genKum

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π is the Hilbert-Chow morphism $X^{[n]} \to X^{(n)}$. For any partition $\alpha = 1^{a_1} \cdots n^{a_n}$ of n, let $|\alpha| = \sum_{k=1}^{n} a_k$, $\mathsf{gcd}(\alpha) = \mathsf{gcd}(\{i : a_i \neq 0\})$ and i_α is the morphism $X^{(\alpha)} = \prod_{k=1}^{n} X^{(a_k)} \to \overline{X_\alpha^{(n)}} \hookrightarrow X^{(n)}$ for $X = A$ or C.

$\mathsf{pr} : A \to C$ is the projection of $A = C \times C'$ to the first component and we use the same symbol for the maps $A^{(\alpha)} \to C^{(\alpha)}$ and $K^{n-1}(A) \to K^{n-1}(C)$. In addition, we have $\mathsf{pr}_* \mathbb{Q}_A = \mathbb{Q}_C \oplus \mathbb{Q}_C^2[-1] \oplus \mathbb{Q}_C[-2]$.

1 local systems on A

For any $\sigma \in \pi_1(A)^\vee = \mathsf{Hom}(\pi_1(A), \mathbb{C}^\times)$, we define $L_{A,\sigma}$ the local system with respect to σ. For any morphism $f : X \to A$, let $L_{X,\sigma}$ denote the local system $f^* L_{A,\sigma}$. Using the identity $\pi_1(A)/n\pi_1(A) = \pi_0(A[n]) = A[n]$ due to the topological fibration $A[n] \hookrightarrow A \xrightarrow{\cdot n} A$, we associate $\sigma \in A[n]^\vee$ with the element $\sigma \in \pi_1(A)^\vee$ such that $\sigma^n = 1$.

Then we determine the sheaf $(\cdot n)_* \mathbb{C}_A$. Since $\cdot n$ is a covering map, it is a locally constant sheaf with $\pi_1(A)$'s representation $\mathrm{Ind}_H^G(\mathbb{C})$ where $\cdot n : H = \pi_1(A) \hookrightarrow \pi_1(A) = G$. It turns out that this is just the regular representation of $G/H = A[n]$ on $\bigoplus_{x \in A[n]} \mathbb{C}_x$ by $g(\mathbb{C}_x) = \mathbb{C}_{gx}$. Since $A[n]$ is an Abelian group, the representation splits into all its 1-dim representations. This is to say, $(\cdot n)_* \mathbb{C}_A = \bigoplus_{\sigma \in A[n]^\vee} L_{A,\sigma}$. Here, $x \in \pi_1(A)$ acting on $L_{A,\sigma}$ is by multiplying $\sigma(x)$.

Now we consider the cup product on A. Recall Leray spectral sequence $H^p(B, R^q f_* \mathbb{C}_X) \Rightarrow H^{p+q}(X, \mathbb{C})$ for a fibration $F \hookrightarrow X \xrightarrow{f} B$. It is equipped with a cup product structure $R^q f_* \mathbb{C}_X \times R^{q'} f_* \mathbb{C}_X \to R^{q+q'} f_* \mathbb{C}_X$ coming from $H^q(F, \mathbb{C}) \times H^{q'}(F, \mathbb{C}) \to H^{q+q'}(F, \mathbb{C})$. In our case, $(\cdot n)_* \mathbb{C}_A = R^0(\cdot n)_* \mathbb{C}_A$ and the cup product is

$$\bigoplus_{x \in A[n]} \mathbb{C}_x \times \bigoplus_{x \in A[n]} \mathbb{C}_x \to \bigoplus_{x \in A[n]} \mathbb{C}_x$$
$$(\mathbb{C}_x, \mathbb{C}_y) \mapsto \delta_{xy} \mathbb{C}_x$$

Under the identification $\bigoplus\limits_{x \in A[n]} \mathbb{C}_x = \bigoplus\limits_{\sigma \in A[n]^\vee} L_\sigma$, we find the above map is $(L_{A,\sigma}, L_{A,\tau}) \mapsto L_{A,\sigma\tau}$.

Example. *Assume the group $G = \mathbb{Z}/n\mathbb{Z}$, then the identification $\bigoplus\limits_{0 \le x \le n-1} \mathbb{C}_x = \bigoplus_{\sigma \in G^\vee} L_\sigma$ is $k^\vee = \sum\limits_{0 \le x \le n-1} \zeta^{-kx} x$, where $k^\vee(x) = \zeta^{kx}$, ζ is the n-th root of unity. Thus*

$$(k^\vee, l^\vee) = \sum_{0 \le x, y \le n-1} (\zeta^{-kx} x, \zeta^{-ly} y) = \sum_{0 \le x \le n-1} \zeta^{-kx-lx} x = (k+l)^\vee.$$

On the other hand, since $x \in \pi_1(A)$ acting on $L_{A,\sigma}$ is by multiplying $\sigma(x)$, $x \in \pi_1(A)$ acting on $L_{A,\sigma} \otimes L_{A,\tau}$ is by multiplying $\sigma(x)\tau(x) = (\sigma\tau)(x)$. Thus we have $L_{A,\sigma} \otimes L_{A,\tau} = L_{A,\sigma\tau}$.

Now we study the cohomology behavior of such local systems.

Let T be a complex torus, $\sigma \in \pi_1(T)^\vee, n = \mathsf{ord}(\tau) < \infty$. Then we have $H^i(T, L_{T,\tau}) \neq 0 \iff \tau = 1 \iff L_{T,\tau}$ is trivial.

Proof. According to the previous discussion, we have the morphism of $\cdot n : T \to T$ and $(\cdot n)_* \mathbb{C}_T = \bigoplus_{\sigma \in T[n]^\vee} L_{T,\sigma}$. Taking cohomology, we have $H^i(T, \mathbb{C}_T) = H^i(T, L_{T,1}) \oplus \bigoplus_{\sigma \in T[n]^\vee, \sigma \neq 1} H^i(T, L_{T,\sigma})$. Since $L_{T,1} = \mathbb{C}_T$, counting the dimensions, we get the results. $\square$

Let $q : A^{|\alpha|} \to A^{(\alpha)}$ be the quotient map by the finite group $\mathfrak{S}_\alpha = \prod_{k=1}^n \mathfrak{S}_{a_k}$. Let $\sigma \in A[n]^\vee$, then $H^i(A^{(\alpha)}, L_{A^{(\alpha)},\sigma}) = H^i(A^{|\alpha|}, q^* L_{A^{(\alpha)},\sigma})^{\mathfrak{S}_\alpha}$. Since $A^{|\alpha|}$ is a complex torus, this group is non-zero if and only if $L_{A^{|\alpha|},\sigma} = q^* L_{A^{(\alpha)},\sigma}$ is trivial.

Consider the following commutative diagram

$$\begin{tikzcd}
A^{|\alpha|} \arrow[r, "q"] \arrow[d, "\dotplus"'] & A^{(\alpha)} \arrow[d, "+"] \arrow[dl, "\dotplus"'] \\
A \arrow[r, "\cdot \mathsf{gcd}(\alpha)"] & A
\end{tikzcd}$$

where $\dotplus : A^{(\alpha)} \to A$ is the map $(x_{ij}, 1 \leq i \leq n, 1 \leq j \leq a_i) \mapsto \sum_{i=1}^n \frac{i}{\mathsf{gcd}(\alpha)} \sum_{j=1}^{a_i} x_{ij}$.

We claim that the map $\dotplus$ induce the surjective map $\pi_1(A^{|\alpha|}) \to \pi_1(A)$. In fact, accroding to the definition of greatest common divisor, there exist intergers b_i such that $\sum_{i=1}^n i b_i = \mathsf{gcd}(\alpha)$. Thus if $f : I \to A$ is a loop, we have $\tilde{f} : I \to A^{|\alpha|}$ defined by $\tilde{f} = (f_{ij}, 1 \leq i \leq n, 1 \leq j \leq a_i)$ where

$$x_{ij} = \begin{cases} f & 1 \leq j \leq a_i - 1 \\ (\cdot(b_i - a_i + 1)) \circ f & j = a_i \end{cases}$$

Thus the composition of $\tilde{f}$ and $\dotplus$ is just $\sum_{i=1}^n \frac{i}{\mathsf{gcd}(\alpha)} (\cdot b_i) \circ f = f$.

Hence, if the local system $(\cdot \mathsf{gcd}(\alpha))^* L_{A,\sigma} = L_{A,\mathsf{gcd}(\alpha)\sigma}$ is not trivial, the pull back to $A^{|\alpha|}$ is also not trivial. In conclusion, $L_{A^{|\alpha|},\sigma}$ is trivial if and only if $\mathsf{gcd}(\alpha)\sigma = 1$, i.e. $\mathsf{ord}(\sigma) | \mathsf{gcd}(\alpha)$. In this case, $L_{A^{(\alpha)},\sigma}$ is also trivial.

2 perverse filtration

$$\begin{tikzcd}
K^{n-1}(A) \arrow[rr, hook, "i_0"] \arrow[dd, "\mathsf{pr}"] & & A \times K^{n-1}(A) \arrow[rr, "r"] \arrow[dl, "p"] \arrow[dd, dashed, "\mathsf{pr} \times \mathsf{pr}"] \arrow[dr, phantom, "\square"] & & A^{[n]} \arrow[dl, "+"] \arrow[dd, "\mathsf{pr} \circ \pi"] \\
& A \arrow[rr, "\cdot n"] \arrow[dd, "\mathsf{pr}"] & & A \arrow[dd, "\mathsf{pr}"] & \\
K^{n-1}(C) \arrow[rr, hook, "i_0"] & & C \times K^{n-1}(C) \arrow[rr, "r"] \arrow[dl, "p"] \arrow[dr, phantom, "\square"] & & C^{[n]} = C^{(n)} \arrow[dl, "+"] \\
& C \arrow[rr, "\cdot n"] & & C &
\end{tikzcd}$$

Our aim is to determine the perverse decomposition for $A \times K^{n-1}(A) \to C^{(n)}$. After that, according to r is finite, we know the perverse number of the element in $H^*(A \times K^{n-1}(A))$ is the same for the two morphisms $A \times K^{n-1}(A) \to C \times K^{n-1}(C)$ and $A \times K^{n-1}(A) \to C^{(n)}$.

Using decomposition theorem, we have the result that $\pi_* \mathbb{Q}_{A^{[n]}}[2n] = \bigoplus_\alpha i_{\alpha *} \mathbb{Q}_{A^{(\alpha)}}[2|\alpha|]$.

According to proper base change, the Cartesian diagram shows that $r_*\mathbb{C}_{A\times K^{n-1}(A)} = r_*p^*\mathbb{Q}_A = (+)^*(\cdot n)_*\mathbb{Q}_A = \bigoplus_{\sigma\in A[n]^\vee} L_{A^{[n]},\sigma}$. The same method as the previous section for the topological fibration $A[n] \hookrightarrow A\times K^{n-1}(A) \xrightarrow{r} A^{[n]}$ shows that the image of $(L_{A^{[n]},\sigma}, L_{A^{[n]},\tau})$ under the cup product morphism $r_*\mathbb{C}_X \times r_*\mathbb{C}_X \to r_*\mathbb{C}_X$ is their tensor product $L_{A^{[n]},\sigma\tau}$ where $X = A\times K^{n-1}(A)$.

Now we can calculate $\pi_*r_*\mathbb{C}_{A\times K^{n-1}(A)}$ as follows:

$$\begin{aligned}
&\pi_*r_*\mathbb{C}_{A\times K^{n-1}(A)}\\
=&\pi_*\left(\bigoplus_{\sigma\in A[n]^\vee} L_{A^{[n]},\sigma}\right)\\
=&\bigoplus_{\sigma\in A[n]^\vee} \pi_*\pi^*L_{A^{(n)},\sigma}\\
=&\bigoplus_{\sigma\in A[n]^\vee} \pi_*\mathbb{Q}_{A^{[n]}}\otimes L_{A^{(n)},\sigma}\\
=&\bigoplus_{\sigma\in A[n]^\vee}\bigoplus_{\alpha} i_{\alpha *}\mathbb{Q}_{A^{(\alpha)}}[2|\alpha|-2n]\otimes L_{A^{(n)},\sigma}\\
=&\bigoplus_{\sigma\in A[n]^\vee}\bigoplus_{\alpha} i_{\alpha *}L_{A^{(\alpha)},\sigma}[2|\alpha|-2n]
\end{aligned}$$

Note that $H^*(A^{(\alpha)}, L_{A^{(\alpha)},\sigma}) \neq 0 \iff L_{A^{(\alpha)},\sigma}$ is trivial $\iff \mathsf{ord}(\sigma)|\,\mathsf{gcd}(\alpha)$. Hence the only components that contribute to the cohomology of $A\times K^{n-1}(A)$ are $\bigoplus\limits_{\alpha}\bigoplus\limits_{\sigma\in A[\mathsf{gcd}(\alpha)]^\vee} i_{\alpha *}L_{A^{(\alpha)},\sigma}[2|\alpha|-2n]$.

Applying pr, we find $\mathsf{pr}_*\pi_*r_*\mathbb{C}_{A\times K^{n-1}(A)}[2n] = \bigoplus\limits_{\alpha}\bigoplus\limits_{\sigma\in A[n]^\vee} i_{\alpha *}\mathsf{pr}_*L_{A^{(\alpha)},\sigma}[2|\alpha|]$.

When $\mathsf{ord}(\sigma)|\,\mathsf{gcd}(\alpha)$, $L_{A^{(\alpha)},\sigma} = \mathbb{C}_{A^{(\alpha)}}$, we have

$$\mathsf{pr}_*\mathbb{C}_{A^{(\alpha)}} = (\mathsf{pr}_*\mathbb{C}_A)^{(\alpha)} = \bigoplus_{\alpha_0+\alpha_1+\alpha_2=\alpha} q_*(\mathbb{C}_C^{(\alpha_0)}\boxtimes(\mathbb{C}_C^2)^{\{\alpha_1\}}\boxtimes\mathbb{C}_C^{(\alpha_2)})[-|\alpha_1|-2|\alpha_2|]$$

where $q: C^{(\alpha_0)}\times C^{(\alpha_1)}\times C^{(\alpha_2)} \to C^{(\alpha)}$ is a finite map.

In conclusion, there is a decomposition:

$$\mathsf{pr}_*\pi_*r_*\mathbb{C}_{A\times K^{n-1}(A)}[2n] = \left(\bigoplus_{\alpha}\bigoplus_{\sigma\in A[\mathsf{gcd}(\alpha)]^\vee} i_{\alpha *}\bigoplus_{\alpha_0+\alpha_1+\alpha_2=\alpha} q_*(\mathbb{C}_C^{(\alpha_0)}\boxtimes(\mathbb{C}_C^2)^{\{\alpha_1\}}\boxtimes\mathbb{C}_C^{(\alpha_2)})[2|\alpha_0|+|\alpha_1|]\right)\oplus K'$$

where K' has no contribution to the cohomology of $A\times K^{n-1}(A)$ and every direct component on the left is the shift of a perverse sheaf and hence is direct summand of perverse cohomologies of $\mathsf{pr}_*\pi_*r_*\mathbb{C}_{A\times K^{n-1}(A)}[2n]$.

3 semismall morphism

Let $f: X\to Y$ denote a semismall morphism from a smooth variety with relative strata Y_α such that $f: X_\alpha = f^{-1}(Y_\alpha)\to Y_\alpha$ is smooth. Let Λ be the set of those strata α such that $2\dim X_\alpha = \dim X + \dim Y_\alpha$.

To make things simpler, we assume every fiber of $y_\alpha \in Y_\alpha$ is irreducible. Let $i_\alpha : Z_\alpha \to \overline{Y_\alpha}$ be a small resolution of $\overline{Y_\alpha} \subseteq Y$ such that Z_α is the quotient of a smooth variety by a finite group. Thus $i_{\alpha *}\mathbb{Q}_{Z_\alpha}[\operatorname{dim} Y_\alpha] = IC_{\overline{Y_\alpha}}$.

The decomposition shows that we have $f_*\mathbb{Q}_X[\operatorname{dim} X] = f_* IC_X = \bigoplus\limits_{\alpha\in\Lambda} IC_{\overline{Y_\alpha}} = \bigoplus\limits_{\alpha\in\Lambda} i_{\alpha *}\mathbb{Q}_{Z_\alpha}[\operatorname{dim} Y_\alpha]$.

$$\begin{tikzcd}
Z_\alpha \times_Y X \arrow[r] \arrow[d, "f"] & \overline{X_\alpha} \arrow[r, hook] \arrow[d, "f"] & X \arrow[d, "f"] \\
Z_\alpha \arrow[r, "i_\alpha"] & \overline{Y_\alpha} \arrow[r, hook] & Y
\end{tikzcd}$$

The direct summand gives the morphism of sheaves $i_{\alpha *}\mathbb{Q}_{Z_\alpha}[\operatorname{dim} Y_\alpha] \to f_*\mathbb{Q}_X[\operatorname{dim} X]$. On the other hand, we have

$$\begin{aligned}
&\operatorname{Hom}(i_{\alpha *}\mathbb{Q}_{Z_\alpha}[\operatorname{dim} Y_\alpha], f_*\mathbb{Q}_X[\operatorname{dim} X])\\
=&\operatorname{Hom}(i_{\alpha *}\mathbb{Q}_{Z_\alpha}[\operatorname{dim} Y_\alpha], f_*\omega_X[-\operatorname{dim} X])\\
=&\operatorname{Hom}(\mathbb{Q}_{Z_\alpha}[\operatorname{dim} Y_\alpha], i_\alpha^! f_*\omega_X[-\operatorname{dim} X])\\
=&\operatorname{Hom}(\mathbb{Q}_{Z_\alpha}[\operatorname{dim} Y_\alpha], f_* i_\alpha^! \omega_X[-\operatorname{dim} X])\\
=&\operatorname{Hom}(f^*\mathbb{Q}_{Z_\alpha}[\operatorname{dim} Y_\alpha], \omega_{Z_\alpha\times_Y X}[-\operatorname{dim} X])\\
=&\operatorname{Hom}(\mathbb{Q}_{Z_\alpha\times_Y X}, \omega_{Z_\alpha\times_Y X}[-\operatorname{dim} X - \operatorname{dim} Y_\alpha])\\
=&H^{\mathrm{BM}}_{\operatorname{dim} X+\operatorname{dim} Y_\alpha}(Z_\alpha \times_Y X, \mathbb{Q})
\end{aligned}$$

However, $\operatorname{dim} X + \operatorname{dim} Y_\alpha = 2\operatorname{dim} X_\alpha$, which is the dimension of $Z_\alpha \times_Y X$. Hence the above group is just $A_{\operatorname{dim} Z_\alpha\times_Y X}(Z_\alpha \times_Y X)$. In the case of our assumption, $Z_\alpha \times_Y X$ only has one irreducible component which is itself.

In conclusion, the above morphism is induced by $[Z_\alpha \times_Y X] \in A_*(Z_\alpha \times X)$. Such correspondence induces the morphism $H^*(Z_\alpha, \mathbb{Q}) \to H^{*+\operatorname{dim} X-\operatorname{dim} Z_\alpha}(X, \mathbb{Q})$. Here, I give a explicit description of such morphism. For $x \in H^*(Z_\alpha, \mathbb{Q})$, we have the pull-back $f^*x \in H^*(Z_\alpha \times_Y X, \mathbb{Q})$. Then the cap product with $H^{\mathrm{BM}}_{2\operatorname{dim} Z_\alpha\times_Y X}(Z_\alpha \times_Y X, \mathbb{Q})$ gives $H^{\mathrm{BM}}_{2\operatorname{dim} Z_\alpha\times_Y X-*}(Z_\alpha \times_Y X, \mathbb{Q})$ by $\mathbb{Q} \otimes \omega = \omega$. Next the push-forward of $Z_\alpha \times_Y X \to X$ gives the resulting element $i_{\alpha *}(f^*x \cap [Z_\alpha \times_Y X]) \in H^{\mathrm{BM}}_{2\operatorname{dim} Z_\alpha\times_Y X-*}(X, \mathbb{Q})$. Finally, the Poincaré duality gives the corresponding element in $H^{*+2\operatorname{dim} X-2\operatorname{dim} Z_\alpha\times_Y X}(X, \mathbb{Q}) = H^{*+\operatorname{dim} X-\operatorname{dim} Y_\alpha}(X, \mathbb{Q})$.

Similarly, Z_α is the quotient of a manifold by a finite group, we have $IC_{Z_\alpha} = \mathbb{Q}_{Z_\alpha}[\operatorname{dim} Z_\alpha]$ and hence $\omega_{Z_\alpha} = D\mathbb{Q}_{Z_\alpha} = D(IC_{Z_\alpha}[-\operatorname{dim} Z_\alpha]) = (DIC_{Z_\alpha})[\operatorname{dim} Z_\alpha] = \mathbb{Q}_{Z_\alpha}[2\operatorname{dim} Z_\alpha]$. The same formula shows $\operatorname{Hom}(f_*\mathbb{Q}_X[\operatorname{dim} X], i_{\alpha *}\mathbb{Q}_{Z_\alpha}[\operatorname{dim} Y_\alpha]) = H^{\mathrm{BM}}_{\operatorname{dim} X+\operatorname{dim} Y_\alpha}(Z_\alpha \times_Y X, \mathbb{Q})$. This implies the projection $f_*\mathbb{Q}_X[\operatorname{dim} X] \to i_{\alpha *}\mathbb{Q}_{Z_\alpha}[\operatorname{dim} Y_\alpha]$ is also induced by the correspondence $[Z_\alpha \times_Y X] \in A_*(Z_\alpha \times X)$.

According to the above discussion, the decomposition $H^*(X, \mathbb{Q}) = \bigoplus_{\alpha\in\Lambda} H^*(Z_\alpha, \mathbb{Q})[\operatorname{dim} Z_\alpha - \operatorname{dim} X]$ is canonical, while the morphisms are given as above.

4 cohomology ring structure

For a given $x \in H^*(A^{(\alpha)}, \mathbb{C}) \subset H^{*+2n-2|\alpha|}(A^{[n]}, \mathbb{C})$, if $\operatorname{ord}(\sigma) \not| \operatorname{gcd}(\alpha)$, write $x_\sigma = 0$; if $\operatorname{ord}(\sigma) | \operatorname{gcd}(\alpha)$, write x_σ the corresponding element in $H^*(A^{(\alpha)}, L_{A^{(\alpha)},\sigma}) \subset H^{*+2n-2|\alpha|}(A^{[n]}, L_{A^{[n]},\sigma}) \subset H^{*+2n-2|\alpha|}(A\times K^{n-1}(A), \mathbb{C})$ under the trivialization $L_{A^{(\alpha)},\sigma} = \mathbb{C}_{A^{(\alpha)}}$.

Similarly, for $0 \neq x_\sigma \in H^*(A^{(\alpha)}, L_{A^{(\alpha)},\sigma})$, we write x the corresponding element in $H^*(A^{(\alpha)}, \mathbb{C}) \subset H^{*+2n-2|\alpha|}(A^{[n]}, \mathbb{C})$.

Theorem 1. *If $x_\sigma \in H^*(A^{(\alpha)}, L_{A^{(\alpha)},\sigma})$ and $y_\tau \in H^*(A^{(\beta)}, L_{A^{(\beta)},\tau})$, their product $x_\sigma \cup y_\tau \in H^*(A \times K^{n-1}(A), \mathbb{C})$ is $\sum_\gamma z^\gamma_{\sigma\tau}$, where $z^\gamma \in H^*(A^{(\gamma)}, \mathbb{C})$ is non-zero such that $x \cup y = \sum_\gamma z^\gamma$.*

Proof. Using the result in the previous section, where $X = A^{[n]}, Y = A^{(n)}$, Λ is the partition of n and $Z_\alpha = A^{(\alpha)}$, we find x is actually coming from $H^{\mathrm{BM}}_*(A^{(\alpha)} \times_{A^{(n)}} A^{[n]}, \mathbb{C})$, and hence comes from $H^{\mathrm{BM}}_*(\overline{A^{[n]}_\alpha}, \mathbb{C})$.

$$\begin{tikzcd} A^{(\alpha)} \times_{A^{(n)}} A^{[n]} \arrow[r] \arrow[d] & \overline{A^{[n]}_\alpha} \arrow[r, hook] \arrow[d] & A^{[n]} \arrow[d] \\ A^{(\alpha)} \arrow[r, "i_\alpha"] & \overline{A^{(n)}_\alpha} \arrow[r, hook] & A^{(n)} \end{tikzcd}$$

Since we have the cup product in $A^{[n]}$:

$$\cup : H^{\mathrm{BM}}_i(\overline{A^{[n]}_\alpha}, \mathbb{C}) \times H^{\mathrm{BM}}_j(\overline{A^{[n]}_\beta}, \mathbb{C}) \to H^{\mathrm{BM}}_{i+j-2\,\mathsf{dim}\,A^{[n]}}(\overline{A^{[n]}_\alpha} \cap \overline{A^{[n]}_\beta}, \mathbb{C})$$

we have $x \cup y$ must be in the image of $H^{\mathrm{BM}}_*(\overline{A^{[n]}_\alpha} \cap \overline{A^{[n]}_\beta}, \mathbb{C}) \to H^{\mathrm{BM}}_*(A^{[n]}, \mathbb{C})$. Thus γ must satisfy $\overline{A^{[n]}_\gamma} \subseteq \overline{A^{[n]}_\alpha} \cap \overline{A^{[n]}_\beta}$.

Now we give a explicit description of $z^\gamma_{\sigma\tau}$ by showing that the construction in the previous section is also valid for L_σ-coefficients.

In fact, the decomposition comes from the changing-of-coefficient formula $\pi_* L_{A^{[n]},\sigma} = \pi_*(\pi^* L_{A^{(n)},\sigma} \otimes \mathbb{C}_{A^{[n]}}) = L_{A^{(n)}} \otimes \pi_* \mathbb{C}_{A^{[n]}}$ and $i_{\alpha *} L_{A^{(\alpha)},\sigma} = i_* \alpha * \mathbb{C}_{A^{(\alpha)}} \otimes L_{A^{(n)},\sigma}$ similarily. As a result, the correspondence works the same for L_σ-coefficients. That is, the element in $H^*(A^{(\alpha)}, L_\sigma)$ gives the pull-back in $H^*(A^{(\alpha)} \times_{A^{(n)}} A^{[n]}, L_\sigma)$. Cupping with $[A^{(\alpha)} \times_{A^{(n)}} A^{[n]}] \in H^{-2(n+|\alpha|)}(A^{(\alpha)} \times_{A^{(n)}} A^{[n]}, \omega)$ gives $H^{*-2(n+|\alpha|)}(A^{(\alpha)} \times_{A^{(n)}} A^{[n]}, \omega \otimes L_\sigma)$. According to $i_{\alpha *}(\omega \otimes L_\sigma) = (i_{\alpha *}\omega) \otimes L_\sigma$ The push-forward gives the element in $H^{*+2n-2|\alpha|}(A^{[n]}, L_\sigma)$. It is clear that this procedure and its inverse are all independent of L_σ. This is to say, applying the cup product and the projection, $H^*(A^{(\alpha)}, L_\sigma) \times H^*(A^{(\beta)}, L_\tau) \to H^*(A^{(\gamma)}, L_{\sigma\tau})$ is independent of σ and τ.

Finally, the trivialization of $L_{A^{[n]},\sigma}|_{A^{[n]}_\alpha} \simeq \mathbb{C}_{A^{[n]}_\alpha}$, $L_{A^{[n]},\tau}|_{A^{[n]}_\beta} \simeq \mathbb{C}_{A^{[n]}_\beta}$ and $L_{A^{[n]},\sigma\tau}|_{A^{[n]}_\gamma} \simeq \mathbb{C}_{A^{[n]}_\gamma}$ is capatible with the tensor product $L_{A^{[n]},\sigma} \otimes L_{A^{[n]},\tau} \xrightarrow{\sim} L_{A^{[n]},\sigma\tau}$ leading to the disired result. $\square$

5 perverse multiplicativity

For a morphism $f : X \to Y$, we use the notation $P_{\leq p} H^*(X, \mathbb{C})$ to mean the perverse filtration of $H^*(X, \mathbb{C})$ by $\mathsf{im}(H^*(Y, {}^{\mathfrak{p}}\tau_{\leq p} f_* \mathbb{C}_X[\mathsf{dim}\, X - r(f)]) \to H^*(Y, f_* \mathbb{C}_X[\mathsf{dim}\, X - r(f)]))$. For $x \in H^*(X, \mathbb{C})$, let $P(x)$ denote the number p such that $x \in P_{\leq p} H^*(X, \mathbb{C})$ and $x \notin P_{\leq p-1} H^*(X, \mathbb{C})$. In the case that f is the morphism $\mathsf{pr} \circ \pi : A^{[n]} \to C^{(n)}$. Then $\mathsf{dim}\, X - r(f) = n$, and $P(x)$ ranges from 0 to $2n$.

In [zzl], he already shows that $A^{[n]} \to C^{(n)}$ has perverse multiplicativity, i.e. we have $P(x \cup y) \leq P(x) + P(y)$ for all $x, y \in H^*(A^{[n]}, \mathbb{C})$. This is to say, If $x \in H^*(A^{(\alpha)}, \mathbb{C}), y \in H^*(A^{(\beta)}, \mathbb{C})$ and we write $x \cup y = \sum z^\gamma$ where $z^\gamma \in H^*({}^{(\gamma)}, \mathbb{C})$, then $P(z) \leq P(x) + P(y)$.

For $x \in H^*(A \times K^{n-1}(A), \mathbb{C})$, we write $P(x)$ for the number associated with the morphism $\mathsf{pr} \circ \pi \circ r : A \times K^{n-1}(A) \to C^{(n)}$. According to the calculation in section 2, we know that $P(x_\sigma)$ is independent of $\sigma \in A[n]^\vee$. Since when $\tau = 1$, $L_{A^{[n]},\tau} = \mathbb{C}_{A^{[n]}}$, we have $P(x_\sigma) = P(x_1) = P(x)$.

Thus, Theorem 1 shows that $P(x_\sigma \cup y_\tau) = \mathsf{max}_\gamma\, P(z^\gamma_{\sigma\tau}) = \mathsf{max}_\gamma\, P(z^\gamma) = P(x \cup y) \leq P(x) + P(y) = P(x_\sigma) + P(y_\tau)$. In another word, the morphism $A \times K^{n-1}(A) \to C^{(n)}$ has perverse multiplicativity.

Now, we consider the morphism $\mathsf{pr} \times \mathsf{pr} : A \times K^{n-1}(A) \to K^{n-1}(C)$. Write $\mathsf{pr}_* \mathbb{C}_{K^{n-1}(A)}[n-1] = \sum_{i=0}^{2n-2} {}^{\mathfrak{p}}\mathcal{H}^i(\mathsf{pr}_* \mathbb{C}_{K^{n-1}(A)}[n-1])[-i]$. Then we have

$$
\begin{aligned}
& r_*(\mathsf{pr} \times \mathsf{pr})_* \mathbb{C}_{A \times K^{n-1}(A)}[n] \\
=& r_* \left(\mathsf{pr}_* \mathbb{C}_A[1] \boxtimes \mathsf{pr}_* \mathbb{C}_{K^{n-1}(A)}[n-1] \right) \\
=& r_* \left(\bigoplus_{i=0}^{2} {}^{\mathfrak{p}}\mathcal{H}^i(\mathsf{pr}_* \mathbb{C}_A[1])[-i] \boxtimes \bigoplus_{j=0}^{2n-2} {}^{\mathfrak{p}}\mathcal{H}^j(\mathsf{pr}_* \mathbb{C}_{K^{n-1}(A)}[n-1])[-j] \right) \\
=& r_* \left(\bigoplus_{k=0}^{2n} \bigoplus_{i+j=k} {}^{\mathfrak{p}}\mathcal{H}^i(\mathsf{pr}_* \mathbb{C}_A[1]) \boxtimes {}^{\mathfrak{p}}\mathcal{H}^j(\mathsf{pr}_* \mathbb{C}_{K^{n-1}(A)}[n-1])[-k] \right) \\
=& \bigoplus_{k=0}^{2n} \bigoplus_{i+j=k} r_* \left({}^{\mathfrak{p}}\mathcal{H}^i(\mathsf{pr}_* \mathbb{C}_A[1]) \boxtimes {}^{\mathfrak{p}}\mathcal{H}^j(\mathsf{pr}_* \mathbb{C}_{K^{n-1}(A)}[n-1]) \right) [-k]
\end{aligned}
$$

Since r is a finite morphism, r_* is t-exact, hence every summand in the above is indeed the shift of a perverse sheaf. As a result, this gives another description of the perverse decomposition for $r \circ (\mathsf{pr} \times \mathsf{pr}) = \mathsf{pr} \circ \pi \circ r$.

If $x \in H^*(K^{n-1}(A), \mathbb{C})$, we write $P(x)$ for the number associated with the morphism $\mathsf{pr} : K^{n-1}(A) \to K^{n-1}(C)$. Then according to the above description, we have $P(1 \otimes x) = P(1) + P(x) = P(x)$.

In conclusion, if $x, y \in H^*(K^{n-1}(A), \mathbb{C})$ we have $P(x \cup y) = P(1 \otimes (x \cup y)) = P((1 \otimes x) \cup (1 \otimes y)) \geq P(1 \otimes x) + P(1 \otimes y) = P(x) + P(y)$. This shows that the morphism $K^{n-1}(A) \to K^{n-1}(C)$ has perverse multiplicativity.