notes on intersection complex

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1 definition

Assume X is a projective complex variety of complex dimension n, we have it's singularities points X_i of complex dimension $\leq i$. Which means we have a filtration(stratification):

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_1 \supseteq X_0$$

such that $X_i \backslash X_{i-1} (1 \leq i \leq n)$ is non-singular complex variety(complex manifold).

Now we can define the *intersection complex* of X to be the complex of sheaves:

$$IC_X = (\tau_{\leq n-1} Ri_{n*} \cdots \tau_{\leq 0} Ri_{1*} \mathbb{Q}_{X - X_{n-1}})[n]$$

where $i_k : X - X_{n-k} \to X - X_{n-k-1}$ is the inclusion.

From the definition, we know the degree this complex range from $-n$ to -1.

We can define the intersection cohomology of this complex variety as $IH^{n+i}(X) = H^i(X, IC_X)$.

2 proposition

More specifically, we can show that when $i > -n$, we have:

$$\mathsf{dim}_{\mathbb{C}} \, \mathsf{supp} \, \mathcal{H}^i(IC_X) < -i$$

Actually, we know every X_k is closed in X. Thus Ri_{k*} doesn't change $X - X_{n-k}$ and we have:

$$IC_X|_{X - X_{n-k}} = (\tau_{\leq k-2} Ri_{k-1*} \cdots \tau_{\leq 0} Ri_{1*} \mathbb{Q}_{X - X_{n-1}})[n]$$

which shows that $\mathcal{H}^{k-1-n}(IC_X)|_{X - X_{n-k}} = 0$ when $k \geq 2$, i.e. $\mathsf{dim}_{\mathbb{C}} \, \mathsf{supp} \, \mathcal{H}^i(IC_X) < -i$ when $i > -n$. And we also get $IC_X|_{X - X_{n-1}} = \mathbb{Q}_{X - X_{n-1}}[n]$.

For a specific integer k, set $j : F = X_{n-k} - X_{n-k-1} \to X - X_{n-k-1}$ be the closed inclusion, $i : U = X - X_{n-k} \to X - X_{n-k-1}$ be the open inclusion. $S_{k+1} = IC_X|_{X - X_{n-k-1}}, S_k = IC_X|_{X - X_{n-k}}$, which means $i^* S_{k+1} = S_k$ and $S_{k+1} = \tau_{\leq k-1-n} Ri_* S_k$.

We have a distinguished triangle:

$$R\Gamma_F(S_{k+1}) \to S_{k+1} \to R\Gamma_U(S_{k+1}) \xrightarrow{+1}$$

applying the functor j^*, we have:

$$j^! S_{k+1} \to j^* S_{k+1} \to j^* Ri_* i^* S_{k+1} \xrightarrow{+1}$$

i.e.

$$j^! S_{k+1} \to j^* \tau_{\le k-1-n} Ri_* S_k \to j^* Ri_* S_k \xrightarrow{+1}$$

However, we have another distinguished triangle:

$$j^* \tau_{\le k-1-n} Ri_* S_k \to j^* Ri_* S_k \to j^* \tau_{\ge k-n} Ri_* S_k \xrightarrow{+1}$$

which shows that $j^! S_{k+1} = (j^* \tau_{\ge k-n} Ri_* S_k)[-1]$.

Now, for any point $u_y : y \to X_{n-k} - X_{n-k-1}$, since $X_{n-k} - X_{n-k-1}$ is a real $2(n-k)$ dimension manifold, we have

$$u_y^! j^! S_{k+1} = u_y^* j^! S_{k+1}[-2(n-k)]$$

thus

$$\mathcal{H}^*_{c,y}(IC_X) = \mathcal{H}^*(u_y^! j^! S_{k+1}) = \mathcal{H}^{*-1-2n+2k}(u_y^* j^* \tau_{\ge k-n} Ri_* S_k) = \mathcal{H}_y^{*-1-2n+2k}(\tau_{\ge k-n} Ri_* S_k).$$

When $* \le n-k$, we have $\mathcal{H}^*_{c,y}(IC_X) = 0$.

For $*$ fixed, $k \ge n - *$, $y \in \bigcup_{k \ge n-*} (X_{n-k} - X_{n-k-1}) = X - X_{*-1}$, we have $\mathcal{H}^*_{c,y}(IC_X) = 0$, i.e. $\mathsf{supp}\, \mathcal{H}^*_c(IC_X) \subseteq X_{*-1}$. This shows that $\mathsf{dim}_{\mathbb{C}}\, \mathsf{supp}\, \mathcal{H}^i_c(IC_X) < i$ when $i < n$.

3 characterize

For a projective complex variety X, the intersection complex IC_X is uniquely determined by the following proposition:

1. IC_X is constructible and $IC_X|_{X-X_{n-1}} = \mathbb{Q}_{X-X_{n-1}}[n]$;
2. $\mathsf{dim}_{\mathbb{C}}\, \mathsf{supp}\, \mathcal{H}^i(IC_X) < -i$ when $i > -n$;
3. $\mathsf{dim}_{\mathbb{C}}\, \mathsf{supp}\, \mathcal{H}^i_c(IC_X) < i$ when $i < n$.

Actually, if we set $S_k = IC_X|_{X-X_{n-k}}$, from the proof above, we can see that S_{k+1} must be the form $\tau_{\le k-1-n} Ri_* S_k$, where $i : X - X_{n-k} \to X - X_{n-k-1}$ is the inclusion. Thus IC_X must be the form we constructed at the beginning.

As we can see, the condition 2 and 3 are dual to each other by Verdier duality. Thus by the uniqueness, we can see the dual of IC_X is precisely IC_X itself. Taking cohomology, we find $IH^i(X) = IH^{2n-i}(X)$.

4 example

Let C be a non-singular curve in $\mathbb{P}^n$ of genus g. It's a topological manifold with cohomology groups:

$$H^0(C) = \mathbb{Q}, \quad H^1(C) = \mathbb{Q}^{2g}, \quad H^2(C) = \mathbb{Q}$$

Now we consider it's projective cone X. Topologically, we know $X = C \times S^2 / C \times \{1\}$, whose cohomology groups are:

$$H^0(X) = \mathbb{Q}, \quad H^1(X) = 0, \quad H^2(X) = \mathbb{Q}, \quad H^3(X) = \mathbb{Q}^{2g}, \quad H^4(X) = \mathbb{Q}$$

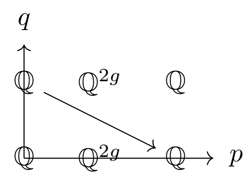
Of course X is not a topological manifold, thus Poincaré duality doesn't hold.

However, X is a complex variety, we have its stratification $X_1 = X_0 = P$ being the cone vertex and its intersection complex $(\tau_{\leq 1} Ri_{2*}\mathbb{Q}_{X-X_0})[2]$.

We know $Ri_{2*}\mathbb{Q}_{X-X_0}|_{X-X_0} = \mathbb{Q}_{X-X_0}$ and

$$(Ri_{2*}\mathbb{Q}_{X-X_0})_P = \varinjlim_{P\in U} \Gamma(U, Ri_{2*}\mathbb{Q}_{X-X_0}) = \varinjlim_{P\in U} R\Gamma(U-P, \mathbb{Q}_{X-X_0})$$

We can choose a well-behaved neighbourhood U to represent this colimit, i.e. $(Ri_{2*}\mathbb{Q}_{X-X_0})_P = R\Gamma(U-P, \mathbb{Q}_{U-P})$ where $U-P$ is homptopy to C with a non-trivial S^1-fibration. Using Serre spectral sequence, we can calculate it's cohomology:



$E_2^{p,q}$

Thus we know $(Ri_{2*}\mathbb{Q}_{X-X_0})_P = (\mathbb{Q} \to \mathbb{Q}^{2g} \to \mathbb{Q}^{2g} \to \mathbb{Q})$ and $\mathcal{H}^{-2}(IC_X) = \mathbb{Q}_X, \mathcal{H}^{-1}(IC_X) = \mathbb{Q}_P^{2g}$.

And the other dimension cohomology sheaves are all zero, which actually satisfies the support condition. Then we can calculate X's intersection cohomology group by the spectral sequence $E_2^{p,q} = H^p(X, \mathcal{H}^q(IC_X)) \Rightarrow H^{p+q}(X, IC_X)$:

$$IH^0(X) = \mathbb{Q}, \quad IH^1(X) = \mathbb{Q}^{2g}, \quad IH^2(X) = \mathbb{Q}, \quad IH^3(X) = \mathbb{Q}^{2g}, \quad IH^4(X) = \mathbb{Q}$$

Now the Poincaré duality holds. Let $f : \tilde{X} \to X$ be the blow-up of X, it's easy to see that $\tilde{X}$ is the $\mathbb{P}^1$-bundle on C, thus $\tilde{X}$ is a complex manifold and $IC_{\tilde{X}} = \mathbb{Q}_{\tilde{X}}[2]$ Since f is proper we know $Rf_*\mathbb{Q}_{\tilde{X}} = (\mathbb{Q}_X \to \mathbb{Q}_P^{2g} \to \mathbb{Q}_P)$ and we have a splitting exact sequence of perverse sheaves:

$$0 \to IC_X \to Rf_* IC_{\tilde{X}} \to (H^2(C))_P \to 0$$

where this exact sequence comes from the distinguished triangle

$$\tau_{\leq 1} R\Gamma(C, \mathbb{Q}_C)[2] \to R\Gamma(C, \mathbb{Q}_C)[2] \to \tau_{\geq 2} R\Gamma(C, \mathbb{Q}_C)[2] \xrightarrow{+1}$$

Because this tiangle splits, we know the previous one also split.

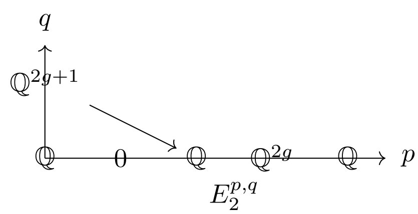
5 another example

Let C be the same as before. let $Y = C \times S^2 / C \times \{1\}$. We have same stratification as before. However, the case is different because $U - P$ is homotopy to C with a trivial S^1-fibration.

Thus we have $(Ri_*\mathbb{Q}_{Y-P})_P = (\mathbb{Q} \to \mathbb{Q}^{2g+1} \to \mathbb{Q}^{2g+1} \to \mathbb{Q})$. After truncated, we know

$$(\mathcal{H}^{-1}(IC_Y))_P = \mathbb{Q}^{2g+1} = H^1(C) \oplus H^0(S^1).$$

In this case, the spectral sequence of $E_2^{p,q} = H^p(Y, \mathcal{H}^q(IC_Y)) \Rightarrow H^{p+q}(Y, IC_Y)$ changed:



Now the intersection cohomology becomes:

$$IH^0(Y) = \mathbb{Q}, \quad IH^1(Y) = \mathbb{Q}^{2g}, \quad IH^2(Y) = 0, \quad IH^3(Y) = \mathbb{Q}^{2g}, \quad IH^4(Y) = \mathbb{Q}$$

From this, we can also find the Poincaré duality remains true.

Now, let $\tilde{Y} = C \times S^2$, we have the quotient map $f : \tilde{Y} \to Y$. Since f is proper and $IC_{\tilde{Y}} = \mathbb{Q}[2]$, we have $Rf_*\mathbb{Q}_{\tilde{Y}} = (\mathbb{Q}_Y \to \mathbb{Q}_P^{2g} \to \mathbb{Q}_P)$.

In this time, we do not have a splitting exact sequence. Instead, we only have $Rf_*IC_{\tilde{Y}}$ be an extension of IC_Y and other sheaves:

$$0 \to \mathsf{ker} \to Rf_*IC_{\tilde{Y}} \to (H^2(C))_P \to 0$$

$$0 \to IC_Y \to \mathsf{ker} \to \mathbb{Q}_P \to 0$$

where the second exact sequence does not split.