K \to K$. We will prove this is the largest sub-object coming from $P(Y)$.

For any element $A \in P(Y)$, we have $\mathsf{Hom}(u^*A, K) = \mathsf{Hom}(A, u_*K) = \mathsf{Hom}(u^*A, u^*u_*K)$. So if a morphism $u^*A \to u^*u_*K$ satisfies the composite with $u^*u_*K \to K$ is zero, then it is also zero. This shows that $u^*u_*K \to K$ is injective since the kernel is a sub-object of u^*u_*K, which must has the form u^*A for some A.

Then for any sub-object of K in the form u^*A, the above argument shows that it pass through u^*u_*K. Thus u^*A is sub-object of u^*u_*K, showing that u^*u_*K is the largest.

$$ \begin{tikzcd} u^*u_*K \arrow[r] & K \\ u^*A \arrow[u] \arrow[ur] & \end{tikzcd} $$

From this, we call u^*u_*K the largest sub-object of K coming from Y. For the same reason, we know $u^*u_!K$ is the largest quotient object of K coming from Y.

1.6 universal hyperplane section and weak Lefschetz

Assume X is a quasi-projective variety which embeds in a projective space $\mathbb{P}$ with dimension d. We denote $\mathbb{P}^\vee$ the dual space of $\mathbb{P}$. Then let $\mathcal{X} = \{(x,s) : s(x) = 0\} \subseteq X \times \mathbb{P}^\vee$ be the universal hyperplane section.

Since X embeds into $\mathbb{P}$, we know that $X \times \mathbb{P}^\vee$ embeds into $\mathbb{P} \times \mathbb{P}^\vee$, which is a projective space. The tautological bundle $\mathcal{O}_{\mathbb{P}\times\mathbb{P}^\vee}(1)$ give the very ample bundle $\mathcal{O}(1)$ on $X \times \mathbb{P}^\vee$, which contains a section $S : (x,s) \mapsto s(x)$. Thus the open space $X \times \mathbb{P}^\vee - \mathcal{X}$ is just the space $(X \times \mathbb{P}^\vee)_S$.

Now, if $f : X \to Y$ is a proper morphism, $f' : X \times \mathbb{P}^\vee \to Y \times \mathbb{P}^\vee$ is also proper. For any affine open subset $Y' = \mathsf{Spec} A$ of $Y \times \mathbb{P}^\vee$, we have a very ample bundle L on $X' = f'^{-1}(Y')$. This makes X' embeds into $P = \mathsf{Proj} \bigoplus_n \Gamma(X', L^{\otimes n})$ which is a projective space over $\mathsf{Spec} A$. Because of the properness of f', we obtain X' is equal to P. Thus X'_S is just the basic open sets of P, which is affine. This shows that the map $X \times \mathbb{P}^\vee - \mathcal{X} \to Y \times \mathbb{P}^\vee$ is affine.

If we set $\mathcal{P} = \{(x,s) : s(x) = 0\} \subseteq \mathbb{P} \times \mathbb{P}^\vee$, then $\mathcal{P}$ is smooth over $\mathbb{P}$. Since $\mathcal{P} \cap (X \times \mathbb{P}^\vee) = \mathcal{X}$, we have $\mathcal{X}$ is smooth over X, which shows that the embedding $i : \mathcal{X} \to X \times \mathbb{P}^\vee$ is transversal. Thus $i^*[-1] = i^![1]$ is t-exact. Furthermore, it implies

$$IC_{\mathcal{X}} = i^*[-1]IC_{X\times\mathbb{P}^\vee} = i^*p^*IC_X[d-1] \tag{3}$$

Now, let's look at a general case. Assume we have a diagram of morphisms:

$$\begin{tikzcd} X_1 \arrow[r, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] & X - X_1 \arrow[l, "j"'] \arrow[dl, "u"] \\ & Y & \end{tikzcd}$$

where i is closed embedding while j is open embedding. f is proper and $u = f \circ j$ is affine. Then we have our theorem:

Theorem 7. *Let K be a perverse sheaf on X. Then*

(i) the map ${}^{\mathsf{p}}\mathcal{H}^p(f_*K) \to {}^{\mathsf{p}}\mathcal{H}^p(g_*i^*K)$ induced by $\mathsf{id} \to i_*i^*$ is isomorphism for $p \leq -2$ and monomorphism for $p = -1$.

(ii) the map ${}^{\mathsf{p}}\mathcal{H}^p(g_*i^!K) \to {}^{\mathsf{p}}\mathcal{H}^p(f_*K)$ induced by $i_!i^! \to \mathsf{id}$ is isomorphism for $p \geq 2$ and epimorphism for $p = 1$.

Proof. Consider the distinguished triangle $j_!j^!K \to K \to i_*i^*K \xrightarrow{+1}$ and apply $f_* = f_!$ we obtain the following distinguished triangle:

$$u_!j^!K \to f_*K \to g_*i^*K \xrightarrow{+1} .$$

Taking the perverse cohomology, we have the exact sequence:

$${}^{\mathsf{p}}\mathcal{H}^p(u_!j^!K) \to {}^{\mathsf{p}}\mathcal{H}^p(f_*K) \to {}^{\mathsf{p}}\mathcal{H}^p(g_*i^*K) \to {}^{\mathsf{p}}\mathcal{H}^{p+1}(u_!j^!K)$$

Since $j^! = j^*$ is exact and $u_!$ is left t-exact from corollary 5, we get $u_!j^!K \in {}^{\mathsf{p}}\mathsf{D}^{\geq 0}(Y)$. This means ${}^{\mathsf{p}}\mathcal{H}^p(u_!j^!K) = 0$ for $p < 0$. Therefore the first statement is proved. The other statement can be proved in the same way or just use Verdier duality. $\square$

Here is the reason why this theorem is called weak Lefschetz theorem. When we take Y to be a point and X_1 to be a hyperplane section of the projective variety X, the results just shows that the restriction map $\mathbb{H}^p(X,K) \to \mathbb{H}^p(X_1, i^*K)$ is isomorphism for $p \leq -2$ and monomorphism for $p = -1$, and that the map $\mathbb{H}^p(X_1, i^!K) \to \mathbb{H}^p(X,K)$ is isomorphism for $p \geq 2$ and epimorphism for $p = 1$.

Return to our special case, assume $f : X \to Y$ is a proper morphism where X is quasi-projective. Take X, Y, X_1 in the theorem to be $X \times \mathbb{P}^\vee$, $Y \times \mathbb{P}^\vee$, $\mathcal{X}$, respectively. We have the maps:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "i"] \arrow[dr, "g"'] & X \times \mathbb{P}^\vee \arrow[r, "p"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ & Y \times \mathbb{P}^\vee \arrow[r, "p"] & Y \end{tikzcd}$$

For any perverse sheaf K on X, set $K' = p^*K[d] \in P(X \times \mathbb{P}^\vee)$ and $M = i^*K'[-1] \in P(\mathcal{X})$. According to the t-exactness of $p^*[d]$ we have ${}^{\mathsf{p}}\mathcal{H}^p(f'_*K') = {}^{\mathsf{p}}\mathcal{H}^p(p^*f_*K[d]) = p^*{}^{\mathsf{p}}\mathcal{H}^p(f_*K)[d]$. Then the theorem says that:

(i) $p^{*}{}^{\mathfrak{p}}\mathcal{H}^{p}(f_*K)[d] \to {}^{\mathfrak{p}}\mathcal{H}^{p+1}(g_*M)$ is isomorphism for $p \le -2$ and monomorphism for $p = -1$.

(ii) ${}^{\mathfrak{p}}\mathcal{H}^{p-1}(g_*M) \to p^{*}{}^{\mathfrak{p}}\mathcal{H}^{p}(f_*K)[d]$ is isomorphism for $p \ge 2$ and epimorphism for $p = 1$.

Now we will prove the additional results that ${}^{\mathfrak{p}}\mathcal{H}^{-1}(f'_*K')$ is the largest sub-object of ${}^{\mathfrak{p}}\mathcal{H}^{0}(g_*M)$ coming from Y and that ${}^{\mathfrak{p}}\mathcal{H}^{1}(f'_*K')$ is the largest quotient object of ${}^{\mathfrak{p}}\mathcal{H}^{0}(g_*M)$ coming from Y.

We only prove the first assertion and the second is deduced for the similar reason. We only need to prove ${}^{\mathfrak{p}}\mathcal{H}^{-d}p_*{}^{\mathfrak{p}}\mathcal{H}^{-1}(f'_*K') \xrightarrow{\sim} {}^{\mathfrak{p}}\mathcal{H}^{-d}p_*{}^{\mathfrak{p}}\mathcal{H}^{0}(g_*M)$.

First, from $\mathsf{id} \to i_*i^*$, we have a morphism $f'_*K'[-1] \to g_*M$. The previous result shows the morphism ${}^{\mathfrak{p}}\tau_{\le -1}f'_*K'[-1] \to {}^{\mathfrak{p}}\tau_{\le -1}g_*M$ is isomorphism. Thus we obtain the following morphism of distinguished triangle:

$$\begin{tikzcd}
{}^{\mathfrak{p}}\tau_{\le -1}f'_*K'[-1] \arrow[r] \arrow[d, "\sim"] & f'_*K'[-1] \arrow[r] \arrow[d] & {}^{\mathfrak{p}}\tau_{\ge 0}f'_*K'[-1] \arrow[r, "+1"] \arrow[d] & {} \\
{}^{\mathfrak{p}}\tau_{\le -1}g_*M \arrow[r] & g_*M \arrow[r] & {}^{\mathfrak{p}}\tau_{\ge 0}g_*M \arrow[r, "+1"] & {}
\end{tikzcd}$$

Apply the cohomological functor ${}^{\mathfrak{p}}\mathcal{H}^{\bullet}p_*$, using the fact that ${}^{\mathfrak{p}}\mathcal{H}^{-d}p_*{}^{\mathfrak{p}}\tau_{\ge 0} = {}^{\mathfrak{p}}\mathcal{H}^{-d}p_*{}^{\mathfrak{p}}\mathcal{H}^{0}$ because of the left exactness of $p_*[-d]$.

$$\begin{tikzcd}
{}^{\mathfrak{p}}\mathcal{H}^{-d}p_*{}^{\mathfrak{p}}\tau_{\le -1}N \arrow[r] \arrow[d, "\sim"] & {}^{\mathfrak{p}}\mathcal{H}^{-d}p_*N \arrow[r] \arrow[d] & {}^{\mathfrak{p}}\mathcal{H}^{-d}p_*{}^{\mathfrak{p}}\mathcal{H}^{0}N \arrow[r] \arrow[d] & {}^{\mathfrak{p}}\mathcal{H}^{-d+1}p_*{}^{\mathfrak{p}}\tau_{\le -1}N \arrow[r] \arrow[d, "\sim"] & {}^{\mathfrak{p}}\mathcal{H}^{-d+1}p_*N \arrow[d] \\
{}^{\mathfrak{p}}\mathcal{H}^{-d}p_*{}^{\mathfrak{p}}\tau_{\le -1}g_*M \arrow[r] & {}^{\mathfrak{p}}\mathcal{H}^{-d}p_*g_*M \arrow[r] & {}^{\mathfrak{p}}\mathcal{H}^{-d}p_*{}^{\mathfrak{p}}\mathcal{H}^{0}g_*M \arrow[r] & {}^{\mathfrak{p}}\mathcal{H}^{-d+1}p_*{}^{\mathfrak{p}}\tau_{\le -1}g_*M \arrow[r] & {}^{\mathfrak{p}}\mathcal{H}^{-d+1}p_*g_*M
\end{tikzcd}$$

where $N = f'_*K'[-1]$. It is sufficient to show ${}^{\mathfrak{p}}\mathcal{H}^{l}p_*N \to {}^{\mathfrak{p}}\mathcal{H}^{l}p_*g_*M$ are isomorphisms for $l = -d$ and $-d+1$.

Lemma. *If $q : X \to Y$ is a morphism such that every fiber are projective space of dimension d and η is the first Chern class of $\mathcal{O}(1)$ and $K \in \mathsf{D}^b(Y)$, we have*

$$\sum \eta^i : \bigoplus_{i=0}^{d} K[-2i] \to q_*q^*K$$

is an isomorphism.

Proof. Passing to the fiber, we can assume X is a point. Then the result is just the property of the projective space. $\square$

Since $p_*N = f_*p_*p^*K[d-1]$ and $p_*g_*M = f_*(pi)_*(pi)^*K[d-1]$, it suffices to prove ${}^{\mathfrak{p}}\mathcal{H}^{l}(f_*p_*p^*K) \to {}^{\mathfrak{p}}\mathcal{H}^{l}(f_*(pi)_*(pi)^*K)$ is isomorphism for $l = -1$ and 0. Because p and $p \circ i$ both satisfy the condition of the lemma, we have $f_*p_*p^*K = \bigoplus_{i=0}^{d} f_*K[-2i]$ and $f_*(pi)_*(pi)^*K = \bigoplus_{i=0}^{d-1} f_*K[-2i]$. Thus it suffices to prove ${}^{\mathfrak{p}}\mathcal{H}^{l}(f_*K[-2d]) = 0$ for $l = -1$ and 0. Indeed, since every fiber of f has dimension $\le d$, we have $f^{*}{}^{\mathfrak{p}}\mathsf{D}^{\le 0}(Y) \subseteq {}^{\mathfrak{p}}\mathsf{D}^{\le d}(X)$. Using the duality, we obtain $f_*{}^{\mathfrak{p}}\mathsf{D}^{\ge 0}(X) \subseteq {}^{\mathfrak{p}}\mathsf{D}^{\ge -d}(Y)$, which shows $f_*K[-2d] \in {}^{\mathfrak{p}}\mathsf{D}^{\ge d}(Y)$. Then the above result holds for $d \ge 1$.

1.7 the defect of semismallness

For any morphism $f : X \to Y$, the defect of semismallness of f is defined by

$$r(f) = \dim X \times_Y X - \dim X.$$

If we set $Y^i = \{y \in Y : \dim f^{-1}(y) = i\}$, then $\dim X \times_Y X = \max_{i:Y^i \neq \emptyset}\{2i + \dim Y^i\}$. Caution $\dim X \geq \dim f(X)$ implies $r(f) \geq 0$. If $r(f)$ is equal to zero, we call f is semismall. If furthermore $r(f) = \max_{i:Y^i \neq \emptyset}\{2i + \dim Y^i - \dim X\}$ take the maximal value only at $i = 0$, i.e. $\dim(X \times_Y X - \Delta(X)) < \dim X$, we call f is small.

Note that if f is semismall and proper, X is nonsingular of dimension n, we have $f_*\mathbb{Q}_X[n] = {}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])$. Indeed, we just need to check the supp condition and the co-supp condition is satisfied by duality.

It suffices to prove $\dim \mathcal{H}^{n-i}(f_*\mathbb{Q}_X) \leq i$. For any $y \in Y$, $\mathcal{H}^{n-i}(f_*\mathbb{Q}_X)_y \neq 0$ is equivalent to $\mathbb{H}^{n-i}(f^{-1}(y), \mathbb{Q}_{f^{-1}(y)}) \neq 0$. It implies $2\dim f^{-1}(y) \geq n - i$, hence $y \in \cup_{2j \geq n-i} Y^j$. Since $\dim Y^j \leq n - 2j \leq i$, we obtain $\dim \mathcal{H}^{n-i}(f_*\mathbb{Q}_X) \leq \dim \cup_{2j \geq n-i} Y^j \leq i$. This proves our result.

Theorem 8. *Let f, g be the same as the previous section, then we have*

(i) If $r(f) > 0$, then $r(g) < r(f)$.

(ii) If $r(f) = 0$, then g is small.

Proof. We separate $\mathcal{Y}^i = \{(y, s) \in Y \times \mathbb{P}^\vee : \dim g^{-1}(y, s) = i\}$ into two parts and estimate their dimension separately.

Note that $g^{-1}(y, s) = f^{-1}(y) \cap X_s$, we have $\dim g^{-1}(y, s)$ is equal to $\dim f^{-1}(y)$ or $\dim f^{-1}(y) - 1$. Let $\mathcal{Y}'^i = \{(y, s) \in Y \times \mathbb{P}^\vee : \dim g^{-1}(y, s) = i = \dim f^{-1}(y)\}$ and $\mathcal{Y}''^i = \{(y, s) \in Y \times \mathbb{P}^\vee : \dim g^{-1}(y, s) = i = \dim f^{-1}(y) - 1\}$.

As for $\mathcal{Y}'^i$, note that the condition implies X_s contains a certain subvariety of dimension i, hence contains $i + 1$ points with general position. These sections forms a linear system of dimension at most $d - i - 1$. That is $\dim \mathcal{Y}'^i \leq \dim Y^i + d - i - 1$. Thus we obtain

$$2i + \dim \mathcal{Y}'^i - \dim \mathcal{X} \leq 2i + \dim Y^i + d - i - 1 - (\dim X + d - 1) \leq r(f) - i \tag{4}$$

As for $\mathcal{Y}''^i$, it is obvious that $\dim \mathcal{Y}''^i \leq \dim Y^{i+1} + d$. Thus

$$2i + \dim \mathcal{Y}''^i - \dim \mathcal{X} \leq 2i + \dim Y^{i+1} + d - (\dim X + d - 1) \leq r(f) - 1$$

Since $\mathcal{Y}^i = \mathcal{Y}'^i \cup \mathcal{Y}''^i$, we conclude that $2i + \dim \mathcal{Y}^i - \dim \mathcal{X} \leq \min\{r(f) - i, r(f) - 1\}$. So $r(g) \leq r(f)$ and equality satisfies only if (4) satisfies when $i = 0$, which suggest that $r(f) = \dim Y^0 - \dim X \leq 0$ and $Y^0 \neq \emptyset$. In this case, $r(f) = 0$ and $2i + \dim \mathcal{Y}^i - \dim \mathcal{X} = 0$ only when $i = 0$, hence g is small. $\square$

2 the proof for projective case

In this section $f : X \to Y$ is a projective morphism between two projective varieties, where X is smooth. We use induction on $r(f)$ and $\dim X$ to prove the following list of theorems, where $H_i^{n+i+j}(X) = \mathbb{H}^j(Y, {}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n]))$, η is a general hyperplane of X and $L = f^*A$ is the pullback of a general hyperplane of Y.

(i) the maps $\eta^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n]) \to {}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n])$ are isomorphisms and ${}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n])$ is semisimple for all $i \neq 0$.

(ii) the maps $\eta^i : H^j_{-i}(X) \to H^{j+2i}_i(X)$ and $L^k : H^{n+b-k}_b(X) \to H^{n+b+k}_b(X)$ are isomorphisms.

(iii) the bilinear map S_{jk} is a polarization of the (η, L)-decomposition on $H^{n-j-k}_{-j}(X)$.

(iv) the perverse sheaf ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])$ are direct sum of intersection complexes.

(v) every local system of the decomposition of ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])$ are semisimple.

Now suppose the above theorems are satisfied for $g : X' \to Y'$ where $r(g) < r(f)$ or $r(g) = r(f)$ and $\dim X' < \dim X$. we will prove the result for $f : X \to Y$.

2.1 proof of (i)

Using the results in section 1.6 with $K = \mathbb{Q}_X[n]$, we have $M = \mathbb{Q}_{\mathcal{X}}[n+d-1]$. Let $\eta' = i^*p^*\eta$. Since $r(g) < r(f)$, according to the inductive hypotheses, there are isomorphisms $\eta'^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1]) \to {}^{\mathfrak{p}}\mathcal{H}^i(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1])$ and ${}^{\mathfrak{p}}\mathcal{H}^0(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1])$ is semisimple.

Then, the morphism $\eta^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n]) \to {}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n])$ is the map below:

$$\begin{tikzcd} p^*[d]{}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n]) \arrow[r, "p^*[d](\eta^i)"] \arrow[d] & p^*[d]{}^{\mathfrak{p}}\mathcal{H}^{i}(f_*\mathbb{Q}_X[n]) \\ {}^{\mathfrak{p}}\mathcal{H}^{1-i}(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1]) \arrow[r, "\eta'^{i-1}"] & {}^{\mathfrak{p}}\mathcal{H}^{i-1}(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1]) \arrow[u] \end{tikzcd}$$

Then when $i \geq 2$, the two vertical morphism are isomorphisms, hence the top morphism is isomorphism. The fully faithfulness gives the result.

As for the case $i = 0$, we need to prove that the morphism $p^*[d]\eta : p^*[d]{}^{\mathfrak{p}}\mathcal{H}^{-1}(f_*\mathbb{Q}_X[n]) \xrightarrow{\alpha^*} {}^{\mathfrak{p}}\mathcal{H}^0(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1]) \xrightarrow{\alpha_*} p^*[d]{}^{\mathfrak{p}}\mathcal{H}^1(f_*\mathbb{Q}_X[n])$ is isomorphism, where α^* is monomorphism and α_* is epimorphism.

Since $\alpha^* \ker p^*[d]\eta \subseteq \ker \alpha_*$ and the semisimplicity of ${}^{\mathfrak{p}}\mathcal{H}^0(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1])$, we have decomposition:

$${}^{\mathfrak{p}}\mathcal{H}^0(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1]) = \alpha^* \ker p^*[d] \oplus R \oplus S$$

where S is isomorphism to $p^*[d]{}^{\mathfrak{p}}\mathcal{H}^1(f_*\mathbb{Q}_X[n])$. But $\alpha^* \ker p^*[d]\eta = \alpha^*p^*[d] \ker \eta$ is also comes from Y, The maximal of $p^*[d]{}^{\mathfrak{p}}\mathcal{H}^1(f_*\mathbb{Q}_X[n])$ shows that $\alpha^* \ker p^*[d]\eta$ is zero, which means η is monomorphism. For the same reason, we know that η is epimorphism and hence η is isomorphism.

The semisimplicity of ${}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n])$ for $i \neq 0$ comes from the morphisms with ${}^{\mathfrak{p}}\mathcal{H}^{i\pm1}(g_*\mathbb{Q}_{\mathcal{X}}[n+d-1])$ and the fully faithfulness of $p^*[d]$.

2.2 proof of (ii)

The isomorphism for $\eta^i : H^j_{-i}(X) \to H^{j+2i}_i(X)$ is trivial from (i). We only prove the second result.

Let X_1 be a smooth general hyperplane section of X. It satisfies the condition of Theorem 7 and we know ${}^{\mathfrak{p}}\mathcal{H}^{l-1}(f_*\mathbb{Q}_X[n]) \to {}^{\mathfrak{p}}\mathcal{H}^l(g_*\mathbb{Q}_{X^1}[n-1])$ is isomorphism for $l < 0$ and monomorphism for $l = 0$. Since ${}^{\mathfrak{p}}\mathcal{H}^l(g_*\mathbb{Q}_{X^1}[n-1])$ is semisimple, ${}^{\mathfrak{p}}\mathcal{H}^{l-1}(f_*\mathbb{Q}_X[n])$ is direct summand of it for $l \leq 0$. Similarly, ${}^{\mathfrak{p}}\mathcal{H}^{l+1}(f_*\mathbb{Q}_X[n])$ is direct summand of ${}^{\mathfrak{p}}\mathcal{H}^l(g_*\mathbb{Q}_{X^1}[n-1])$ for $l \geq 0$. Also, such decomposition is capable with the action of L. Hence, when $b \neq 0$, the isomorphism for L^k can always be checked in the case of X^1.

Next, we consider the case when $b = 0$ and we need to prove $L^k : H^{n-k}_0(X) \to H^{n+k}_0(X)$ is isomorphism.

Choose a general hyperplane Y_1 on Y such that $X_1 = f^{-1}(Y_1)$ is smooth, i.e. we get such diagram:

$$\begin{tikzcd} X_1 \arrow[r, hook, "i"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y_1 \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

The proper base change shows that $i^*[-1]{}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n]) = {}^{\mathfrak{p}}\mathcal{H}^0(i^*[-1]f_*\mathbb{Q}[n]) = {}^{\mathfrak{p}}\mathcal{H}^0(g_*\mathbb{Q}_{X_1}[n-1])$. According to the weak Lefschetz in the absolute case, we know $H_0^{n+j}(X) = \mathbb{H}^j(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])) \to \mathbb{H}^{j+1}(Y_1, i^*[-1]{}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])) = H_0^{n+j}(X_1)$ is isomorphism for $j \le -2$ and similarly $H_0^{n+j-2}(X_1) \to H_0^{n+j}(X)$ is isomorphism for $j \ge 2$. Thus from the commutative diagram below and the inductive hypotheses for $g : X_1 \to Y_1$ we know the result holds for $k \ge 2$.

$$\begin{tikzcd} H_0^{n-k}(X) \arrow[r, "L^k"] \arrow[d] & H_0^{n+k}(X) \\ H_0^{n-k}(X_1) \arrow[r, "L_{|X_1}^{k-1}"] & H_0^{n+k-2}(X_1) \arrow[u] \end{tikzcd}$$

Now, we only need to consider the case for $L : H_0^{n-1}(X) \to H_0^{n+1}(X)$. Verdier duality shows that these two spaces have the same rank, so it suffices to prove the injectivity.

$$\begin{tikzcd} & H_2^{n-1} \arrow[r, tail, "L"] & H_2^{n+1} \arrow[r, "\sim"] & H_2^{n+3} \\ & H_0^{n-1} \arrow[r, "L"] \arrow[ur, "\eta"'] & H_0^{n+1} \arrow[ur, "\eta"] & \\ H_{-2}^{n-3} \arrow[r, "\sim"] \arrow[ur, tail, "\eta"'] \arrow[uurr, tail, bend left, "\sim"] & H_{-2}^{n-1} \arrow[r, two heads, "L"] \arrow[ur, tail, "\eta"] \arrow[uurr, tail, bend right, "\sim"'] & H_{-2}^{n+1} & \end{tikzcd}$$

Choose $\alpha \in \mathsf{ker}(L : H_0^{n-1} \to H_0^{n+1})$. From the η-decomposition, we can write $\alpha = \sum_{j \ge 0} \eta^j \alpha_j$ where $\alpha_j \in \mathsf{ker}(\eta^{2j+1} : H_{-2j}^{n-1-2j} \to H_{2j+2}^{n+1+2j})$. Then $L\alpha_j \in \mathsf{ker}(\eta^{2j+1} : H_{-2j}^{n+1-2j} \to H_{2j+2}^{n+3+2j})$. This means that $L\alpha = \sum_{j \ge 0} \eta^j L\alpha_j$ is the η-decomposition and hence $L\alpha_j = 0$ for all j. Since we already proved the injectivity of L on H_{-2j}^{n-1-2j} for $j \le -1$, we know $\alpha_j = 0$ for $j \le -1$, which means $\alpha = \alpha_0 \in \mathsf{ker}(\eta : H_0^{n-1} \to H_2^{n+1})$.

Recall we have monomorphism $i^* : H_0^{n-1}(X) \to H_0^{n-1}(X_1)$, we only need to prove $i^*\alpha$ is zero where $i^*\alpha \in \mathsf{ker}\,\eta \cap \mathsf{ker}\,L = P_0^0(X_1)$. Since all these maps are capable with the Hodge decomposition, we can assume α is pure (p,q)-type and hence $i^*\alpha$ is also pure (p,q)-type. Then we have $\int_{X_1} i^*\alpha \wedge \overline{i^*\alpha} = \int_X L \wedge \alpha \wedge \overline{\alpha} = 0$. From the polarization of S_{00} on X_1, we get $i^*\alpha = 0$, which proves our result.

2.3 proof of (iii)

Set $P_{-j}^{-k} = \mathsf{ker}(\eta^{j+1} : H_{-j}^{n-j-k} \to H_{j+2}^{n+j+2-k}) \cap \mathsf{ker}(L^{k+1} : H_{-j}^{n-j-k} \to H_{-j}^{n-j+k+2})$. From (ii), we have the (η, L)-decomposition $H_{-j}^{n-j-k} = \oplus_{l,m \ge 0} \eta^l L^m P_{-j-2l}^{-k-2m}$.

Because S_{jk} restricting to $\eta^l L^m P_{-j-2l}^{-k-2m}$ is just $S_{j+2l,k+2m}$ restricting to P_{-j-2l}^{-k-2m}. We only need to check every polarization of S_{jk} on these P_{-j}^{-k}.

When $j > 0$, choose a general hyperplane X^1 on X. We already know $i^* : H_{-j}^{n-j-k}(X) \to H_{-j+1}^{n-j-k}(X^1)$ is injective for all j. Since i^* is commutative with η and L, we have $i^* P_{-j}^{-k}(X) \subseteq P_{-j+1}^{-k}(X_1)$ and S_{jk} restricting to P_{-j}^{-k} is $S_{j-1,k}$ restricting to $i^* P_{-j}^{-k}(X)$ and is polarized according to the induction hypotheses.

When $j = 0$ and $k > 0$, choose a general hyperplane Y_1 on Y and $X_1 = f^{-1}(Y_1)$ is smooth. We already know $i^* : H_0^{n-k}(X) \to H_0^{n-k}(X_1)$ is injective for $k \geq 1$. Since i^* is commutative with η and L, we have $i^* P_0^{-k}(X) \subseteq P_0^{-k+1}(X_1)$ and similarly, the result holds according to the induction hypotheses.

Finally, we prove the polarization of S_0 on P_0^0. We can pass to the real coefficient.

Let $\Lambda_\varepsilon = \mathsf{ker}(\varepsilon\eta + L) \subseteq \mathsf{H}^n(X)$. When ε is sufficient small, $\eta + \dfrac{1}{\varepsilon} L$ is ample and the classical Hodge-Lefschetz theorem holds. In particular, $\mathsf{dim}\,\Lambda_\varepsilon = b_n - b_{n-2}$, where b_m is the Betti number of X. Let $\Lambda = \lim\limits_{\varepsilon \to 0} \Lambda_\varepsilon$ where the limit is taken in the subspace of $H^n(X)$, then $\mathsf{dim}\,\Lambda = b_n - b_{n-2}$.

For any $u \in \Lambda$, assume $u = \lim\limits_{\varepsilon \to 0} u_\varepsilon$, where $Lu_\varepsilon = -\varepsilon\eta u_\varepsilon$, we have $Lu = \lim\limits_{\varepsilon \to 0} \varepsilon\eta u_\varepsilon = 0$, so $\Lambda \subseteq \mathsf{ker}\, L$.

$$\begin{tikzcd}
 & & H_2^{n-2} \arrow[r, "L"] & H_2^n \arrow[r, "L"] \arrow[rr, bend left, "\sim"] & H_2^{n+2} \arrow[r, "L"] & H_2^{n+4} \\
 & H_0^{n-4} \arrow[r, "L"] \arrow[ur, "\eta"] & H_0^{n-2} \arrow[r, "L"] \arrow[ur, "\eta"] \arrow[rr, bend left, "\sim"] & H_0^n \arrow[r, "L"] \arrow[ur, "\eta"] & H_0^{n+2} \arrow[ur, "\eta"] & \\
H_{-2}^{n-6} \arrow[r, "L"] \arrow[ur, "\eta"] \arrow[rrrr, bend right, "\sim"'] & H_{-2}^{n-4} \arrow[r, "L"] \arrow[ur, "\eta"] \arrow[rr, bend right, "\sim"'] \arrow[uurr, bend left, "\sim"] & H_{-2}^{n-2} \arrow[r, "L"] \arrow[ur, "\eta"] & H_{-2}^n \arrow[r, "L"] \arrow[ur, "\eta"] \arrow[uurr, bend right, "\sim"'] & H_{-2}^{n+2} &
\end{tikzcd}$$

Lemma. *Let L_r^k denote the morphism $L^k : \mathsf{H}^{n-r}(X) \to \mathsf{H}^{n-r+2k}(X)$. We have the equality*

$$\eta\,\mathsf{ker}\, L_2^1 \cap (\eta\,\mathsf{ker}\, L_2^1)^\perp \cap (\eta^2\,\mathsf{ker}\, L_4^2)^\perp \cap \cdots \cap (\eta^i\,\mathsf{ker}\, L_{2i}^i)^\perp = \eta\,\mathsf{ker}\, L_2^1 \cap H_{\leq -i}^n(X).$$

Proof. From (ii), we know $\mathsf{ker}\, L_{2i}^i \subseteq H_{\leq -i-1}^{n-2i}(X)$ and hence $\eta\,\mathsf{ker}\, L_2^1 \subseteq H_{\leq 0}^n(X)$. This is the result for $i = 0$. Now we use induction on i, which means we need to prove for all $i \geq 0$

$$\eta\,\mathsf{ker}\, L_2^1 \cap H_{\leq -i}^n(X) \cap (\eta^{i+1}\,\mathsf{ker}\, L_{2i+2}^{i+1})^\perp = \eta\,\mathsf{ker}\, L_2^1 \cap H_{\leq -i-1}^n(X).$$

Let $\alpha = \eta\lambda \in \eta\,\mathsf{ker}\, L_2^1 \cap H_{\leq -i}^n(X)$ where $L\lambda = 0$. Then $\lambda \in H_{\leq -2}^{n-2}(X)$. The injectivity of $\eta : H_{-j-2}^{n-2} \to H_{-j}^n$ for $j \geq 0$ shows that $\lambda \in H_{\leq -i-2}^{n-2}$. The isomorphism for $L^j : H_{-j-2}^{n-2j-2} \to H_{-j-2}^{n-2}$ shows that we can write $\lambda = L^i \lambda'$ where $\lambda' \in H_{\leq -i-2}^{n-2i-2}$.

So we can write $\alpha = \eta L^i \lambda'$ where $L^{i+1}\lambda' = 0$. Then $\alpha \in (\eta^{i+1}\,\mathsf{ker}\, L_{2i+2}^{i+1})^\perp$ shows that $S(\alpha, \eta^{i+1}\beta) = 0$ for all $\beta \in \mathsf{ker}\, L_{2i+2}^{i+1}$. This is just $S_{i+2,i}$ restricting to $H_{\leq -i-2}^{n-2i-2}$. The polarization of $S_{i+2,i}$ on H_{-i-2}^{n-2i-2} implies that the component of λ' in H_{-i-2}^{n-2i-2} is zero, i.e. $\lambda' \in H_{\leq -i-3}^{n-2i-2}$. Thus $\alpha \in H_{\leq -i-1}^n$. □

Lemma. $\Lambda = \mathsf{ker}\, L_0^1 \cap \left(\bigcap_{i \geq 1} (\eta^i\,\mathsf{ker}\, L_{2i}^i)^\perp\right)$ *and* $\mathsf{ker}\, L_0^1 = \Lambda \oplus \eta\,\mathsf{ker}\, L_2^1$ *is orthogonal decomposition.*

Proof. First, we prove $\Lambda_\varepsilon \subseteq (\eta^i\,\mathsf{ker}\, L_{2i}^i)^\perp$ for all $i \geq 1$. In fact, for any $u_\varepsilon \in \Lambda_\varepsilon$ and $\lambda \in \mathsf{ker}\, L_{2i}^i$, we have $\eta u_\varepsilon = -\dfrac{1}{\varepsilon} L u_\varepsilon$ and hence $\int u_\varepsilon \wedge \eta^i \lambda = \int (-\dfrac{1}{\varepsilon})^i L^i u_\varepsilon \wedge \lambda = 0$. This shows that $\Lambda \subseteq \left(\bigcap_{i \geq 1} (\eta^i\,\mathsf{ker}\, L_{2i}^i)^\perp\right)$.

As for the second statement, we note from the previous lemma that $\Lambda \cap \eta\,\mathsf{ker}\, L_2^1 = 0$. Then we have a inclusion $\Lambda \oplus \eta\,\mathsf{ker}\, L_2^1 \subseteq \mathsf{ker}\, L_0^1$. Now, we count the dimension of them.

From the injectivity and surjectivity of L, we have:

$$\dim \ker L_0^1 = \sum_{i\le 0} \dim H_i^n - \dim H_i^{n+2}$$
$$\dim \ker L_2^1 = \sum_{i\le -2} \dim H_i^{n-2} - \dim H_i^n$$

Then use the isomorphisms and $H_{-1}^{n-1} \xrightarrow{\sim} H_{-1}^{n+1}$, we obtain

$$\begin{aligned}\dim \ker L_0^1 - \dim \eta \ker L_2^1 &= \left(\sum_{i\ge 0} \dim H_i^n - \dim H_i^{n-2}\right) - \left(\sum_{i\le -2} \dim H_i^{n-2} - \dim H_i^n\right)\\ &= \sum \dim H_i^n - \sum \dim H_i^{n-2} = b_n - b_{n-2} = \dim \Lambda\end{aligned}$$

This proves the second equality. As for the first one, we only need to note that $\Lambda \oplus \eta \ker L_2^1 \subseteq (\ker L_0^1 \cap \left(\bigcap_{i\ge 1} (\eta^i \ker L_{2i}^i)^\perp\right)) \oplus \eta \ker L_2^1 \subseteq \ker L_0^1$ and the equality has to be hold. $\square$

Consider the integral restricting to $\ker L_0^1$, for any $b \in H_{\le -1}^n$, we can find a $b' \in H_{\le -1}^{n-2}$ such that $b = Lb'$, then $\int a \wedge b = 0$ for all $a \in \ker L_0^1$. So $\ker L_0^1 \cap H_{\le -1}^n$ is in the radical of S restricting to $\ker L_0^1$. Thus we have the orthogonal decomposition $\ker L_0^1/(\ker L_0^1 \cap H_{\le -1}^n) = \Lambda/(\Lambda \cap H_{\le 1}^n) \oplus \eta \ker L_2^1/(\eta \ker L_2^1 \cap H_{-1}^n)$.

Since $P_0^0 \subseteq \ker \eta$, we have $P_0^0 \subseteq (\eta \ker L_2^1)^\perp/((\eta \ker L_2^1)^\perp \cap H_{\le -1}^n) = \Lambda/(\Lambda \cap H_{\le 1}^n)$. From the classical Hodge theory shows that S_{00} is polarized on every Λ_ε and hence is semipositive on Λ. On the another hand, we know S_{00} is nondegenerate on the summand P_0^0 of the (η, L)-decomposition, so it has to be positive, i.e. it is a polarization on P_0^0.

2.4 proof of (iv)

Let S_s denote the stratum of Y of dimension s and $S_s \xrightarrow{\alpha_s} U_s \xleftarrow{\beta_s} U_{s+1}$ denote the corresponding inclusions.

We would like to prove the splitting

$${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])|_{U_s} = \beta_{s!*}({}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])|_{U_{s+1}}) \oplus \mathcal{H}^{-s}({}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])|_{U_s})[s],$$

then we can get the decomposition:

$${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n]) = \bigoplus_s IC_{\overline{S_s}}(\mathcal{H}^{-s}({}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n]))|_{s_s}).$$

When $s = \dim f(X)$, $U_{s+1} = \emptyset$, there is nothing to prove.

When $0 < s < \dim f(X)$, choose s general hyperplanes on Y such they intersect transversally and $X_s = f^{-1}(Y_s)$ is smooth and Y_s denotes the intersection of them which intersects S_s the isolated finite points T. Apply the inductive hypotheses to $f_s : X_s \to Y_s$, we know ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])|_{Y_s}[-s] = {}^{\mathfrak{p}}\mathcal{H}^0(f_{s*}\mathbb{Q}_{X_s}[n-s])$ is semisimple. Then the splitting criteria for ${}^{\mathfrak{p}}\mathcal{H}^0(f_{s*}\mathbb{Q}_{X_s}[n-s])$ is satisfied for the points in T. Since every point $y \in S_s$ has neighborhood in Y homeomorphic to $\mathbb{C}^s \times N$, the splitting criteria for ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])|_{U_s}$ is satisfied for the points in S_s.

It remains to prove the case $s = 0$. That means, $\mathcal{H}^0(\alpha_{0!}\alpha_0^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])) \to \mathcal{H}^0({}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n]))$ is isomorphism. The support of them are isolated points, so we can check it at every point $y \in S_0$, i.e. the induced map $H_{n,0}^{\mathrm{BM}}(f^{-1}(y)) \to H_0^n(f^{-1}(y))$.

Lemma. *Let U be an affine variety, $K \in {}^{\mathfrak{p}}\mathsf{D}^{\leq 0}(U)$, $\alpha : T \to U$ be the inclusion of $\mathsf{supp}\,\mathcal{H}^0(K)$. Then the map $\mathbb{H}^0(U, K) \to \mathbb{H}^0(U, \alpha_*\alpha^*K)$ is surjective.*

Proof. Consider the two spectral sequences

$$I_2^{pq} = \mathbb{H}^p(U, \mathcal{H}^q(K)) \Rightarrow \mathbb{H}^{p+q}(U, K)$$

$$II_2^{pq} = \mathbb{H}^p(U, \mathcal{H}^q(\alpha_*\alpha^*K)) \Rightarrow \mathbb{H}^{p+q}(U, \alpha_*\alpha^*K).$$

We know $\mathsf{dim\,supp}\,\mathcal{H}^q(K) \leq -q$, Artin vanishing theorem shows that $\mathbb{H}^p(U, \mathcal{H}^q(K)) = 0$ if $p \geq -q$. This implies $I_2^{pq} = I_\infty^{pq}$ since all $d_r^{00} : I_r^{00} \to I_r^{r,-r+1}$ are zero.

On the other hand, $\mathsf{dim\,supp}\,\mathcal{H}^q(\alpha_*\alpha^*K) = 0$ for all $q \leq 0$, which means $II_2^{pq} = 0$ if $p \neq 0$.

From the fact that $\mathcal{H}^0(K) = \mathcal{H}^0(\alpha_*\alpha^*K)$ we know the morphism $\mathbb{H}^0(U, K) \to I_\infty^{00} = I_2^{00} = II_2^{00} = \mathbb{H}^0(U, \alpha_*\alpha^*K)$ is surjective. □

Corollary 9. *Let K denote ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])$ and $\alpha : T \to Y$ is the inclusion of $\mathsf{supp}\,K$. Then the map $H_0^n(X) = \mathbb{H}^0(Y, K) \to \mathbb{H}^0(Y, \alpha_*\alpha^*K) = \bigoplus_{y\in T} H_0^n(f^{-1}(y))$ is surjective.*

Proof. Choose an affine open subset U of Y such that it covers T and $U' = f^{-1}(U)$ is smooth. Hodge III Proposition 8.2.6 shows that $\mathsf{H}^n(X)$ and $\mathsf{H}^n(U')$ have same image in $\mathsf{H}^n(f^{-1}(y))$. Apply to the perverse filtration, we get $H_0^n(X)$ and $H_0^n(U')$ have same image in $H_0^n(f^{-1}(y))$. The previous lemma shows the latter one is surjective, so the former one is also surjective. □

Apply to the dual case, we obtain $H_{n,0}^{\mathrm{BM}}(f^{-1}(y)) \to H_0^n(X)$ is injective.

Return to the problem, we need to show the morphism

$$\begin{tikzcd}
H_{n,0}^{\mathrm{BM}}(f^{-1}(y)) \arrow[r, equal] & \mathcal{H}^0(\alpha_!\alpha^{!\,\mathfrak{p}}\mathcal{H}^0(f_*\omega_X[-n]))_y \arrow[d, rightarrowtail] & \mathcal{H}^0(\alpha_*\alpha^{*\,\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n]))_y \arrow[r, equal] & H_0^n(f^{-1}(y)) \\
 & \mathcal{H}^0({}^{\mathfrak{p}}\mathcal{H}^0(f_*\omega_X[-n]))_y \arrow[r, "\sim"] & \mathcal{H}^0({}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n]))_y \arrow[u, twoheadrightarrow] &
\end{tikzcd}$$

is isomorphism. This comes from the fact that S_{00} is a polarization of $H_0^n(X)$ and $H_0^n(f^{-1}(y))$ is a Hodge substructure of it, which implies S_{00} is nondegenerate on $H_0^n(f^{-1}(y))$.

2.5 proof of (v)

We need to prove the local system $\mathcal{H}^{-s}({}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n]))|_{S_s}$ is semisimple.

Lemma. *Let*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\Phi"] \arrow[dr, "F"'] & \mathcal{Y} \arrow[d, shift right, "\pi"'] \\
 & T \arrow[u, shift right, "\theta"']
\end{tikzcd}$$

be projective maps of quasi-projective varieties such that

(i) $\mathcal{X}$ is nonsingular of dimension n, T is nonsingular of dimension s;

(ii) $F = \pi \circ \Phi$ is surjective of relative dimension $n - s$;

(iii) every strata of $\mathcal{Y}$ map smoothly and surjectively onto T;

(iv) θ *is a section of* π*, i.e.* $\pi \circ \theta = \mathsf{id}$*.*

(v) there is an isomorphism $\Phi_*\mathbb{Q}_{\mathcal{X}}[n] = \sum_l {}^{\mathfrak{p}}\mathcal{H}^l(\Phi_*\mathbb{Q}_{\mathcal{X}}[n])[-l]$*.*

Then there is a surjective map of local systems on T*:*

$$R^{n-s}F_*\mathbb{Q}_{\mathcal{X}} \to \mathcal{H}^{-s}(\theta^{*\mathfrak{p}}\mathcal{H}^0(\Phi_*\mathbb{Q}_{\mathcal{X}}[n])).$$

Furthermore, the local system $R^{n-s}F_*\mathbb{Q}_{\mathcal{X}}$ *being semisimple indicates that the latter is also semisimple.*

Proof. The map is induced by $\pi_* \to \pi_*\theta_*\theta^*$ acting to $\Phi_*\mathbb{Q}_{\mathcal{X}}[n]$ and taking the cohomology. The result is local on T, so we can check it at every point $t \in T$, for which we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{X}_t \arrow[r, "\Phi_t"] \arrow[d] & \mathcal{Y}_t \arrow[r, shift left, "\pi_t"] \arrow[d, "j"] & \{t\} \arrow[l, shift left, "\theta_t"] \arrow[d, "i"] \\
\mathcal{X} \arrow[r, "\Phi"] & \mathcal{Y} \arrow[r, shift left, "\pi"] & T \arrow[l, shift left, "\theta"]
\end{tikzcd}$$

Then we find $(R^{n-s}F_*\mathbb{Q}_{\mathcal{X}})_y = H^{n-s}(\mathcal{X}_t)$. On the other hand, the embedding $j : \mathcal{Y}_t \to \mathcal{Y}$ is transversal to all the strata of Y and hence $j^*[-s]$ is t-exact. Then we have $\mathcal{H}^{-s}(\theta^{*\mathfrak{p}}\mathcal{H}^0(\Phi_*\mathbb{Q}_{\mathcal{X}}[n]))_y = \mathcal{H}^{-s}(i^*\theta^{*\mathfrak{p}}\mathcal{H}^0(\Phi_*\mathbb{Q}_{\mathcal{X}}[n])) = \mathcal{H}^{-s}(\theta_t^{*\mathfrak{p}}\mathcal{H}^0(\Phi_{t*}\mathbb{Q}_{\mathcal{X}_t}[n-s])[s]) = \mathcal{H}^0({}^{\mathfrak{p}}\mathcal{H}^0(\Phi_{t*}\mathbb{Q}_{\mathcal{X}_t}[n-s]))_{\theta_t(t)}$, which is just $H_0^{n-s}(\Phi_t^{-1}(\theta_t(t)))$. so the induced map at t is just $H^{n-s}(\mathcal{X}_t) \to H_0^{n-s}(\Phi_t^{-1}(\theta_t(t)))$, which is surjective in the sense of the corollary in the previous subsection. □

As for the local system on S_s, we only need to prove it is semisimple on a Zariski-dense open subset T of it.

Let $Y \hookrightarrow \mathbb{P}$ and $\Pi = (\mathbb{P}^\vee)^s$ which means s hyperplanes of $\mathbb{P}$. So we can set $\mathcal{Y} = \{(y, H_1, \cdots, H_s) : y \in \cap_{i=1}^s H_i\} \subseteq Y \times \Pi$. Then for any inclusion $T \to Y$, there is a Cartesian diagram:

$$\begin{tikzcd}
\mathcal{X}_T \arrow[r] \arrow[d] & \mathcal{X} \arrow[r] \arrow[d] & X \times \Pi \arrow[r, "p"] \arrow[d] & X \arrow[d, "f"] \\
\mathcal{Y}_T \arrow[r] \arrow[d] & \mathcal{Y} \arrow[r] \arrow[d] & Y \times \Pi \arrow[r, "p"] & Y \\
T \arrow[r] & Y & &
\end{tikzcd}$$

From the condition, we know there is a Zariski-dense open subset $\Pi^0 \subseteq \Pi$ such that

(i) the surjective map $\mathcal{X} \to \Pi$ is smooth over Π^0;

(ii) the complete intersection Y_s of s hyperplanes that associated with the points of Π^0 meet all strata of Y transversally;

(iii) the restriction of $\mathcal{Y} \to \Pi$ over Π^0 is stratified so that every stratum maps surjectively and smoothly to Π^0.

Since general s hyperplanes intersect with $S = S_s$ a non-empty and finite set, the map $b : \mathcal{Y}_S \to \Pi$ is dominant, so $b^{-1}(\Pi^0)$ is Zariski-dense open in $\mathcal{Y}_S$.

Since $\mathcal{Y}_S \to S$ is Zariski-locally trivial, there exists a Zariski-dense open subset $T \subseteq S$ such that $\mathcal{Y}_T \to T$ admits a section $\mu : T \to \mathcal{Y}_T$ with the property that $\mu(T) \subseteq b^{-1}\Pi^0$. By shrinking T, we may assume the map $b \circ \mu : T \to \Pi^0$ is smooth.

The map $b \circ \mu : T \to \Pi$ and the inclusion $T \to Y$ inherit a morphism $T \to \mathcal{Y}$, which gives the morphism θ from T to the fiber product $\mathcal{Y}_T$. It gives the commutative diagram:

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "p"] \arrow[d, "\Phi"] & & X \arrow[d, "f"] \\
\mathcal{Y} \arrow[rr, "p"] \arrow[dr, shift left, "\pi"] & & Y \\
& T \arrow[ul, shift left, "\theta"] \arrow[ur, "\alpha"] &
\end{tikzcd}$$

For p is the composition of a normal inclusion and a projection, we know p^* is t-exact and hence $\Phi_*\mathbb{Q}_{\mathcal{X}_T}[n]$ splits together with the isomorphism $p^{*\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n]) = {}^{\mathfrak{p}}\mathcal{H}^0(\Phi_*\mathbb{Q}_{\mathcal{X}_T}[n])$. Apply the lemma, we deduce the semisimplicity of

$$\mathcal{H}^{-s}(\theta^{*\mathfrak{p}}\mathcal{H}^0(\Phi_*\mathbb{Q}_{\mathcal{X}_T}[n])) = \mathcal{H}^{-s}(\theta^* p^{*\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])) = \mathcal{H}^{-s}(\alpha^{*\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])).$$

This is what we need.

3 the algebraic case

Lemma. *The decomposition theorem remains true when f is a projective morphism between two quasi-projective varieties and $\mathbb{Q}_X[n]$ is replaced by IC_X.*

Proof. Take a projective compactification of f to be $\bar{f} : \bar{X} \to \bar{Y}$.

Choose a projective resolution of singularities of $\bar{X}$, which is a morphism $p : X' \to \bar{X}$ such that X' is smooth. According to the theorem, we know $IC_{\bar{X}}$ is a direct summand of $p_*\mathbb{Q}_{X'}$. and $IC_{\bar{X}}|_X = IC_X$. Then applying the theorem to the map $f \circ p : X' \to Y$, we get ${}^{\mathfrak{p}}\mathcal{H}^i(f_*p_*\mathbb{Q}_{X'})$ is semisimple and $f_*p_*\mathbb{Q}_{X'} = \oplus {}^{\mathfrak{p}}\mathcal{H}^i(f_*p_*\mathbb{Q}_{X'})$. This shows that ${}^{\mathfrak{p}}\mathcal{H}^i(f_*IC_{\bar{X}})$ is semisimple and $f_*IC_{\bar{X}} = \oplus {}^{\mathfrak{p}}\mathcal{H}^i(f_*IC_{\bar{X}})$ since direct sum commutes with perverse cohomology. Hence the restriction to Y, which is ${}^{\mathfrak{p}}\mathcal{H}^i(f_*IC_X)$ is semisimple and $f_*IC_X = \oplus {}^{\mathfrak{p}}\mathcal{H}^i(f_*IC_X)$. □

Theorem 10. *The decomposition theorem remains true when f is a proper morphism between two varieties and $\mathbb{Q}_X[n]$ is replaced by IC_X.*

Proof. Since the results for sheaves on Y hold if they hold when restricted to Zariski open covers of Y, we can assume Y is quasi-projective.

Applying Chow lemma, there is a projective morphism $p : X' \to X$ such that X' is quasi-projective and p is birational. Using the same method as the lemma above, since $f \circ p$ is also projective, we get ${}^{\mathfrak{p}}\mathcal{H}^i(f_*IC_X)$ is semisimple. □

Theorem 11. *The relative hard Lefschetz theorem remains true when f is a projective morphism and η is an f-ample line bundle.*

Proof. Since the results for sheaves on Y hold if they hold when restricted to Zariski open covers of Y, we can assume Y is quasi-projective. Thus f is a projective morphism between two quasi-projective varieties. Now the prove of (i) remains true thanks to (3) when $\mathbb{Q}_X[n]$ and $\mathbb{Q}_{\mathcal{X}}[n+d-1]$ are replaced by IC_X and $IC_{\mathcal{X}}$, where the semisimplicity holds a priori due to the lemma. This shows the hard Lefschetz theorem holds. $\square$