numerical

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1 notation

π is the Hilbert-Chow morphism $X^{[n]} \to X^{(n)}$. For any partition $\alpha = 1^{a_1} \cdots n^{a_n}$ of n, let $|\alpha| = \sum_{k=1}^{n} a_k$, $\mathsf{gcd}(\alpha) = \mathsf{gcd}(\{i : a_i \neq 0\})$ and $n = ||\alpha|| = \sum ia_i$. The set of all partitions of n is denoted by $P(n)$. i_α is the morphism $X^{(\alpha)} = \prod_{k=1}^{n} X^{(a_k)} \to \overline{X_\alpha^{(n)}} \hookrightarrow X^{(n)}$.

Let $p : S \to C$ be a proper morphism from a smooth surface to a curve. We use the same symbol for the morphism $S^n \to C^n$ and $S^{(\alpha)} \to C^{(\alpha)}$ for any partition $\alpha = 1^{a_1} \cdots n^{a_n}$.

Let $p_* S[2] = \mathcal{P}_{-1}[1] \oplus \mathcal{P}_0 \oplus \mathcal{P}_1[-1]$ be the perverse cohomology decomposition.

For any rational homology manifold X of dimension $2n$, its k-th cohomology has pure Hodge structure of weight k. Let $P_{x,y}(X)$ denote the Hodge number polynomial of middle degree, i.e. $P_{x,y}(X) = \sum_{-n \leq i,j \leq n} h^{n+i,n+j}(X)x^i y^j$. Thus we know $P_{x,y}(X \times Y) = P_{x,y}(X)P_{x,y}(Y)$. For any perverse sheaf K, let $P_x(K)$ denote the Poincaré polynomial, i.e. $P_t(K) = \sum_i \mathsf{dim}\, H^i(X, K)t^i$. Hence, there is an identity $P_t(\mathbb{C}_X[n]) = P_{t,t}(X)$.

2 Hilbert case

2.1 one side

The decomposition theorem shows that $\pi_* \mathbb{C}_{S^{[n]}}[2n](n) = \bigoplus_{\alpha \in P(n)} i_{\alpha *} \mathbb{C}_{S^{(\alpha)}}[2|\alpha|](\alpha)$, which implies that $P_{x,y}(S^{[n]}) = \sum_\alpha P_{x,y}(S^{(\alpha)})$.

In this case, we have

$$\sum_{n \geq 0} P_{x,y}(S^{[n]})t^n = \sum_{n \geq 0} \sum_{\alpha \in P(n)} P_{x,y}(S^{(\alpha)})t^{||\alpha||} = \sum_{a_i \geq 0} \prod_{i \geq 1} P_{x,y}(S^{(a_i)})t^{ia_i} = \prod_{i \geq 1} \left(\sum_{n \geq 0} P_{x,y}(S^{(n)})t^{in} \right).$$

Recall the relation $H^*(X^{(n)}, K^{(n)}) = H^*(X^n, K^n)^{\mathfrak{S}_n} = \bigoplus_{i+j=n} S^i H^{ev}(X, K) \otimes \wedge^j H^{odd}(X, K)$. It is obvious that this formular is capatible with the Hodge decomposition when $K = \mathbb{C}_X$. Hence, we have $P_{x,y}(X^{(n)}) = \sum_{i+j=n} (S^i P_{x,y}^{ev}(X))(\wedge^j P_{x,y}^{odd}(X))$. Summing with n, we obtain

$$\sum_{n \geq 0} P_{x,y}(X^{(n)})t^n = \sum_{i,j \geq 0} (S^i P_{x,y}^{ev}(X))(\wedge^j P_{x,y}^{odd}(X))t^{i+j} = \left(\sum_{i \geq 0} S^i P_{x,y}^{ev}(X)t^i \right) \left(\sum_{j \geq 0} \wedge^j P_{x,y}^{odd}(X)t^j \right).$$

Furthermore,

$$\sum_{i \geq 0} S^i P_{x,y}^{ev}(X)t^i = \prod_{\substack{f \in P_{x,y}^{ev}(X) \\ f \text{ is monomial}}} \frac{1}{1 - tf}, \quad \sum_{j \geq 0} \wedge^j P_{x,y}^{odd}(X)t^j = \prod_{\substack{f \in P_{x,y}^{odd}(X) \\ f \text{ is monomial}}} (1 + tf).$$

In conclusion,

$$\sum_{n\geq 0} P_{x,y}(S^{(n)})t^n = \frac{(1+y^{-1}t)^{h^{1,0}}(1+x^{-1}t)^{h^{0,1}}(1+xt)^{h^{2,1}}(1+yt)^{h^{1,2}}}{(1-x^{-1}y^{-1}t)^{h^{0,0}}(1-xy^{-1}t)^{h^{2,0}}(1-t)^{h^{1,1}}(1-xy^{-1}t)^{h^{0,2}}(1-xyt)^{h^{2,2}}}$$

and

$$\sum_{n\geq 0} P_{x,y}(S^{[n]})t^n = \prod_{i\geq 1} \frac{(1+y^{-1}t^i)^{h^{1,0}}(1+x^{-1}t^i)^{h^{0,1}}(1+xt^i)^{h^{2,1}}(1+yt^i)^{h^{1,2}}}{(1-x^{-1}y^{-1}t^i)^{h^{0,0}}(1-xy^{-1}t^i)^{h^{2,0}}(1-t^i)^{h^{1,1}}(1-xy^{-1}t^i)^{h^{0,2}}(1-xyt^i)^{h^{2,2}}}.$$

2.2 another side

For the morphism $p\circ\pi : S^{[n]} \to C^{(n)}$, let $k_n^{i,j}$ be the dimension of $H^j(C^{(n)}, {}^{\mathfrak{p}}\mathcal{H}^i(p_*\pi_*\mathbb{C}_{S^{[n]}}[2n]))$. We consider the polynomial $\sum_{n\geq 0}\sum_{i,j} k_n^{i,j}x^iy^jt^n$.

According to the decomposition theorem, we have

$$p_*\pi_*\mathbb{Q}_{S^{[n]}}[2n] = \bigoplus_{\alpha\in P(n)} p_*i_{\alpha *}\mathbb{Q}_{S^{(\alpha)}}[2|\alpha|] = \bigoplus_{\substack{\alpha\in P(n)\\ \alpha_{-1}+\alpha_0+\alpha_1=\alpha}} i_{\alpha *}q_*((\mathcal{P}_{-1}[1])^{(\alpha_{-1})}\boxtimes(\mathcal{P}_0)^{(\alpha_0)}\boxtimes(\mathcal{P}_1[-1])^{(\alpha_1)}),$$

where $q : C^{(\alpha_0)}\times C^{(\alpha_1)}\times C^{(\alpha_2)} \to C^{(\alpha)}$ is a finite map and every summand is a shift of perverse sheaves. Hence, we have

$${}^{\mathfrak{p}}\mathcal{H}^i(p_*\pi_*\mathbb{Q}_{S^{[n]}}[2n]) = \bigoplus_{\substack{\alpha\in P(n)\\ \alpha_{-1}+\alpha_0+\alpha_1=\alpha\\ |\alpha_1|-|\alpha_{-1}|=i}} i_{\alpha *}q_*((\mathcal{P}_{-1})^{\{\alpha_{-1}\}}\boxtimes(\mathcal{P}_0)^{(\alpha_0)}\boxtimes(\mathcal{P}_1)^{\{\alpha_1\}}),$$

which implies

$$\sum_j k_n^{i,j}y^j = P_y({}^{\mathfrak{p}}\mathcal{H}^i(p_*\pi_*\mathbb{C}_{S^{[n]}}[2n])) = \sum_{\substack{\alpha\in P(n)\\ \alpha_{-1}+\alpha_0+\alpha_1=\alpha\\ |\alpha_1|-|\alpha_{-1}|=i}} P_y((\mathcal{P}_{-1})^{\{\alpha_{-1}\}}\boxtimes(\mathcal{P}_0)^{(\alpha_0)}\boxtimes(\mathcal{P}_1)^{\{\alpha_1\}}).$$

Summing with i and furthermore with n, we obtain

$$\sum_{i,j} k_n^{i,j}x^iy^j = \sum_{\substack{\alpha\in P(n)\\ \alpha_{-1}+\alpha_0+\alpha_1=\alpha}} P_y((\mathcal{P}_{-1})^{\{\alpha_{-1}\}})P_y((\mathcal{P}_0)^{(\alpha_0)})P_y((\mathcal{P}_1)^{\{\alpha_1\}})x^{|\alpha_1|-|\alpha_{-1}|}$$

and

$$\sum_{n\geq 0}\sum_{i,j} k_n^{i,j}x^iy^jt^n = \sum_{\alpha_{-1},\alpha_0,\alpha_1} P_y((\mathcal{P}_{-1})^{\{\alpha_{-1}\}})P_y((\mathcal{P}_0)^{(\alpha_0)})P_y((\mathcal{P}_1)^{\{\alpha_1\}})x^{|\alpha_1|-|\alpha_{-1}|}t^{||\alpha_{-1}||+||\alpha_0||+||\alpha_1||},$$

which is equal to

$$\begin{aligned}
&\left(\sum_{\alpha} P_y((\mathcal{P}_{-1})^{\{\alpha\}})x^{-|\alpha|}t^{||\alpha||}\right)\left(\sum_{\alpha} P_y((\mathcal{P}_0)^{(\alpha)})t^{||\alpha||}\right)\left(\sum_{\alpha} P_y((\mathcal{P}_1)^{\{\alpha\}})x^{|\alpha|}t^{||\alpha||}\right)\\
=&\prod_{i\geq 1}\left(\sum_{n\geq 0} P_y((\mathcal{P}_{-1})^{\{n\}})x^{-n}t^{in}\right)\left(\sum_{n\geq 0} P_y((\mathcal{P}_0)^{(n)})t^{in}\right)\left(\sum_{n\geq 0} P_y((\mathcal{P}_1)^{\{n\}})x^{n}t^{in}\right).
\end{aligned}$$

Recall $H^*(X^{(n)}, K^{\{n\}}) = \bigoplus_{i+j=n} \wedge^i H^{ev}(X,K) \otimes S^j H^{odd}(X,K)$. We have

$$P_t(K^{\{n\}}) = \sum_{i+j=n} (\wedge^i P_t^{ev}(K))(S^j P_t^{odd}(K)).$$

Summing with n, we obtain

$$\sum_{n\geq 0} P_y(K^{\{n\}})t^n = \left(\sum_{i\geq 0} \wedge^i P_y^{ev}(K)t^i\right)\left(\sum_{j\geq 0} S^j P_y^{odd}(K)t^j\right).$$

In conclusion, this case implies

$$\sum_{n\geq 0} P_y((\mathcal{P}_{-1})^{\{n\}})t^n = \frac{(1+t)^{k_1^{-1,0}}}{(1-y^{-1}t)^{k_1^{-1,-1}}(1-yt)^{k_1^{-1,1}}},$$

$$\sum_{n\geq 0} P_y((\mathcal{P}_{0})^{(n)})t^n = \frac{(1+y^{-1}t)^{k_1^{0,-1}}(1+yt)^{k_1^{0,1}}}{(1-t)^{k_1^{0,0}}},$$

$$\sum_{n\geq 0} P_y((\mathcal{P}_{1})^{\{n\}})t^n = \frac{(1+t)^{k_1^{1,0}}}{(1-y^{-1}t)^{k_1^{1,-1}}(1-yt)^{k_1^{1,1}}}$$

and finally

$$\sum_{\substack{n\geq 0\\ i,j}} k_n^{i,j}x^iy^jt^n = \prod_{i\geq 1} \frac{(1+x^{-1}t^i)^{k_1^{-1,0}}(1+y^{-1}t^i)^{k_1^{0,-1}}(1+yt^i)^{k_1^{0,1}}(1+xt^i)^{k_1^{1,0}}}{(1-x^{-1}y^{-1}t^i)^{k_1^{-1,-1}}(1-x^{-1}yt^i)^{k_1^{-1,1}}(1-t^i)^{k_1^{0,0}}(1-xy^{-1}t^i)^{k_1^{1,-1}}(1-xyt^i)^{k_1^{1,1}}}.$$

2.3 conclusion

For a proper morphism $X \to Y$ form a smooth variety of dimension $2n$ to a variety of dimension n, we say *the two diamonds are equal* if the Hodge number is equal to the decomposition number, i.e. $h^{n+i,n+j} = k^{i,j}$.

Example. *The elliptic fibration of a K3 surface $S \to \mathbb{P}^1$.*

Example. *The surjective map from an Abelian surface to an elliptic curve $A \to E$.*

Theorem 1. *If the morphism $S \to C$ satisfies that the two diamonds are equal, then the morphism $S^{[n]} \to C^{(n)}$ satisfies that the two diamonds are equal.*

Proof. Comparing the two polynomials, since $h^{1+i,1+j} = k_1^{i,j}$, they are equal. Hence every coeffcients are equal, which means that $h^{n+i,n+j}(S^{[n]}) = k_n^{i,j}$. This is the result. □

In fact, from the calculation, we see easily the following theorem.

Theorem 2. *If the morphism $S \to C$ satisfies that the two diamonds are equal, then the morphism $S^{(n)} \to C^{(n)}$ satisfies that the two diamonds are equal.*

Proof. There generating polynomial is just the product component for $i = 1$. □

3 Kummer case

3.1 one side

For $n \geq 1$, we have the finite map $r : A \times K^{n-1}(A) \to A^{[n]}$. Furthermore, the following formular holds:

$$\begin{aligned} h^{n+p,n+q}(A \times K^{n-1}(A)) &= \sum_{\sigma \in A[n]^\vee} h^{n+p,n+q}(A^{[n]}, L_\sigma) = \sum_{\sigma \in A[n]^\vee} \sum_{\alpha \in P(n)} h^{|\alpha|+p,|\alpha|+q}(A^{(\alpha)}, L_\sigma) \\ &= \sum_{\alpha \in P(n)} \sum_{\sigma \in A[\mathsf{gcd}(\alpha)]^\vee} h^{|\alpha|+p,|\alpha|+q}(A^{(\alpha)}, L_\sigma) = \sum_{\alpha \in P(n)} \mathsf{gcd}(\alpha)^4 h^{|\alpha|+p,|\alpha|+q}(A^{(\alpha)}) \end{aligned}$$

This is to say, $P_{x,y}(A \times K^{n-1}(A)) = \sum_{\alpha \in P(n)} \mathsf{gcd}(\alpha)^4 P_{x,y}(A^{(\alpha)})$. Summing with n, we have

$$\sum_{n \geq 1} P_{x,y}(A \times K^{n-1}(A)) t^n = \sum_{\alpha \neq 0} \mathsf{gcd}(\alpha)^4 P_{x,y}(A^{(\alpha)}) t^{||\alpha||}.$$

3.2 another side

Recall the diagram where r are finite morphism:

$$\begin{tikzcd} A \times K^{n-1}(A) \arrow[r, "r"] \arrow[d, "p \times p"] & A^{[n]} \arrow[r, "\pi"] & A^{(n)} \arrow[d, "p"] \\ E \times K^{n-1}(E) \arrow[rr, "r"] & & E^{(n)} \end{tikzcd}$$

Thus $h^j(E \times K^{n-1}(E), {}^{\mathfrak{p}}\mathcal{H}^i((p \times p)_* \mathbb{C}_{A \times K^{n-1}(A)}[2n])) = h^j(E^{(n)}, {}^{\mathfrak{p}}\mathcal{H}^i(r_*(p \times p)_* \mathbb{C}_{A \times K^{n-1}(A)}[2n]))$.

Recall the decomposition

$$\mathsf{pr}_* \pi_* r_* \mathbb{C}_{A \times K^{n-1}(A)}[2n] = K' \oplus \bigoplus_{\substack{\alpha \in P(n) \\ \alpha_{-1}+\alpha_0+\alpha_1=\alpha}} \bigoplus_{\sigma \in A[\mathsf{gcd}(\alpha)]^\vee} i_{\alpha *} q_* (\mathcal{P}_{-1}^{\{\alpha_{-1}\}} \boxtimes \mathcal{P}_0^{(\alpha_0)} \boxtimes \mathcal{P}_1^{\{\alpha_1\}})[|\alpha_{-1}| - |\alpha_1|]$$

where K' has no contribution to the cohomology of $A \times K^{n-1}(A)$ and every direct component on the right is the shift of a perverse sheaf.

Hence if we let $k_n^{i,j}$ denote the similar number for the morphism $A \times K^{n-1}(A) \to R \times K^{n-1}(E)$, we have

$$k_n^{i,j} = \sum_{\substack{\alpha \in P(n) \\ \alpha_0+\alpha_1+\alpha_2=\alpha \\ |\alpha_1|-|\alpha_{-1}|=i}} \mathsf{gcd}(\alpha)^4 h^j(E^{(\alpha_{-1})} \times E^{(\alpha_0)} \times E^{(\alpha_1)}, \mathcal{P}_{-1}^{\{\alpha_{-1}\}} \boxtimes \mathcal{P}_0^{(\alpha_0)} \boxtimes \mathcal{P}_1^{\{\alpha_1\}})$$

and summing with i, j,

$$\sum_{i,j} k_n^{i,j} x^i y^j = \sum_{\substack{\alpha \in P(n) \\ \alpha_{-1}+\alpha_0+\alpha_1=\alpha}} \mathsf{gcd}(\alpha)^4 P_y((\mathcal{P}_{-1})^{\{\alpha_{-1}\}}) P_y((\mathcal{P}_0)^{(\alpha_0)}) P_y((\mathcal{P}_1)^{\{\alpha_1\}}) x^{|\alpha_1|-|\alpha_{-1}|}.$$

3.3 conclusion

Theorem 3. *Let $A \to E$ be a surjective morphism from an Abelian surface to an elliptic curve, then the morphism $A \times K^{n-1}(A) \to E \times K^{n-1}(E)$ satisfies that the two diamonds are equal.*

Proof. From theorem 2, we know that $A^{(\alpha)} \to E^{(\alpha)}$ satisfies that the two diamonds are equal, which means

$$P_{x,y}(A^{(\alpha)}) = \sum_{\alpha_{-1}+\alpha_0+\alpha_1=\alpha} P_y((\mathcal{P}_{-1})^{\{\alpha_{-1}\}})P_y((\mathcal{P}_0)^{(\alpha_0)})P_y((\mathcal{P}_1)^{\{\alpha_1\}})x^{|\alpha_1|-|\alpha_{-1}|}.$$

Adding them after multiplied by $\mathsf{gcd}(\alpha)^4$, we get the desired result. □

Theorem 4. *Let $A \to E$ be a surjective morphism from an Abelian surface to an elliptic curve, then the morphism $K^{n-1}(A) \to K^{n-1}(E)$ satisfies that the two diamonds are equal.*

Proof. On the one hand, $P_{x,y}(A \times K^{n-1}(A)) = P_{x,y}(A)P_{x,y}(K^{n-1}(A))$. On the other hand,

$$\begin{aligned} k_n^{i,j} &= h^j(E \times K^{n-1}(E), \bigoplus_{i_1+i_2=i} {}^{\mathfrak{p}}\mathcal{H}^{i_1}(p_*\mathbb{C}_A[2]) \boxtimes {}^{\mathfrak{p}}\mathcal{H}^{i_2}(p_*\mathbb{C}_{K^{n-1}(A)}[2n-2])) \\ &= \sum_{\substack{i_1+i_2=i \\ j_1+j_2=j}} h^{j_1}(E, {}^{\mathfrak{p}}\mathcal{H}^{i_1}(p_*\mathbb{C}_A[2]))h^{j_2}(K^{n-1}(E), {}^{\mathfrak{p}}\mathcal{H}^{i_2}(p_*\mathbb{C}_{K^{n-1}(A)}[2n-2])). \end{aligned}$$

After summing with i, j, it decomposes into a product. One component is the polynomial associated with $A \to E$, while the other one is the polynomial associated with $K^{n-1}(A) \to K^{n-1}(E)$.

The first is precisely $P_{x,y}(A)$. Hence the remaining parts are equal. □

Problem. I could not find a beautiful expression for $\sum_{n\geq 1} P_{x,y}(A \times K^{n-1}(A))t^n$.