multiplicity

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1 the simple one

Recall the definition of cup product:

$$\cup : \mathsf{H}^l(X,\mathbb{Q}) \times \mathsf{H}^m(X,\mathbb{Q}) \to \mathsf{H}^{l+m}(X\times X, \mathbb{Q}\otimes\mathbb{Q}) \xrightarrow{\Delta^*} \mathsf{H}^{l+m}(X,\mathbb{Q}).$$

For the proper morphism between algebraic varieties $f : X \to Y$ where X is smooth projective of dimension n. Decomposition theorem gives the filtration of $\mathsf{H}^l(X,\mathbb{Q}) = \mathbb{H}^l(Y, f_*\mathbb{Q}_X)$ into perverse cohomology:

$$\mathsf{H}^{n+l}_{\le a}(X) = \mathsf{im}(\mathbb{H}^l(Y, {}^{\mathfrak{p}}\tau_{\le a}(f_*\mathbb{Q}_X[n])) \to \mathbb{H}^l(Y, f_*\mathbb{Q}_X[n]))$$

by ${}^{\mathfrak{p}}\tau_{\le a} \to {}^{\mathfrak{p}}\tau_{\le a+1}$.

Besides, we have natural morphism

$$\begin{aligned}
&\mathbb{H}^l(Y, {}^{\mathfrak{p}}\tau_{\le a}(f_*\mathbb{Q}_X[n])) \times \mathbb{H}^m(Y, {}^{\mathfrak{p}}\tau_{\le b}(f_*\mathbb{Q}_X[n]))\\
&= \mathsf{Hom}_{\mathsf{D}(Y)}(\mathbb{Q}_Y, {}^{\mathfrak{p}}\tau_{\le a}(f_*\mathbb{Q}_X[n])[l]) \times \mathsf{Hom}_{\mathsf{D}(Y)}(\mathbb{Q}_Y, {}^{\mathfrak{p}}\tau_{\le b}(f_*\mathbb{Q}_X[n])[m])\\
&\to \mathsf{Hom}_{\mathsf{D}(Y\times Y)}(\mathbb{Q}_Y \boxtimes \mathbb{Q}_Y, {}^{\mathfrak{p}}\tau_{\le a}(f_*\mathbb{Q}_X[n])[l] \boxtimes {}^{\mathfrak{p}}\tau_{\le b}(f_*\mathbb{Q}_X[n])[m])\\
&= \mathsf{Hom}_{\mathsf{D}(Y\times Y)}(\mathbb{Q}_{Y\times Y}, {}^{\mathfrak{p}}\tau_{\le a}(f_*\mathbb{Q}_X[n]) \boxtimes {}^{\mathfrak{p}}\tau_{\le b}(f_*\mathbb{Q}_X[n])[l+m])
\end{aligned}$$

From $\boxtimes : {}^{\mathfrak{p}}\mathsf{D}^{\le a}(Y) \times {}^{\mathfrak{p}}\mathsf{D}^{\le b}(Y) \to {}^{\mathfrak{p}}\mathsf{D}^{\le a+b}(Y\times Y)$, we obtain the morphism

$${}^{\mathfrak{p}}\tau_{\le a}(f_*\mathbb{Q}_X[n]) \boxtimes {}^{\mathfrak{p}}\tau_{\le b}(f_*\mathbb{Q}_X[n]) \to f_*\mathbb{Q}_X[n] \boxtimes f_*\mathbb{Q}_X[n]$$

passes through ${}^{\mathfrak{p}}\tau_{\le a+b}(f_*\mathbb{Q}_X[n] \boxtimes f_*\mathbb{Q}_X[n])$.

Note that f is proper, we have $f_*\mathbb{Q}_X[n] \boxtimes f_*\mathbb{Q}_X[n] = (f\times f)_*\mathbb{Q}_{X\times X}[2n]$ and hece a commutative diagram

$$\begin{tikzcd}
\mathbb{H}^l(Y, {}^{\mathfrak{p}}\tau_{\le a}(f_*\mathbb{Q}_X[n])) \times \mathbb{H}^m(Y, {}^{\mathfrak{p}}\tau_{\le b}(f_*\mathbb{Q}_X[n])) \arrow[r] \arrow[d] & \mathsf{Hom}_{\mathsf{D}(Y\times Y)}(\mathbb{Q}_{Y\times Y}, {}^{\mathfrak{p}}\tau_{\le a}(f_*\mathbb{Q}_X[n]) \boxtimes {}^{\mathfrak{p}}\tau_{\le b}(f_*\mathbb{Q}_X[n])[l+m]) \arrow[d] \\
\mathbb{H}^l(Y, f_*\mathbb{Q}_X[n]) \times \mathbb{H}^m(Y, f_*\mathbb{Q}_X[n]) \arrow[d] & \mathsf{Hom}_{\mathsf{D}(Y\times Y)}(\mathbb{Q}_{Y\times Y}, {}^{\mathfrak{p}}\tau_{\le a+b}((f\times f)_*\mathbb{Q}_{X\times X}[2n])[l+m]) \arrow[d] \\
\mathbb{H}^{l+m}(Y\times Y, (f\times f)_*\mathbb{Q}_{X\times X}[2n]) \arrow[r, equal] & \mathsf{Hom}_{\mathsf{D}(Y\times Y)}(\mathbb{Q}_{Y\times Y}, (f\times f)_*\mathbb{Q}_{X\times X}[2n][l+m])
\end{tikzcd}$$

i.e., there is a morphism $\mathsf{H}^{n+l}_{\le a}(X) \times \mathsf{H}^{n+m}_{\le b}(X) \to \mathsf{H}^{2n+l+m}_{\le a+b}(X\times X)$.

Let Δ_X and Δ_Y denote the diagonal map $X \to X \times X$ and $Y \to Y \times Y$, respectively. Then the cup product given by $\mathbb{Q}_{X\times X} \to \Delta_{X*}\Delta_X^*\mathbb{Q}_{X\times X} = \mathbb{Q}_X$ gives the morphism:

$$
\begin{aligned}
&\mathbb{H}^{l+m}(Y \times Y, {}^{\mathfrak{p}}\tau_{\leq a+b}(f \times f)_*\mathbb{Q}_{X\times X}[2n]) \\
\to&\mathbb{H}^{l+m}(Y \times Y, {}^{\mathfrak{p}}\tau_{\leq a+b}(f \times f)_*\Delta_{X*}\mathbb{Q}_X[2n]) \\
=&\mathbb{H}^{l+m}(Y \times Y, {}^{\mathfrak{p}}\tau_{\leq a+b}\Delta_{Y*}f_*\mathbb{Q}_X[2n]) \\
=&\mathbb{H}^{l+m}(Y \times Y, \Delta_{Y*}{}^{\mathfrak{p}}\tau_{\leq a+b}(f_*\mathbb{Q}[n][n])) \\
=&\mathbb{H}^{l+m}(Y, {}^{\mathfrak{p}}\tau_{\leq a+b+n}(f_*\mathbb{Q}[n])[n]) \\
=&\mathbb{H}^{n+l+m}(Y, {}^{\mathfrak{p}}\tau_{\leq a+b+n}f_*\mathbb{Q}[n])
\end{aligned}
$$

which gives $\mathsf{H}^{2n+l+m}_{\leq a+b}(X \times X) \to \mathsf{H}^{2n+l+m}_{\leq a+b+n}(X)$.

In conclusion, we obtain the cup product $\mathsf{H}^{n+l}_{\leq a}(X) \times \mathsf{H}^{n+m}_{\leq b}(X) \to \mathsf{H}^{2n+l+m}_{\leq a+b+n}(X)$.

Now consider an example when f is the map $X = Bl_{\mathsf{pt}}\mathbb{P}^3 \to \mathbb{P}^3 = Y$ with $E = f^{-1}(\mathsf{pt}) = \mathbb{P}^2$. Then

$$
{}^{\mathfrak{p}}\mathcal{H}^{-1}(f_*\mathbb{Q}_X[3]) = (\mathsf{H}_4(E))_{\mathsf{pt}}, \quad {}^{\mathfrak{p}}\mathcal{H}^{0}(f_*\mathbb{Q}_X[3]) = \mathbb{Q}_Y[3], \quad {}^{\mathfrak{p}}\mathcal{H}^{1}(f_*\mathbb{Q}_X[3]) = (\mathsf{H}^4(E))_{\mathsf{pt}}
$$

and

$$
\begin{tikzcd}
 & & \mathsf{H}^4_1 = \mathsf{H}^4(E) = \mathbb{Q} & \\
\mathsf{H}^0_0 = \mathsf{H}^0(Y) = \mathbb{Q} \arrow[r, "L"] & \mathsf{H}^2_0 = \mathsf{H}^2(Y) = \mathbb{Q} \arrow[r, "L"] & \mathsf{H}^4_0 = \mathsf{H}^4(Y) = \mathbb{Q} \arrow[r, "L"] & \mathsf{H}^6_0 = \mathsf{H}^6(Y) = \mathbb{Q} \\
 & \mathsf{H}^2_{-1}(X) = \mathsf{H}_4(E) = \mathbb{Q} \arrow[uur, bend left, "\eta"] & &
\end{tikzcd}
$$

where L is a hyperplane in Y and η is a hyperplane in X.

In this case, we have the cup product of $\mathsf{H}^2_{-1}(X)$ and itself lies in $\mathsf{H}^4_1(X)$.

This is the case when $a = b = -1$ and $n = 3$, which shows in general this bound is tight.

In the case when $f : X \to Y$ is smooth morphism between two smooth varieties, we have ${}^{\mathfrak{p}}\tau_{\leq 0}f_*\mathbb{Q}_X = \tau_{\leq -\dim Y}f_*\mathbb{Q}_X$ and then

$$
\begin{aligned}
&\mathbb{H}^{l}(Y, {}^{\mathfrak{p}}\tau_{\leq a}(f_*\mathbb{Q}_X[\dim Y])) \times \mathbb{H}^{m}(Y, {}^{\mathfrak{p}}\tau_{\leq b}(f_*\mathbb{Q}_X[\dim Y])) \\
\to&\mathbb{H}^{l+m}(Y \times Y, {}^{\mathfrak{p}}\tau_{\leq a}(f_*\mathbb{Q}_X[\dim Y]) \boxtimes {}^{\mathfrak{p}}\tau_{\leq b}(f_*\mathbb{Q}_X[\dim Y])) \\
=& \mathbb{H}^{l+m}(Y \times Y, \tau_{\leq a-\dim Y}(f_*\mathbb{Q}_X[\dim Y]) \boxtimes \tau_{\leq b-\dim Y}(f_*\mathbb{Q}_X[\dim Y])) \\
=& \mathbb{H}^{l+m}(Y \times Y, \tau_{\leq a}(f_*\mathbb{Q}_X)[\dim Y] \boxtimes \tau_{\leq b}(f_*\mathbb{Q}_X)[\dim Y]) \\
\to&\mathbb{H}^{l+m}(Y \times Y, (\tau_{\leq a+b}(f \times f)_*\mathbb{Q}_{X\times X})[2\dim Y]) \\
\to&\mathbb{H}^{l+m+\dim Y}(Y \times Y, (\tau_{\leq a+b}(f \times f)_*\Delta_{X*}\mathbb{Q}_X)[\dim Y]) \\
=& \mathbb{H}^{l+m+\dim Y}(Y \times Y, \Delta_{Y*}(\tau_{\leq a+b}f_*\mathbb{Q}_X)[\dim Y]) \\
=& \mathbb{H}^{l+m+\dim Y}(Y, {}^{\mathfrak{p}}\tau_{\leq a+b}(f_*\mathbb{Q}_X[\dim Y]))
\end{aligned}
$$

This is stronger than previous result.

2 the complicate one

We use another way to calculate the cup product.

$$\begin{tikzcd}
X \arrow[dr, "\Delta_f"] \arrow[drr, "\Delta_X"] \arrow[ddr, "f"'] & & \\
& X \times_Y X \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & X \times X \arrow[d, "f \times f"] \\
& Y \arrow[r, "\Delta_Y"] & Y \times Y
\end{tikzcd}$$

We have the cup product on Y at first, which gives

$$\mathbb{H}^l(Y, {}^{\mathfrak{p}}\tau_{\leq a}(f_*\mathbb{Q}_X[n])) \times \mathbb{H}^m(Y, {}^{\mathfrak{p}}\tau_{\leq b}(f_*\mathbb{Q}_X[n])) \to \mathbb{H}^{l+m}(Y, {}^{\mathfrak{p}}\tau_{\leq a}(f_*\mathbb{Q}_X[n]) \otimes {}^{\mathfrak{p}}\tau_{\leq b}(f_*\mathbb{Q}_X[n]))$$

In fact, we know $f_*\mathbb{Q}_X \otimes f_*\mathbb{Q}_X = \Delta_Y^*(f_*\mathbb{Q}_X \boxtimes f_*\mathbb{Q}_X) = \Delta_Y^*(f \times f)_*\mathbb{Q}_{X \times X} = g_*\mathbb{Q}_{X \times_Y X}$. However, the map $\otimes : {}^{\mathfrak{p}}\mathsf{D}^{\leq a} \times {}^{\mathfrak{p}}\mathsf{D}^{\leq b} \to {}^{\mathfrak{p}}\mathsf{D}^{\leq a+b}$ gives the map

$${}^{\mathfrak{p}}\tau_{\leq a}(f_*\mathbb{Q}_X[n]) \otimes {}^{\mathfrak{p}}\tau_{\leq b}(f_*\mathbb{Q}_X[n]) \to {}^{\mathfrak{p}}\tau_{\leq a+b}(g_*\mathbb{Q}_{X \times_Y X}[2n]) = {}^{\mathfrak{p}}\tau_{\leq a+b+n-r(f)}(g_*\mathbb{Q}_{X \times_Y X}[\mathsf{dim}\, X \times_Y X])[n - r(f)].$$

This gives the morphism $\mathsf{H}^{n+l}_{\leq a}(X) \times \mathsf{H}^{n+m}_{\leq b}(X) \to \mathsf{H}^{2n+l+m}_{\leq a+b+n-r(f)}(X \times_Y X)$.

Then we use the diagonal morphism $\Delta_f : X \to X \times_Y X$ and $\mathbb{Q}_{X \times_Y X} \to \Delta_{f*}\mathbb{Q}_X$ to obtain the morphism $\mathsf{H}^{2n+l+m}_{\leq a+b+n-r(f)}(X \times_Y X) \to \mathsf{H}^{2n+l+m}_{\leq a+b+n}(X)$ and the composition gives the cup product on X.

Now if we assume $\mathsf{dim}\, Y \geq 2$ and every perverse cohomology of $f_*\mathbb{Q}_X$ has support Y, i.e. has the form $IC_Y(L_U)$ for an open subset of Y and the local system L_U on U, the result can be strengthened as follows. Since $IC_Y(L_U) \otimes IC_Y(L'_V) \in {}^{\mathfrak{p}}\mathsf{D}^{\leq -2}(Y)$, we have the image of $\mathsf{H}^{n+l}_{\leq a}(X) \times \mathsf{H}^{n+m}_{\leq b}(X) \to \mathsf{H}^{2n+l+m}_{\leq a+b+n-r(f)}(X \times_Y X)$ actually lies in $\mathsf{H}^{2n+l+m}_{\leq a+b+n-r(f)-2}(X \times_Y X)$ and hence the cup product lies in $\mathsf{H}^{2n+l+m}_{\leq a+b+n-2}(X)$.

Another special case is when f has pure relative dimension d. Goresky-MacPherson inequality ([LF théorème 7.3.1]) tells that $\mathsf{codim}(Z) \leq d$ for all support of intersection complex in the decomposition of $f_*\mathbb{Q}_X$. So we get the bound of dimensions of supports for every perverse cohomology of $f_*\mathbb{Q}_X$:

$$\mathsf{dim}\, \mathcal{H}^i({}^{\mathfrak{p}}\mathcal{H}^*(f_*\mathbb{Q}_X)) \leq \begin{cases} -i-1 & i > d - \mathsf{dim}\, Y \\ -i & i \leq d - \mathsf{dim}\, Y \end{cases}$$

This is to say, choose a suitable stratification S_s such that $\mathsf{dim}\, S_s = s$, we have

$$\mathcal{H}^i({}^{\mathfrak{p}}\mathcal{H}^*(f_*\mathbb{Q}_X)|_{S_s}) = 0, i > \begin{cases} -s-1 & s < \mathsf{dim}\, Y - d \\ -s & s \geq \mathsf{dim}\, Y - d \end{cases}$$

Thus for any a, b, we have

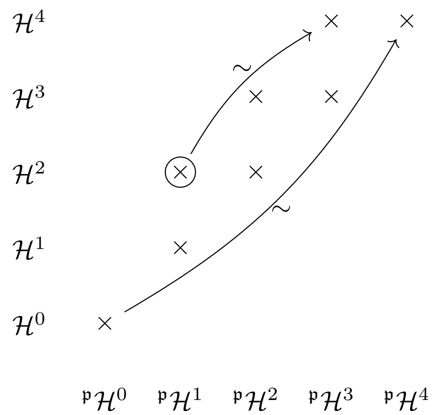
$$\mathcal{H}^i({}^{\mathfrak{p}}\mathcal{H}^a(f_*\mathbb{Q}_X) \otimes {}^{\mathfrak{p}}\mathcal{H}^b(f_*\mathbb{Q}_X)|_{S_s}) = 0, i > \begin{cases} -2s-2 & s < \mathsf{dim}\, Y - d \\ -2s & s \geq \mathsf{dim}\, Y - d \end{cases}$$

This implies when $\mathsf{dim}\, Y - d \geq 2$, such as $\mathsf{dim}\, X = 4$ and $\mathsf{dim}\, Y = 3$, we have $-2s \leq -s-2$, so ${}^{\mathfrak{p}}\mathcal{H}^a(f_*\mathbb{Q}_X) \otimes {}^{\mathfrak{p}}\mathcal{H}^b(f_*\mathbb{Q}_X) \in {}^{\mathfrak{p}}\mathsf{D}^{\leq -2}$. Hence we obtain the same result as previous one.

3 addition

Although the case when $\mathsf{dim}\, X = 2$ and $\mathsf{dim}\, Y = 1$ is always true, the case when $\mathsf{dim}\, X = 3$ and $\mathsf{dim}\, Y = 1$ could be false. Here is the reason:

Let $f : X \to Y, Y = U \sqcup \mathsf{pt}$.



As long as $\mathcal{H}^0({}^{\mathfrak{p}}\mathcal{H}^1(f_*\mathbb{Q}_X[1])) \neq 0$ and the square of it is non-zero, the image must lies in $\Gamma(Y, \mathcal{H}^4)$. In this case, the multiplicity is false.

For example, when f is $\mathrm{Bl}_{(0,0)}\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$, the value in the circle is the exceptional divisor $[E]$. The self intersection is non-zero and hence lies in $\mathcal{H}^0({}^{\mathfrak{p}}\mathcal{H}^3(f_*\mathbb{Q}_X[1]))$.

We can also consider the case $f : \mathrm{Bl}_C\mathbb{P}^3 \xrightarrow{g} \mathbb{P}^3 \xrightarrow{p} \mathbb{P}^1$ where $C \subset p^{-1}(0)$ is a smooth curve. Then $g_*\mathbb{Q}_X = \mathbb{Q}_{\mathbb{P}^3} \oplus \mathbb{Q}_C[-2]$ so that we can see the mechanism better.

Note in these two case, the morphism f are all flat.