exceptional objects of Hilbert schemes

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1 notations

For a surface S, let $S^{[n]}$ be the Hilbert scheme of n points on this surface, $S^{(n)} = S^n/\mathfrak{S}_n$ be the n-th symmetry product of S, where $\mathfrak{S}_n$ is the symmetry group of n elements. $\pi : S^{[n]} \to S^{(n)}$ is the Hilbert-Chow morphism and $q : S^n \to S^{(n)}$ is the quotient map. For any partition $\alpha = 1^{a_1}2^{a_2}\cdots n^{a_n}$ of n, let $|\alpha| = \sum a_i, ||\alpha|| = \sum ia_i = n$.

2 exceptional objects

Theorem 1. *Let G be a finite group acting on X. Assume that the following elements form a full exceptional collection of the category $\mathsf{D}^b(X)$, where $R\operatorname{Hom}(E_i^j, E_i^l) = 0$ if $j \neq l$ and $R\operatorname{Hom}(E_i^j, E_k^l) = 0$ if $i < k$:*

$$\begin{pmatrix} E_1^1 & E_2^1 & \cdots & E_p^1 \\ \vdots & \vdots & & \vdots \\ E_1^{q_1} & E_2^{q_2} & \cdots & E_p^{q_p} \end{pmatrix}.$$

Assume that the group G acts on the elements $E_i^j, 1 \leq j \leq q_j$ transitively. Let H_i be the stablizer of the element E_i^1. Assume furthermore that there exists an element $\mathcal{E}_i$ such that $\mathsf{Res}(\mathcal{E}_i) = E_i^1$. Then the following elements also form a full exceptional collection of the category $\mathsf{D}_G^b(X)$:

$$\begin{pmatrix} \mathsf{Ind}_{H_1}^G(\mathcal{E}_1 \otimes V_1^1) & \mathsf{Ind}_{H_2}^G(\mathcal{E}_2 \otimes V_2^1) & \cdots & \mathsf{Ind}_{H_p}^G(\mathcal{E}_p \otimes V_p^1) \\ \vdots & \vdots & & \vdots \\ \mathsf{Ind}_{H_1}^G(\mathcal{E}_1 \otimes V_1^{r_1}) & \mathsf{Ind}_{H_2}^G(\mathcal{E}_2 \otimes V_2^{r_2}) & \cdots & \mathsf{Ind}_{H_p}^G(\mathcal{E}_p \otimes V_p^{r_p}) \end{pmatrix},$$

where $V_i^j, 1 \leq j \leq r_i$ are the irreducible representations of H_i.

Proof. Let $F_i^j = \mathsf{Ind}_{H^i}^G(\mathcal{E}_i \otimes V_i^j)$. We will show that $R\operatorname{Hom}_G(F_i^j, F_i^l) = 0$ if $j \neq l$, $R\operatorname{Hom}_G(F_i^j, F_k^l) = 0$ if $i < k$ and $R\operatorname{Hom}_G(F_i^j, F_i^j) = \mathbb{C}$.

In fact, we have

$$R\operatorname{Hom}_G(F_i^j, F_i^l) = R\operatorname{Hom}_{H_i}\left(\mathcal{E}_i \otimes V_i^j, \mathsf{Res}_{H_i}^G \mathsf{Ind}_{H_i}^G(\mathcal{E}_i \otimes V_i^l)\right) = R\operatorname{Hom}_{H_i}\left(\mathcal{E}_i \otimes V_i^j, \bigoplus_{g \in H_i \backslash G} g^*(\mathcal{E}_i \otimes V_i^l)\right).$$

When $g \notin H_i$, $\mathsf{Res}(g^*\mathcal{E}_i) = E_i^q$ for $q \neq 1$ and hence $R\operatorname{Hom}(\mathcal{E}_i \otimes V_i^j, g^*(\mathcal{E}_i \otimes V_i^l)) = 0$. Furthermore,

$$R\operatorname{Hom}_G(F_i^j, F_i^l) = R\operatorname{Hom}_{H_i}(\mathcal{E}_i \otimes V_i^j, \mathcal{E}_i \otimes V_i^l) = \operatorname{Hom}_{H_i}(V_i^j, V_i^l),$$

which is what we want.

If $i < k$, similarly,

$$\begin{aligned} R\,\mathsf{Hom}_G(F_i^j, F_k^l) &= R\,\mathsf{Hom}_{H_i}\left(\mathcal{E}_i \otimes V_i^j, \mathsf{Res}_{H_i}^G \mathsf{Ind}_{H_k}^G(\mathcal{E}_k \otimes V_k^l)\right) \\ &= \left(R\,\mathsf{Hom}\left(\mathsf{Res}(\mathcal{E}_i \otimes V_i^j), \mathsf{Res}\,\mathsf{Ind}_{H_k}^G(\mathcal{E}_k \otimes V_k^l)\right)\right)^{H_i} \\ &= \left(R\,\mathsf{Hom}\left(E_i^1 \otimes V_i^j, \bigoplus_{g \in H_k \backslash G} g^*(E_k^1 \otimes V_k^l)\right)\right)^{H_i} \\ &= \left(R\,\mathsf{Hom}\left(E_i^1 \otimes V_i^j, \bigoplus_{q=1}^{q_k} E_k^q \otimes V_k^l\right)\right)^{H_i} \end{aligned}$$

is zero since every $R\,\mathsf{Hom}$ in the bracket is zero.

It suffices to show that this exceptional collection is full. For any $\mathcal{F} \in \mathsf{D}_G^b(X)$, since $F = \mathsf{Res}\,\mathcal{F}$ is in $\mathsf{D}^b(X)$, there exists i, j, k such that $\mathsf{Hom}(E_i^j, F[k]) \neq 0$. Since G acts on F, one can assume that $j = 1$. With H_i-action, the vector space $\mathsf{Hom}(\mathcal{E}_i, \mathcal{F}[k])$ admits a subspace of irreducible H_i representation V_i^l.

The inclusion gives a non-zero element in

$$\mathsf{Hom}_{H_i}(V_i^l, \mathsf{Hom}(\mathcal{E}_i, \mathcal{F}[k])) = \mathsf{Hom}_{H_i}(\mathcal{E}_i \otimes V_i^l, \mathcal{F}[k]) = \mathsf{Hom}_G(\mathsf{Ind}_{H_i}^G(\mathcal{E}_i \otimes V_i^l), \mathcal{F}[k]).$$

Hence our result follows. □

3 Hilbert schemes

Bridgeland-King-Reid and Haiman's results shows that:

$$\Phi : \mathsf{D}^b(S^{[n]}) \rightleftarrows \mathsf{D}_{\mathfrak{S}_n}^b(S^n) : \Psi$$

are equivalences of categories. Here Φ and Ψ are defined as follows:

Consider the Cartesian diagram of schemes:

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{r} & S^n \\ \downarrow p & \square & \downarrow q \\ S^{[n]} & \xrightarrow{\pi} & S^{(n)} \end{array}$$

Then $\Phi\mathcal{F} = Rr_*p^*\mathcal{F}, \Psi\mathcal{G} = (p_*r^!\mathcal{G})^{\mathfrak{S}_n}$.

As a result, every full exceptional collection of $\mathsf{D}^b(S^{[n]})$ comes from a full exceptional collection of $\mathsf{D}_{\mathfrak{S}_n}^b(S^n)$. When the surface S satisfies that $\mathsf{D}^b(S)$ has a full exceptional collection, Theorem 1 gives a full exceptional collection of $\mathsf{D}_{\mathfrak{S}_n}^b(S^n)$.

Theorem 2. *Assume $E_1, E_2, \ldots, E_m$ form a full exceptional collection of $\mathsf{D}^b(S)$, then the following objects form a full exceptional collection of $D_{\mathfrak{S}_n}^b(S^n)$:*

$$\mathsf{Ind}_{\prod_{i=1}^m \mathfrak{S}_{a_i}}^{\mathfrak{S}_n}\left(\boxtimes_{i=1}^m \left(E_i^{\boxtimes a_i} \otimes V_i^j\right)\right),$$

where $a_1 + \cdots + a_m = n$, V_i^j is an irreducible representation of $\mathfrak{S}_{a_i}$.

Proof. In order to use Theorem 1, one just need to arrange the full exceptional collection of $\mathsf{D}^b(S^n)$ in a correct order.

We assign the lexicographical order on the set of $(a_1, \dots, a_m)$ such that $a_1 + \cdots + a_m = n$. Thus, if $(a_1, \dots, a_m) \prec (b_1, \dots, b_m)$, one of the following is satisfied:

$$\begin{gathered} a_m > b_m, \\ a_{m-1} + a_m > b_{m-1} + b_m, \\ \cdots \\ a_2 + \cdots + a_m > b_2 + \cdots + b_m. \end{gathered}$$

Put exceptional objects $E_{i_1} \boxtimes E_{i_2} \boxtimes \cdots \boxtimes E_{i_n}$ into groups. every group $(a_1, \dots, a_m)$ contains the objects who consist of a_i times E_i.

For any two different objects $F_i = E_{i_1} \boxtimes E_{i_2} \boxtimes \cdots \boxtimes E_{i_n}$ and $F_j = E_{j_1} \boxtimes E_{j_2} \boxtimes \cdots \boxtimes E_{j_n}$ in the same group, there exists a number k such that $i_k > j_k$. Hence $R\,\mathsf{Hom}(F_i, F_j) = 0$.

Consider the object $F_i = E_{i_1} \boxtimes E_{i_2} \boxtimes \cdots \boxtimes E_{i_n}$ belonging to the group $(a_1, \dots, a_m)$ and the object $F_j = E_{j_1} \boxtimes E_{j_2} \boxtimes \cdots \boxtimes E_{j_n}$ belonging to the group $(b_1, \dots, b_m)$. If $(a_1, \dots, a_m) \prec (b_1, \dots, b_m)$, then $a_l + \cdots + a_m > b_l + \cdots + b_m$ for some l and $i_k \geq l > j_k$ for some k. Hence $R\,\mathsf{Hom}(F_i, F_j) = 0$.

Finally, the group $\mathfrak{S}_n$ acts transitively on each group $(a_1, \dots, a_m)$ with the stablizer $\prod_{i=1}^m \mathfrak{S}_{a_i}$. Theorem 1 applies. □

Corollary 3. *If the number of exceptional objects in one of the exceptional collection of $\mathsf{D}^b(S)$ is equal to the dimension of the cohomology ring $H^*(S)$. Then the number of exceptional objects in the exceptional collection of $\mathsf{D}^b(S^{[n]})$ is equal to the dimension of the cohomology ring $H^*(S^{[n]})$.*

Proof. Consider the generating function of both numbers. Note that the condition implies that $H^p(S) = 0$ if p is odd.

On the one hand, Göttsche's formula reads:

$$\sum_{n \geq 0} P_x(S^{[n]}) t^n = \prod_{i \geq 1} \frac{(1 + x^{-1}t^i)^{b^1} (1 + xt^i)^{b^3}}{(1 - x^{-2}t^i)^{b^0} (1 - t^i)^{b^2} (1 - x^2 t^i)^{b^4}},$$

where $P_x(S^{[n]}) = \sum\limits_{i=-2n}^{2n} b^{2n+i}(S^{[n]}) x^i$. Hence

$$\sum_{n \geq 0} \dim H^*(S^{[n]}) t^n = \prod_{i \geq 1} \frac{1}{(1 - t^i)^{\dim H^*(S)}}.$$

On the other hand, the number of exceptional objects in the exceptional collection of $\mathsf{D}^b(S^{[n]})$ is

$$\begin{aligned} &\sum_{a_1 + \cdots + a_m = n} \text{number of irreducible representations of } \prod_{i=1}^m \mathfrak{S}_{a_i} \\ = &\sum_{a_1 + \cdots + a_m = n} P(a_1) \cdots P(a_m), \end{aligned}$$

where $P(a_i)$ is the number of partitions of a_i.

Summing with n, we obtain

$$\sum_{\substack{a_i \geq 0 \\ i=1,\ldots,m}} P(a_1)\cdots P(a_m)t^{a_1+\cdots+a_m} = \left(\sum_{a\geq 0} P(a)t^a\right)^m = \left(\prod_{i\geq 1}\frac{1}{1-t^i}\right)^m.$$

This is just what we want. □

Example. *When S is a Fano surface, then the condition of this corollary satisfies.*