STABILITY OF HALF-DEGREE POINT DEFECT PROFILES FOR 2-D NEMATIC LIQUID CRYSTAL

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Abstract. In this paper, we prove the stability of half-degree point defect profiles in $\mathbb{R}^2$ for the nematic liquid crystal within Landau-de Gennes model.

1. **Introduction.** Defects in liquid crystal are known as the places where the degree of symmetry of the nematic order increases so that the molecular direction cannot be well defined. The most striking feature of liquid crystal is a variety of visual defect patterns. Predicting the profiles of defect as well as stability is thus of great practical importance and theoretical interest. We mention some works [2, 16, 23, 27] on the defects based on the topological properties of the order parameter manifolds.

There exist three commonly used continuum theories describing the nematic liquid crystal: Oseen-Frank model, Ericksen model and Landau-de Gennes model. In the Oseen-Frank model, the state of nematic liquid crystals is described by a unit-vector filed which represents the mean local orientation of molecules, and defects are interpreted as all singularities of this vector field [9, 10, 6, 19]. However, the core structure of defects in nematic liquid crystals, such as the disclination lines observed in experiments, cannot be represented by the usual director field and requires description by Landau-de Gennes model [4]. In this model, the state of nematic liquid crystals is described by a 3×3 order tensor Q belonging to

$$\mathcal{Q} = \left\{ Q : Q \in \mathbb{M}^{3 \times 3}, \, Q = Q^T, \, \mathrm{tr} Q = 0 \right\}.$$

For $Q \in \mathcal{Q}$, one can find $s, b \in \mathbb{R}, \mathbf{n}, \mathbf{m} \in \mathbb{S}^2$ with $\mathbf{n} \cdot \mathbf{m} = 0$ such that

$$Q = s(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}) + b(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3}\mathbf{I}),$$

where $\mathbf{I}$ is a 3×3 identity matrix. The local physical properties of nematic liquid crystals depend on the degree of symmetry of order tensor Q. Specifically, there are three different states:

1. $s = b = 0$, which describes the isotropic distribution;
2. $s \neq 0, b = 0$, which corresponds to the uniaxial distribution;
3. $s \neq 0, b \neq 0$, which describes the biaxial distribution.

Configuration of nematic liquid crystals corresponds to local minimizers of the Landau-de Gennes energy functional, whose simplest form is given by

$$\mathcal{F}_{LG}[Q] = \int_\Omega \left\{ \frac{L}{2}|\nabla Q(x)|^2 + f_B(Q(x)) \right\} \mathrm{d}x, \tag{1}$$

where $L > 0$ is a material-dependent elastic constant, and f_B is the bulk energy density, which can be taken as follows

$$f_B(Q) = -\frac{a^2}{2}\mathrm{tr}(Q^2) - \frac{b^2}{3}\mathrm{tr}(Q^3) + \frac{c^2}{4}\mathrm{tr}(Q^2)^2,$$

where a^2, b^2, c^2 are material-dependent and non-zero constants, which may depend on temperature. A well-known fact is that $f_B(Q)$ attains its minimum on a manifold $\mathcal{N}$ given by

$$\mathcal{N} = \left\{ Q \in \mathcal{Q} : Q = s^+(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}), \mathbf{n} \in \mathbb{R}^3, |\mathbf{n}| = 1 \right\},$$

where $s^+ = \frac{b^2 + \sqrt{b^4 + 4a^2c^2}}{4c^2}$. It is easy to see that $\mathcal{N}$ is a smooth submanifold of $\mathcal{Q}$, homemorphic to the real projective plane $\mathbb{RP}^2$, and contained in the sphere $\left\{ Q \in \mathcal{Q} : |Q| = \sqrt{\frac{2}{3}}s^+ \right\}$. Critical points of Landau-de Gennes functional satisfy the Euler-Lagrange equation

$$L\Delta Q = -a^2 Q - b^2(Q^2 - \frac{1}{3}|Q|^2 I) + c^2 Q|Q|^2. \tag{2}$$

The Landau-de Gennes energy (1) and Euler-Lagrange equation (2) are widely used to study the behavior of defects, see [1, 3, 7, 22] and references therein. However, there still exist many challenging problems in understanding the mechanism which generates defects and predicting their profiles as well as stability, see [11] for many conjectures. The radial symmetric solution in a ball or in $\mathbb{R}^3$, named hedgehog solution, is regarded as a potential candidate profile for the isolated point defect in 3-D region. The property and stability of this solution are well studied and it is shown that the radial symmetric solution are not stable for large a^2 and stable for small a^2 [13]. We also refer [26, 21, 17, 12] and references therein for related works.

In this paper, we are concerned with a class of point defects in $\mathbb{R}^2$, which correspond to "radial" solutions of the Euler-Lagrange equation (2). Here "radial" means that the eigenvectors of Q don't change along the radial direction. Precisely speaking, we study the solution with the form

$$Q(r, \varphi) = u(r)F_1 + v(r)F_2, \tag{3}$$

where (r, φ) is the polar coordinate in $\mathbb{R}^2$, and

$$F_1 = 2\mathbf{nn} - \mathbf{I}_2 = \begin{pmatrix} \cos k\varphi & \sin k\varphi & 0 \\ \sin k\varphi & -\cos k\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

$$F_2 = 3\mathbf{e}_3 \otimes \mathbf{e}_3 - \mathbf{I} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{5}$$

The boundary condition on these solutions is taken to be

$$\lim_{r \to +\infty} Q(r, \varphi) = s_+ (\mathbf{n}(\varphi) \otimes \mathbf{n}(\varphi) - \frac{1}{3}\mathbf{I}), \quad \mathbf{n}(\varphi) = (\cos \frac{k}{2}\varphi, \sin \frac{k}{2}\varphi, 0), \tag{6}$$

which has degree $\frac{k}{2}$ about origin as an $\mathbb{RP}^2$-valued map. Here $k \in \mathbb{Z} \setminus \{0\}$. Note that if we assume the invariance of Q along the defect line, then disclination line in 3-D domain can be ideally treated as a point defect in 2-D domain.

In [14], Ignat, Nguyen, Slastikov and Zarnescu proved the existence of the radial solution for any non-zero integer k. Moreover, the solution is also a local minimizer of the 1-dimensional reduced functional (see (9)). An important question is whether the radical solution they constructed is a local minimizer of the energy $\mathcal{F}_{LG}$. This problem was also partially answered in [14], where the instability result is proved for $|k| > 1$. However, the question of whether the k-radially symmetric solutions (3) subject to (6) for $k = \pm 1$ are stable remains open.

The goal of this paper is to give a positive answer to this question. Precise result will be stated in next section. We remark that this problem is somewhat analogous to the stability of radial solutions of the Ginzburg-Landau equation(see [18, 24, 20, 8, 25] for example).

2. **The stability of radially symmetric solution with $k = \pm 1$.** We make the following rescaling

$$\widetilde{Q} = \frac{c^2}{b^2}Q, \quad \widetilde{x} = \sqrt{\frac{2}{L}}\frac{b^2}{c},$$

and let $t = \frac{a^2 c^2}{b^4}$. Then Landau-de Gennes energy functional (1) is rescaled into the form(drop the tildes):

$$\mathcal{F}_{LG}[Q] = \int_\Omega \left\{ \frac{1}{2}|\nabla Q(x)|^2 - \frac{t}{2}\text{tr}(Q^2) - \frac{1}{3}\text{tr}(Q^3) + \frac{1}{4}\text{tr}(Q^2)^2 \right\} \mathrm{d}x.$$

Therefore, without loss of generality, we may take $L = b = c = 1$ and $a^2 = t > 0$.

In such case, substituting (3) into (2), (u, v) satisifes the following ODE system(see [5, 11]):

$$\begin{cases} u'' + \frac{u'}{r} - \frac{k^2}{r^2}u = u[-t + 2v + (6v^2 + 2u^2)], \\ v'' + \frac{v'}{r} = v[-t - v + (6v^2 + 2u^2)] + \frac{1}{3}u^2, \end{cases} \tag{7}$$

together with the boundary conditions

$$u(0) = 0, \quad v'(0) = 0, \quad u(+\infty) = \frac{s^+}{2}, \quad v(+\infty) = -\frac{s^+}{6}, \tag{8}$$

where $s^+ = \frac{1+\sqrt{1+24t}}{4}$. In [14], it has been constructed a solution to (7)-(8) with $u > 0, v < 0$ and

$$v \gtreqless -\frac{s_+}{6} \gtreqless -\frac{1}{6}, \quad \text{for } t \lesseqgtr \frac{1}{3}.$$

Define the 1-D reduced energy density

$$e_1(u, v; r) = (\frac{\partial u}{\partial r})^2 + 3(\frac{\partial v}{\partial r})^2 + \frac{2k^2 u^2}{r^2} - t(u^2 + 3v^2)$$
$$- 2v(v^2 - u^2) + (u^2 + 3v^2)^2.$$

A solution (u, v) of (7)-(8) is called a local minimizer of the 1-D reduced energy of (1) if

$$\mathcal{J}(\eta, \xi) \triangleq \int_0^\infty \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} \Big(e_1(u + \varepsilon\eta, v + \varepsilon\xi; r) - e_1(u, v; r) \Big) r dr \qquad (9)$$
$$= \int_0^\infty \Big\{ (\partial_r \eta)^2 + (\partial_r \xi)^2 + \eta^2(18v^2 + 2u^2 - t - 2v)$$
$$+ \xi^2(6v^2 + 6u^2 - t + 2v + \frac{k^2}{r^2}) + \frac{4u}{\sqrt{3}}(1 + 6v)\eta\xi \Big\} r dr \geq 0$$

for all $\eta, \xi \in C_c^\infty(0, \infty)$. The definition can be extended to $H^1((0, \infty), rdr) \times H^1((0, \infty), rdr) \cap L^2((0, \infty), \frac{1}{r}dr)$, because $C_c^\infty(0, \infty)$ is dense in $H^1((0, \infty), rdr)$. In particular, if (u, v) is a local minimizer of the 1-D reduced energy of (1), then $\mathcal{J}(\eta, \xi) \geq 0$ for all $(\eta, \xi) \in H^1((0, \infty), rdr) \times H^1((0, \infty), rdr) \cap L^2((0, \infty), \frac{1}{r}dr)$.

For $V \in C_c^\infty(\mathbb{R}^2)$, we define

$$\mathcal{I}_Q(V) \triangleq \frac{d^2}{d\varepsilon^2} \int_{\mathbb{R}^2} \Big\{ \frac{1}{2}|\nabla(Q + \varepsilon V)|^2 - \frac{1}{2}|\nabla Q|^2 - \frac{t}{2}(|Q + \varepsilon V|^2 - |Q|^2) \qquad (10)$$
$$- \frac{1}{3}\big[\text{tr}((Q + \varepsilon V)^3) - \text{tr}(Q^3)\big] + \frac{1}{4}(|Q + \varepsilon V|^4 - |Q|^4) \Big\} dx \Big|_{\varepsilon=0}$$
$$= \int_{\mathbb{R}^2} \Big\{ |\nabla V|^2 - t|V|^2 - 2\text{tr}(QV^2) + |Q|^2|V|^2 + 2(\text{tr}(QV))^2 \Big\} dx.$$

The definition in the last line can be extended to all function $V \in H^1(\mathbb{R}^2, \mathcal{Q})$. We say that a solution Q to the Euler-Lagrange equation is a local minimizer of the Landau-de Gennes energy (1) if $I(V) \geq 0$ for all $V \in H^1(\mathbb{R}^2, \mathcal{Q})$.

The main result of this paper is stated as follows.

Theorem 2.1. *Let (u, v) be a solution to (7)-(8) for $k = \pm 1$ with $v < 0$ and $\mathcal{J}(\eta, \xi) \geq 0$ for all $\eta, \xi \in C_c^\infty(0, \infty)$. Then the solution $Q = u(r)F_1 + v(r)F_2$ is a local minimizer of Landau-de Gennes energy (1). That is, for any perturbation $V \in H^1(\mathbb{R}^2, \mathcal{Q})$, it holds*

$$\mathcal{I}_Q(V) = \int_{\mathbb{R}^2} \Big\{ |\nabla V|^2 - t|V|^2 - 2\text{tr}(QV^2) + |Q|^2|V|^2 + 2(\text{tr}(QV))^2 \Big\} dx \geq 0. \quad (11)$$

The equality holds if and only if

$$V = \eta_0 F_1 + \xi_0 F_2 \qquad (12)$$

for some (η_0, ξ_0) satisfying $\mathcal{J}(\eta_0, \xi_0) = 0$.

This result implies that the solution (3) for $k = \pm 1$ can be regarded as the profile of point defects in $\mathbb{R}^2$ or the local profile of line defects in $\mathbb{R}^3$. The proof is based on the following properties of (u, v) on $(0, \infty)$:

(H1) $u > 0$, $u' > 0$;

(H2) $v'(1 + 6v) < 0$ if $t \neq 1/3$; $v' \equiv 1 + 6v \equiv 0$ for $t = 1/3$.

We will prove these properties in Proposition 1.

Remark 1. Since we have the invariance of the energy $\mathcal{F}_{LG}$ under translation:

$$\mathcal{F}_{LG}\big(Q(x+x_0)\big) = \mathcal{F}_{LG}(Q(x)) \quad \text{for any constant } x_0 \in \mathbb{R}^2,$$

as well as the invariance of the energy $\mathcal{F}_{LG}$ under rotation:

$$\mathcal{F}_{LG}(RQR^T) = \mathcal{F}_{LG}(Q) \quad \text{for any constant } R \in SO(3),$$

one can obtain the following functions in the "null space":

$$\partial_x Q = (u'(r)E_1 + \sqrt{3}v'(r)E_0)\cos\varphi - \frac{ku}{r}E_2\sin\varphi,$$

$$\partial_y Q = (u'(r)E_1 + \sqrt{3}v'(r)E_0)\sin\varphi + \frac{ku}{r}E_2\cos\varphi,$$

$$u(r)E_2, \quad (u(r)\cos k\varphi - 3v(r))E_3 + u(r)\sin k\varphi E_4,$$

$$u(r)\sin k\varphi E_3 - (u(r)\cos k\varphi + 3v(r))E_4.$$

Here $E_0, E_1, \cdots, E_4$ are defined in (13). Although these functions do not belong to $H^1(\mathbb{R}^2)$, they inspire us to construct suitable identities to prove the theorem.

Remark 2. The result (with slight difference) have been independently obtained in the work [15]. They also considered the uniqueness of solution of (7)-(8) for t large.

3. **The second variation of Landau-de Gennes energy.** To prove the stabillity, we need to compute the second variation of $\mathcal{F}_{LG}$ at critical point $Q = u(r)F_1 + v(r)F_2$. For any $V \in H^1(\mathbb{R}^2, \mathcal{Q})$, we have

$$\mathcal{I}(V) = \int_{\mathbb{R}^2} \Big\{|\nabla V|^2 - t|V|^2 - 2\mathrm{tr}(QV^2) + |Q|^2|V|^2 + 2(\mathrm{tr}(QV))^2\Big\}\mathrm{d}x$$

$$= \int_{\mathbb{R}^2} \Big\{|\nabla V|^2 - t|V|^2 - 2\big(u\,\mathrm{tr}(F_1V^2) + v\,\mathrm{tr}(F_2V^2)\big)$$

$$+ (6v^2 + 2u^2)|V|^2 + 2\big(u\,\mathrm{tr}(F_1V) + v\,\mathrm{tr}(F_2V)\big)^2\Big\}\mathrm{d}x.$$

We define

$$E_0 = \sqrt{\frac{3}{2}}\begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix} = \frac{1}{\sqrt{6}}F_2,$$

$$E_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos k\varphi & \sin k\varphi & 0 \\ \sin k\varphi & -\cos k\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}F_1,$$

$$E_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin k\varphi & \cos k\varphi & 0 \\ \cos k\varphi & \sin k\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{13}$$

$$E_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

A straightforward calculation shows

$$\mathrm{tr}(E_iE_j) = \delta_j^i \quad \text{for } 0 \le i, j \le 4,$$

which implies that $\{E_i\}_{0 \le i \le 4}$ is an orthonormal basis in $\mathcal{Q}$. Thus, we can write $V \in H^1(\mathbb{R}^2, \mathcal{Q})$ as a linear combination of this basis in the polar coordinate

$$V(r, \varphi) = \sum_{i=0}^{4} w_i(r, \varphi) E_i(\varphi). \tag{14}$$

Using (14), a direct calculation yields that

$$|V|^2 = \sum_{i=0}^{4} w_i^2, \quad \mathrm{tr}(F_1 V) = \sqrt{2} w_1, \quad \mathrm{tr}(F_2 V) = \sqrt{6} w_0,$$

$$V^2 = w_0^2 E_0^2 + (w_1^2 + w_2^2) E_1^2 + w_3^2 E_3^2 + w_4^2 E_4^2 + (w_1 w_3 + w_2 w_4)(E_1 E_3 + E_3 E_1)$$

$$+ (w_1 w_4 + w_2 w_3)(E_1 E_4 + E_4 E_1) + w_3 w_4 (E_3 E_4 + E_4 E_3) - \frac{\sqrt{6}}{3} w_0 w_1 E_1$$

$$- \frac{\sqrt{6}}{3} w_0 w_2 E_2 + \frac{\sqrt{6}}{6} w_0 w_3 E_3 + \frac{\sqrt{6}}{6} w_0 w_4 E_4,$$

thus we obtain

$$\mathrm{tr}(F_1 V^2) = \frac{\cos k\varphi}{2} (w_3^2 - w_4^2) - \frac{2\sqrt{3}}{3} w_0 w_1 + \sin k\varphi w_3 w_4,$$

$$\mathrm{tr}(F_2 V^2) = w_0^2 - w_1^2 - w_2^2 + \frac{1}{2} w_3^2 + \frac{1}{2} w_4^2.$$

From the fact $|\nabla V|^2 = (\partial_r V)^2 + \frac{1}{r^2}(\partial_\varphi V)^2$, we have

$$|\nabla V|^2 = \sum_{i=0}^{4} w_{ir}^2 + \frac{1}{r^2} \left(w_{0\varphi}^2 + (kw_2 - w_{1\varphi})^2 + (kw_1 + w_{2\varphi})^2 + w_{3\varphi}^2 + w_{4\varphi}^2 \right).$$

Here f_r and f_φ denote $\partial_r f$ and $\partial_\varphi f$ respectively.

In summary, we conclude that

$$\mathcal{I}(V) = \int_0^{+\infty} \int_0^{2\pi} \Big\{ \sum_{i=0}^{4} w_{ir}^2 + \frac{1}{r^2} \big[w_{0\varphi}^2 + (kw_2 - w_{1\varphi})^2 + (kw_1 + w_{2\varphi})^2 \tag{15}$$

$$+ w_{3\varphi}^2 + w_{4\varphi}^2 \big] + (6v^2 + 2u^2 - t) \big(\sum_{i=0}^{4} w_i^2 \big) - 2 \big[u \big(\frac{1}{2} \cos k\varphi (w_3^2 - w_4^2) \big.$$

$$- \frac{2}{\sqrt{3}} w_0 w_1 + \sin k\varphi w_3 w_4 \big) + v(w_0^2 - w_1^2 - w_2^2 + \frac{1}{2} w_3^2 + \frac{1}{2} w_4^2) \big]$$

$$+ 4(uw_1 + \sqrt{3} v w_0)^2 \Big\} r dr d\varphi.$$

In order to prove $\mathcal{I}(V) \ge 0$, it suffices to show that for $|k| = 1$,

$$I^A(w_0, w_1, w_2) \triangleq \int_0^{+\infty} \int_0^{2\pi} \Big\{ w_{0r}^2 + w_{1r}^2 + w_{2r}^2 + \frac{1}{r^2} [w_{0\varphi}^2 + (kw_2 - w_{1\varphi})^2$$

$$+ (kw_1 + w_{2\varphi})^2] + (6v^2 + 2u^2 - t)(w_0^2 + w_1^2 + w_2^2) - \frac{4}{\sqrt{3}} u w_0 w_1$$

$$+ v(w_0^2 - w_1^2 - w_2^2)] + 4(uw_1 + \sqrt{3} v w_0)^2 \Big\} r dr d\varphi \ge 0, \tag{16}$$

and

$$I^B(w_3, w_4) \triangleq \int_0^{+\infty} \int_0^{2\pi} \left\{ w_{3r}^2 + w_{4r}^2 + \frac{1}{r^2}(w_{3\varphi}^2 + w_{4\varphi}^2) \right.$$
$$+ (6v^2 + 2u^2 - t)(w_3^2 + w_4^2) - u(\cos k\varphi(w_3^2 - w_4^2)$$
$$+ 2\sin k\varphi w_3 w_4) - v(w_3^2 + w_4^2) \Big\} r\, dr\, d\varphi \geq 0. \tag{17}$$

The following lemma shows that $C_c^\infty(\mathbb{R}^2\backslash\{0\})$ is dense in $H^1(\mathbb{R}^2)$. Thus, we may assume $V \in C_c^\infty(\mathbb{R}^2\backslash\{0\})$, hence $w_i \in C_c^\infty(\mathbb{R}^2\backslash\{0\})$.

Lemma 3.1. $C_c^\infty(\mathbb{R}^2\backslash\{0\})$ *is dense in* $H^1(\mathbb{R}^2)$.

Proof. Since $C_c^\infty(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, it suffices to show that any $C_c^\infty(\mathbb{R}^2)$ function can be approximated by $C_c^\infty(\mathbb{R}^2\backslash\{0\})$. For this, we introduce a smooth cut-off function $\chi(r)$ defined by

$$\chi(r) = \begin{cases} 0 & r \leq -2 \\ 1 & r \geq -1. \end{cases}$$

For any $u \in C_c^\infty(\mathbb{R}^2)$, let $u_N(x) = u(x)\chi(\frac{\ln|x|}{N})$ for $N \geq 1$. Obviously, $u_N \in C_c^\infty(\mathbb{R}^2\backslash\{0\})$. Moreover,

$$\|u - u_N\|_{H^1} \leq \left\|u(1 - \chi(\frac{\ln|x|}{N}))\right\|_{L^2} + \left\|\nabla u(1 - \chi(\frac{\ln|x|}{N}))\right\|_{L^2}$$
$$+ \frac{1}{N}\left\|\frac{u}{|x|}\chi'(\frac{\ln|x|}{N}))\right\|_{L^2}.$$

It is easy to see that the first two terms on the right hand side tend to zero as $N \to +\infty$. While, the third term is bounded by

$$\frac{C}{N}\left(\int_{e^{-2N} \leq |x| \leq e^{-N}} \frac{|u(x)|^2}{|x|^2}\mathrm{d}x\right)^{\frac{1}{2}} \leq \frac{C}{\sqrt{N}}\|u\|_{L^\infty},$$

which tends to zero as $N \to +\infty$. $\qquad\square$

Remark 3. Using the same argument, we know that $C_c^\infty((0,\infty))$ is dense in $H^1((0,\infty), r\, dr)$.

4. **Some important integral identities.** In this section, let us derive some important integral identities, which will play crucial roles in our proof. In the sequel, we assume that $\eta \in C_c^\infty((0,+\infty), \mathbb{R})$.

Using (7), we deduce that

$$\mathcal{A}(\eta) := \int_0^\infty \left\{ (v\eta)_r^2 + (6v^2 + 2u^2 - t - v)(v\eta)^2 \right\} r\, dr$$
$$= \int_0^\infty \left\{ (v\eta)_r^2 + (v'' + \frac{v'}{r} - \frac{1}{3}u^2)v\eta^2 \right\} r\, dr$$
$$= (vv'\eta^2 r)|_0^\infty + \int_0^\infty (v^2\eta_r^2 - \frac{1}{3}vu^2\eta^2)r\, dr$$
$$= \int_0^\infty \left\{ (v\eta_r)^2 - \frac{1}{3}vu^2\eta^2 \right\} r\, dr, \tag{18}$$

and

$$\mathcal{B}(\eta) := \int_0^\infty \left\{ (u\eta)_r^2 + (6v^2 + 2u^2 - t + 2v + \frac{k^2}{r^2})(u\eta)^2 \right\} r dr$$

$$= \int_0^\infty [(\eta' u + u'\eta)^2 + u\eta^2(u'' + \frac{u'}{r})] r dr$$

$$= (uu'\eta^2 r)|_0^\infty + \int_0^\infty u^2\eta_r^2 r dr$$

$$= \int_0^\infty (u\eta_r)^2 r dr. \tag{19}$$

Taking derivative to (7) gives

$$u''' + \frac{u''}{r} - \frac{2u'}{r^2} + \frac{2u}{r^3} = u'[-t + 2v + 6v^2 + 6u^2] + 2uv'(1 + 6v),$$

$$v''' + \frac{v''}{r} - \frac{v'}{r^2} = v'[-t - 2v + 18v^2 + 2u^2] + \frac{2uu'}{3}(1 + 6v).$$

Therefore, we have

$$\mathcal{C}(\eta) := \int_0^\infty \left\{ (v'\eta)_r^2 + (v'\eta)^2(18v^2 + 2u^2 - t - 2v) \right\} r dr$$

$$= \int_0^\infty \left\{ (v''\eta + v'\eta')^2 + v'\eta^2(v''' + \frac{v''}{r} - \frac{v'}{r^2} - \frac{2uu'}{3}(1 + 6v)) \right\} r dr$$

$$= \int_0^\infty \left\{ (v'\eta')^2 - \frac{(v'\eta)^2}{r^2} - \frac{2uu'v'(1 + 6v)}{3}\eta^2 \right\} r dr + (rv''v'\eta^2)|_0^\infty$$

$$= \int_0^\infty \left\{ (v'\eta')^2 - \frac{(v'\eta)^2}{r^2} - \frac{2uu'v'(1 + 6v)}{3}\eta^2 \right\} r dr, \tag{20}$$

and

$$\mathcal{D}(\eta) := \int_0^\infty \left\{ (u'\eta)_r^2 + (u'\eta)^2(6v^2 + 6u^2 - t + 2v + \frac{k^2}{r^2}) \right\} r dr$$

$$= \int_0^\infty \left\{ (u''\eta + u'\eta')^2 + u'\eta^2\left(u''' + \frac{u''}{r} - \frac{u'}{r^2} + \frac{2u}{r^3} - 2uv'(1 + 6v)\right) \right\} r dr$$

$$= \int_0^\infty \left\{ (u'\eta')^2 - \frac{(u'\eta)^2}{r^2} + \frac{2uu'\eta^2}{r^3} - 2uu'v'(1 + 6v)\eta^2 \right\} r dr + (ru''u'\eta^2)|_0^\infty$$

$$= \int_0^\infty \left\{ (u'\eta')^2 - \frac{(u'\eta)^2}{r^2} + \frac{2uu'\eta^2}{r^3} - 2uu'v'(1 + 6v)\eta^2 \right\} r dr. \tag{21}$$

In addition, we have

$$\mathcal{E}(\eta) := \int_0^\infty \left\{ (\frac{u\eta}{r})_r^2 + (6v^2 + 2u^2 - t + 2v + \frac{k^2}{r^2})\frac{(u\eta)^2}{r^2} \right\} r dr$$

$$= \int_0^\infty [(\frac{u\eta'}{r} + (\frac{u}{r})'\eta)^2 + \frac{u\eta^2}{r^2}(u'' + \frac{u'}{r})] r dr$$

$$= \int_0^\infty \left((\frac{u}{r}\eta_r)^2 + \frac{\eta^2}{r^4}(2ruu' - u^2) \right) r dr + (\frac{u}{r})'u\eta^2|_0^\infty$$

$$= \int_0^\infty \left((\frac{u}{r}\eta_r)^2 + \frac{\eta^2}{r^4}(2ruu' - u^2) \right) r dr. \tag{22}$$

5. **Monotonicity of u and v.** In this section, we will prove that u and v are monotonic if (u, v) is a local minimizer for the 1-D reduced energy functional. The proof is based on several contradiction arguments, which is similar to the work [17] on the monotonicity of the scalar order parameter of uniaxial radial hedgehog solution.

Proposition 1. *If (u, v) is a solution to (7)-(8) and satisfies $\mathcal{J}(\zeta, \xi) \geq 0$ for all $\zeta, \xi \in C_c^\infty(0, \infty)$, then it holds for all $r > 0$ that*

$$u'(r) > 0, \quad u(r) > 0, \quad \mathrm{sgn}(v'(r)) = -\mathrm{sgn}(1 + 6v(r)) = \mathrm{sgn}(t - 1/3).$$

Proof. Let

$$p(r) = uu', \quad q(r) = -v'(1 + 6v).$$

First of all, we will show that $p(r)$ and $q(r)$ are nonnegative. If $\{p(r) < 0\} \cup \{q(r) < 0\} \neq \emptyset$, we let

$$\chi = \mathbf{1}_{\{p(r)<0\}}, \qquad \eta = -\sqrt{3}\mathbf{1}_{\{q(r)<0\}}.$$

Take $\zeta = v'\eta$ and $\xi = u'\chi$. Formally, it follows from (20) and (21) that $\mathcal{J}(\zeta, \xi) \geq 0$ where

$$
\begin{aligned}
\mathcal{J}(\zeta, \xi) &= \int_0^\infty \left\{ (v'\eta')^2 - \frac{(v'\eta)^2}{r^2} - \frac{2uu'v'(1+6v)}{3}\eta^2 \right\} r\,dr + (rv''v'\eta^2)\big|_0^\infty \\
&\quad + \int_0^\infty \left\{ (u'\chi')^2 - \frac{(u'\chi)^2}{r^2} + \frac{2uu'\chi^2}{r^3} - 2uu'v'(1+6v)\chi^2 \right\} r\,dr \\
&\quad + (ru''u'\chi^2)\big|_0^\infty + \int_0^\infty \left\{ \frac{4}{\sqrt{3}}uu'v'(1+6v)\chi\eta \right\} r\,dr \\
&= \int_0^\infty \left\{ (v'\eta')^2 - \frac{(v'\eta)^2}{r^2} - \frac{2uu'v'(1+6v)}{3}(\frac{\eta}{\sqrt{3}} + \chi)^2 \right\} r\,dr + (rv''v'\eta^2)\big|_0^\infty \\
&\quad + \int_0^\infty \left\{ (u'\chi')^2 - \frac{(u'\chi)^2}{r^2} + \frac{2uu'\chi^2}{r^3} \right\} r\,dr + (ru''u'\chi^2)\big|_0^\infty \\
&= \int_0^\infty \left\{ (v'\eta')^2 - \frac{(v'\eta)^2}{r^2} + \frac{2pq}{3}(\frac{\eta}{\sqrt{3}} + \chi)^2 + (u'\chi')^2 \right. \\
&\quad \left. - \frac{(u'\chi)^2}{r^2} + \frac{2p\chi^2}{r^3} \right\} r\,dr.
\end{aligned}
$$

Notice that

$$\int_0^\infty (v'\eta')^2 r\,dr = \int_0^\infty (u'\chi')^2 r\,dr = 0,$$

$$-\int_0^\infty \left\{ \frac{(v'\eta)^2}{r^2} + \frac{(u'\chi)^2}{r^2} \right\} r\,dr < 0,$$

$$\int_0^\infty pq(\frac{\eta}{\sqrt{3}} + \chi)^2 r\,dr = \int_{\{p<0,q<0\}} pq(\frac{\eta}{\sqrt{3}} + \chi)^2 r\,dr + \int_{\{p<0,q>0\}} pq(\frac{\eta}{\sqrt{3}} + \chi)^2 r\,dr$$

$$+ \int_{\{p>0,q<0\}} pq(\frac{\eta}{\sqrt{3}} + \chi)^2 r\,dr \leq 0,$$

$$\int_0^\infty \frac{2p\chi^2}{r^3} r\,dr = \int_{\{p<0\}} \frac{2p\chi^2}{r^3} r\,dr \leq 0,$$

which contradict with $\mathcal{J}(\zeta, \xi) \geq 0$. Thus, we deduce that

$$uu'(r) \geq 0, \quad v'(r)(1 + 6v) \leq 0 \quad \text{for } r > 0.$$

Thanks to the fact that $p(r) = (u^2)'/2 \geq 0$, $q(r) \geq 0$ for r small, and $u' = 0$ on $p(r) = 0$, $v' = 0$ on $q(r) = 0$, the above formal derivation can be justified by a standard smoothing procedure and cutoff argument.

Next, we prove that $u'(r) > 0$ for $r > 0$. Otherwise, $u'(r_0) = 0$ for some $r_0 > 0$. Since $(u^2)' \geq 0$, we have that $u(r) \geq 0$ for all $r > 0$. Thus, it holds $u'(r) \geq 0$ which implies $u''(r_0) = 0$. On the other hand, we have

$$u''' + \frac{u''}{r} - \frac{2u'}{r^2} + \frac{2u}{r^3} = u'\big(-t + 2v + 6v^2 + 6u^2\big) + 2uv'(1 + 6v).$$

Taking $r = r_0$, we get

$$u'''(r_0) = u(-\frac{2}{r^3} + 2v'(1 + 6v)) < 0,$$

which contradicts with $u'(r) \geq 0$. Thus, $u' > 0$ for all $r > 0$. This also implies $u(r) > 0$ for $r > 0$.

We turn to study the sign of v'. Assume that there exists $r_0 > 0$ such that $v'(r_0) = 0$. If $1 + 6v(r_0) > 0$, then we know that $v'(r) \leq 0$ for r near r_0. Thus, we can deduce that $v''(r_0) = 0$, $v'''(r_0) \leq 0$. Consider the equation

$$v''' + \frac{v''}{r} - \frac{v'}{r^2} = v'\big(-t - 2v + 18v^2 + 2u^2\big) + \frac{2uu'}{3}(1 + 6v)$$

on $r = r_0$. The left hand side is non-positive, while the right hand side is positive. This contradiction implies that if $v'(r_0) = 0$ for $r_0 > 0$, then $1 + 6v(r_0)$ can not be positive. Using a similar argument, we can also prove that $1 + 6v(r_0)$ can not be negative. Thus, $1 + 6v(r_0) = 0$.

When $t > 1/3$, we have $v(+\infty) = -s_+/6 = -(1 + \sqrt{1 + 24t})/24 < -1/6$. Choose $r_0 = \sup\{r : v(r) = -1/6\}$. Then $1 + 6v(r) < 0$ on $(r_0, +\infty)$. This implies $v'(r) \geq 0$ on $(r_0, +\infty)$, which contradicts with the fact that $v(+\infty) < v(r_0)$. Thus, v' has no zero point on $(0, +\infty)$. From the fact that $v(+\infty) < -1/6$ and $v'(1 + 6v) \geq 0$, we have $v'(r) > 0$ for $r > 0$. Similarly, we can prove that $v'(r) < 0$ for $r > 0$ when $t < 1/3$.

For $t = 1/3$, we show $v \equiv -1/6$. If not, consider $w = v + 1/6$, then we have that $\sup_{r \in [0,+\infty)} w(r)$, $\inf_{r \in [0,+\infty)} w(r)$ can not both be zero. If $w(r_1) = \sup_{r \in [0,+\infty)} w(r) > 0$, then we have $v'(r_1) = w'(r_1) = 0$ whenever $r_1 > 0$ or $r_1 = 0$. Then by the previous argument, we have $w(r_1) = 0$, which is a contradiction. Similar discussion can be applied to the case of $\inf_{r \in [0,+\infty)} w(r) < 0$. Thus, $w \equiv 0$ and then $v \equiv -1/6$. The proof is finished. $\square$

6. **Proof of Theorem 2.1.** This section is devoted to the proof of Theorem 2.1.

6.1. **Non-negativity of $I^B(w_3, w_4)$.**

Proposition 2. *For any $w_3, w_4 \in C_c^\infty(\mathbb{R}^2 \backslash \{0\})$ and $|k| = 1$, we have*

$$I^B(w_3, w_4) := \int_0^{+\infty} \int_0^{2\pi} \Big\{ w_{3r}^2 + w_{4r}^2 + \frac{1}{r^2}(w_{3\varphi}^2 + w_{4\varphi}^2) + (6v^2 + 2u^2 - t)(w_3^2 + w_4^2)$$

$$- u(\cos k\varphi(w_3^2 - w_4^2) + 2\sin k\varphi w_3 w_4) - v(w_3^2 + w_4^2) \Big\} r \, dr \, d\varphi \geq 0.$$

Proof. Let $z = w_3 + iw_4$, $i = \sqrt{-1}$. Then we have

$$|\partial_r z|^2 = (\partial_r w_3)^2 + (\partial_r w_4)^2, \quad |\partial_\varphi z|^2 = (\partial_\varphi w_3)^2 + (\partial_\varphi w_4)^2, \quad |z|^2 = w_3^2 + w_4^2,$$

$$\cos k\varphi (w_3^2 - w_4^2) + 2\sin k\varphi w_3 w_4$$

$$= Re(\cos k\varphi - i\sin k\varphi)(w_3 + iw_4)^2 = Re(e^{-ik\varphi} z^2).$$

Thus, we can rewrite $I^B(w_3, w_4)$ as

$$I^B(z) = \int_0^{2\pi} \left\{ \int_0^\infty |\partial_r z|^2 + \frac{1}{r^2}|\partial_\varphi z|^2 + (6v^2 + 2u^2 - t - v)|z|^2 - uRe(e^{-ik\varphi} z^2) \right\} r dr d\varphi. \tag{23}$$

Assume that

$$z(r, \varphi) = \sum_{m=-\infty}^{+\infty} z_m(r) e^{im\varphi},$$

we have

$$Re(e^{-ik\varphi} z^2) = Re\Big(e^{-ik\varphi} \sum_{l,m=-\infty}^{+\infty} e^{(m+l)\varphi} z_m z_l\Big) = Re\Big(\sum_{l,m=-\infty}^{+\infty} e^{i(m+l-k)\varphi} z_m z_l\Big).$$

Substituting it into (23), we get

$$I^B(z) = 2\pi \int_0^{+\infty} \left\{ \sum_{m=-\infty}^{+\infty} \left[|\partial_r z_m|^2 + \frac{m^2}{r^2}|z_m|^2 + (6v^2 + 2u^2 - t - v)|z_m|^2 \right] \right.$$

$$\left. - u \sum_{m+l=k} Re(z_m z_l) \right\} r dr.$$

When $k = 1$, we can write

$$I^B(z) = 2\pi \sum_{m=1}^\infty M_m,$$

where

$$M_m = \int_0^\infty \left\{ |\partial_r z_m|^2 + |\partial_r z_{1-m}|^2 + \frac{1}{r^2}(m^2|z_m|^2 + (1-m)^2|z_{1-m}|^2) \right.$$

$$\left. + (6v^2 + 2u^2 - t - v)(|z_m|^2 + |z_{1-m}|^2) - 2uRe(z_m z_{1-m}) \right\} r dr.$$

Noticing that $m^2 \geq 1, (1-m)^2 \geq 0$ for $m \geq 1$, and using the following simple relations

$$|z_m z_{1-m}| \geq Re(z_m z_{1-m}), \quad |\partial_r z_m|^2 \geq (\partial_r |z_m|)^2, \quad |\partial_r z_{1-m}|^2 \geq (\partial_r |z_{1-m}|)^2,$$

we conclude that

$$M_m \geq \int_0^\infty \Big((\partial_r |z_m|)^2 + (\partial_r |z_{1-m}|)^2 + \frac{1}{r^2}|z_m|^2$$

$$+ (6v^2 + 2u^2 - t - v)(|z_m|^2 + |z_{1-m}|^2) - 2u|z_m||z_{1-m}|\Big) r dr.$$

Thus, we only need to show that for any $q_0, q_1 \in C_c^\infty((0, \infty), \mathbb{R})$,

$$\tilde{I}(q_0, q_1) \triangleq \int_0^\infty \left\{ (\partial_r q_0)^2 + (\partial_r q_1)^2 + \frac{q_1^2}{r^2} + (6v^2 + 2u^2 - t - v)(q_0^2 + q_1^2) - 2uq_0 q_1 \right\} r dr > 0.$$

Let $\eta = q_1/u$ and $\zeta = q_0/v$. Then $\eta, \zeta \in C_c^\infty((0, \infty))$. From (18) and (19), it is straightforward to obtain

$$\tilde{I}(q_0, q_1) = \mathcal{A}(\zeta) + \mathcal{B}(\eta) - \int_0^\infty \left\{ 3v(u\eta)^2 + 2vu^2\zeta\eta \right\} r dr$$

$$= \int_0^\infty \left\{ (u\eta')^2 + (v\zeta')^2 - \frac{1}{3} vu^2(3\eta + \zeta)^2 \right\} r dr, \qquad (24)$$

which is non-negative since $v < 0$.

The case of $k = -1$ can be considered similarly. $\qquad\qquad\square$

6.2. **Non-negativity of $I^A(w_0, w_1, w_2)$.** First of all, we expand $w_i(r, \varphi)$ as

$$w_i(r, \varphi) = \sum_{n=0}^\infty \left(\mu_n^{(i)}(r) \cos n\varphi + \nu_n^{(i)} \sin n\varphi \right).$$

Since $\omega_i \in C_c^\infty(\mathbb{R}^2 \backslash \{0\})$, we may assume $\mu_n^{(i)}, \nu_n^{(i)} \in C_c^\infty((0, \infty), \mathbb{R})$ for all n and i. Furthermore, $w_0 \in C_c^\infty(\mathbb{R}^2)$ requires $\mu_n^{(0)} = \nu_n^{(0)} = 0$ for $n \geq 1$.

Direct calculation shows that

$$\int_0^{2\pi} (\frac{\partial w_i}{\partial r})^2 d\varphi = \pi \left[2(\frac{\partial \mu_0^{(i)}}{\partial r})^2 + \sum_{n=1}^\infty ((\frac{\partial \mu_n^{(i)}}{\partial r})^2 + (\frac{\partial \nu_0^{(i)}}{\partial r})^2) \right],$$

$$\int_0^{2\pi} (\frac{\partial w_i}{\partial \varphi})^2 d\varphi = \pi \sum_{n=1}^\infty n^2 ((\mu_n^{(i)})^2 + (\nu_n^{(i)})^2),$$

$$\int_0^{2\pi} \frac{\partial w_i}{\partial \varphi} w_j d\varphi = \pi \sum_{n=1}^\infty n (\mu_n^{(j)} \nu_n^{(i)} - \nu_n^{(j)} \mu_n^{(i)}),$$

$$\int_0^{2\pi} w_i^2 d\varphi = \pi \left[2(\mu_0^{(i)})^2 + \sum_{n=1}^\infty ((\mu_n^{(i)})^2 + (\nu_n^{(i)})^2) \right],$$

$$\int_0^{2\pi} w_i w_j d\varphi = \pi \left[2\mu_0^{(i)} \mu_0^{(j)} + \sum_{n=1}^\infty (\mu_n^{(i)} \mu_n^{(j)} + \nu_n^{(i)} \nu_n^{(j)}) \right].$$

Thus, we can decompose $I^A(w_0, w_1, w_2)$ as

$$I^A(w_0, w_1, w_2) = I_{0,01}^A + I_{0,2}^A + \sum_{n=1}^\infty I_n^A, \qquad (25)$$

where

$$I_{0,01}^A = \int_0^\infty \left\{ (\frac{\partial \mu_0^{(0)}}{\partial r})^2 + (\mu_0^{(0)})^2 (18v^2 + 2u^2 - t - 2v) + (\frac{\partial \mu_0^{(1)}}{\partial r})^2 \right.$$

$$\left. + (\mu_0^{(1)})^2 (6v^2 + 6u^2 - t + 2v + \frac{k^2}{r^2}) + \frac{4u}{\sqrt{3}} (1 + 6v) \mu_0^{(0)} \mu_0^{(1)} \right\} r dr,$$

$$I_{0,2}^A = \int_0^\infty \left\{ (\frac{\partial \mu_0^{(2)}}{\partial r})^2 + (\mu_0^{(2)})^2 (6v^2 + 2u^2 - t + 2v + \frac{k^2}{r^2}) \right\} r dr,$$

and

$$I_n^A = \int_0^\infty \left\{ \sum_{i=0}^2 \left((\frac{\partial \mu_n^{(i)}}{\partial r})^2 + (\frac{\partial \nu_n^{(i)}}{\partial r})^2 \right) + \frac{4kn}{r^2} (\mu_n^{(1)} \nu_n^{(2)} - \mu_n^{(2)} \nu_n^{(1)}) \right.$$

$$+ \sum_{i=0}^{2} \frac{n^2}{r^2}((\mu_n^{(i)})^2 + (\nu_n^{(i)})^2 + ((\mu_n^{(0)})^2 + (\nu_n^{(0)})^2)(18v^2 + 2u^2 - t - 2v)$$

$$+ ((\mu_n^{(1)})^2 + (\nu_n^{(1)})^2)(6v^2 + 6u^2 - t + 2v + \frac{k^2}{r^2})$$

$$+ ((\mu_n^{(2)})^2 + (\nu_n^{(2)})^2)(6v^2 + 2u^2 - t + 2v + \frac{k^2}{r^2})$$

$$+ \frac{4u}{\sqrt{3}}(1 + 6v)(\mu_n^{(0)}\mu_n^{(1)} + \nu_n^{(0)}\nu_n^{(1)})\Big\}rdr.$$

From the assumption of local stability of (u, v), we have

$$I_{0,01}^A = \mathcal{J}(\mu_0^{(0)}, \mu_0^{(1)}) \geq 0. \tag{26}$$

In addition, it follows from (19) that

$$I_{0,2}^A = \mathcal{B}(\mu_0^{(2)}/u) = \int_0^\infty \left(u\partial_r(\mu_0^{(2)}/u)\right)^2 rdr \geq 0. \tag{27}$$

It remains to prove $I_n^A \geq 0$ for all $n \geq 1$, which is a consequence of the following proposition.

Proposition 3. *For any* $\mu_0, \nu_0, \mu_1, \nu_1, \mu_2, \nu_2 \in C_c^\infty((0, \infty))$ *and* $|k| = 1$, *we have*

$$I_n^A(\mu_0, \nu_0, \mu_1, \nu_1, \mu_2, \nu_2)$$

$$\triangleq \int_0^\infty \Big\{ \sum_{i=0}^{2} \left((\frac{\partial \mu_i}{\partial r})^2 + (\frac{\partial \nu_i}{\partial r})^2\right) + \frac{4kn}{r^2}(\mu_1\nu_2 - \mu_2\nu_1) + \sum_{i=0}^{2} \frac{n^2}{r^2}(\mu_i^2 + \nu_i^2)$$

$$+ (\mu_0^2 + \nu_0^2)(18v^2 + 2u^2 - t - 2v) + (\mu_1^2 + \nu_1^2)(6v^2 + 6u^2 - t + 2v + \frac{k^2}{r^2})$$

$$+ (\mu_2^2 + \nu_2^2)(6v^2 + 2u^2 - t + 2v + \frac{k^2}{r^2}) + \frac{4u}{\sqrt{3}}(1 + 6v)(\mu_0\mu_1 + \nu_0\nu_1)\Big\}rdr \geq 0.$$

Proof. From the fact that

$$2(\mu_1\nu_2 - \mu_2\nu_1) \geq -(\mu_1^2 + \nu_1^2 + \mu_2^2 + \nu_2^2),$$

and $n \geq 1$, we get

$$4n(\mu_1\nu_2 - \mu_2\nu_1) + n^2(\mu_1^2 + \nu_1^2 + \mu_2^2 + \nu_2^2) \geq 4(\mu_1\nu_2 - \mu_2\nu_1) + (\mu_1^2 + \nu_1^2 + \mu_2^2 + \nu_2^2).$$

So, it suffices to consider the case of $n = 1$.

On the other hand, we have

$$|\mu_0\mu_1 + \nu_0\nu_1| \leq \sqrt{\mu_0^2 + \nu_0^2}\sqrt{\mu_1^2 + \nu_1^2},$$

$$|\mu_1\nu_2 - \mu_2\nu_1| \leq \sqrt{\mu_2^2 + \nu_2^2}\sqrt{\mu_1^2 + \nu_1^2},$$

and $(\partial_r\mu)^2 + (\partial_r\nu)^2 \geq (\partial_r\sqrt{\mu^2 + \nu^2})^2$. Thus, we only need to prove that for $\alpha_i = \pm\sqrt{\mu_i^2 + \nu_i^2}$,

$$\tilde{I}_1^A(\alpha_0, \alpha_1, \alpha_2) = \int_0^\infty \left((\partial_r\alpha_0)^2 + (\partial_r\alpha_1)^2 + (\partial_r\alpha_2)^2 - \frac{4}{r^2}\alpha_1\alpha_2\right) \tag{28}$$

$$+ \frac{1}{r^2}(\alpha_0^2 + \alpha_1^2 + \alpha_2^2) + \alpha_0^2(18v^2 + 2u^2 - t - 2v)$$

$$+ \alpha_1^2(6v^2 + 6u^2 - t + 2v + \frac{1}{r^2}) + \alpha_2^2(6v^2 + 2u^2 - t + 2v + \frac{1}{r^2})$$

$$+ \frac{4u}{\sqrt{3}}(1 + 6v)\alpha_0\alpha_1\Big)rdr \geq 0.$$

When $t \neq 1/3$, we have $v' \neq 0$ for $r > 0$. Let $\xi = \alpha_0/v'$, $\eta = \alpha_1/u'$, $\zeta = r\alpha_2/u$. Then we infer from (20)-(22) that

$$\tilde{I}_1^A(\alpha_0, \alpha_1, \alpha_2) = \mathcal{C}(\xi) + \mathcal{D}(\eta) + \mathcal{E}(\zeta) + \int_0^\infty \Big(\frac{1}{r^2}((v'\xi)^2 + (u'\eta)^2 + (\frac{u\zeta}{r})^2)$$

$$- \frac{4}{r^3}uu'\eta\zeta + \frac{4uu'v'}{\sqrt{3}}(1 + 6v)\xi\eta \Big)rdr$$

$$= \int_0^\infty \Big((v'\xi')^2 + (u'\eta')^2 + (\frac{u\zeta'}{r})^2 + \frac{2uu'}{r^3}(\eta - \zeta)^2 \qquad (29)$$

$$- \frac{2uu'v'}{\sqrt{3}}(1 + 6v)\big(\eta + \frac{\xi}{\sqrt{3}}\big)^2 \Big)rdr \geq 0.$$

Here we have used the fact that $u' > 0$ and $v'(1 + 6v) < 0$ for $r > 0$.

When $t = 1/3$, we have $v \equiv -1/6$. Let $\eta = \alpha_1/u'$, $\zeta = r\alpha_2/u$. Then

$$\tilde{I}_1^A(\alpha_0, \alpha_1, \alpha_2) = \tilde{I}_1^A(0, \alpha_1, \alpha_2) + \int_0^\infty \Big((\partial_r\alpha_0)^2 + \frac{1}{r^2}\alpha_0^2 + \alpha_0^2(\frac{1}{2} + 2u^2) \Big)rdr$$

$$\geq \tilde{I}_1^A(0, \alpha_1, \alpha_2)$$

$$= \int_0^\infty \Big((u'\eta')^2 + (\frac{u\zeta'}{r})^2 + \frac{2uu'}{r^3}(\eta - \zeta)^2 \Big)rdr \geq 0. \qquad (30)$$

This completes our proof. □

6.3. **Proof of Theorem 2.1.** In order to prove $\mathcal{I}(V) \geq 0$, it suffices to show that

$$I^A(w_0, w_1, w_2) \geq 0 \quad \text{and} \quad I^B(w_3, w_4) \geq 0,$$

which follow from Proposition 2, (26), (27) and Proposition 3.

Now we prove that $I(V) = 0$ for $V \in H^1(\mathbb{R}^2)$ only when (12) holds. Assume $I(V) = 0$ with $V \in H^1(\mathbb{R}^2)$. We can also perform the decomposition (15). Following the discussions in subsection 6.1 and 6.2, it suffices to prove that

(i) For any q_0, q_1 with $q_0, q_1, \partial_r q_0, \partial_r q_1, q_1/r \in L^2((0, \infty), rdr)$, it holds

$$\tilde{I}(q_0, q_1) = \int_0^\infty \Big\{ (\partial_r q_0)^2 + (\partial_r q_1)^2 + \frac{q_1^2}{r^2} + (6v^2 + 2u^2 - t - v)(q_0^2 + q_1^2) - 2uq_0q_1 \Big\}rdr > 0;$$

(ii) For any q_0 with $q_0, \partial_r q_0 \in L^2((0, \infty), rdr)$, it holds

$$I_{0,2}^A(q_0) = \int_0^\infty \Big\{ (\frac{\partial q_0}{\partial r})^2 + q_0^2(6v^2 + 2u^2 - t + 2v + \frac{1}{r^2}) \Big\}rdr > 0;$$

(iii) For any q_0, q_1, q_2 with $q_i, \partial_r q_i, q_i/r \in L^2((0, \infty), rdr)$ for $i = 0, 1, 2$, it holds (see (28))

$$\tilde{I}_1^A(q_0, q_1, q_2) > 0.$$

We first prove (i). Take $q_{0,n}, q_{1,n} \in C_c^\infty(0,\infty)$ so that

$$q_{0,n} \to q_0 \text{ in } H^1((0,+\infty), rdr), \tag{31}$$

$$q_{1,n} \to q_1 \text{ in } L^2((0,+\infty), \frac{1}{r}dr) \cap H^1((0,+\infty), rdr) \tag{32}$$

as $n \to +\infty$. From (24), we have

$$\tilde{I}(q_{0,n}, q_{1,n}) = \int_0^\infty \left\{ (u(\frac{q_{1,n}}{u})')^2 + (v(\frac{q_{0,n}}{v})')^2 - \frac{1}{3}vu^2(3\frac{q_{1,n}}{u} + \frac{q_{0,n}}{v})^2 \right\} rdr$$

$$\triangleq \int_0^\infty f(q_{0,n}, q_{1,n}) rdr.$$

It follows from (31) and (32) that

$$\tilde{I}(q_{0,n}, q_{1,n}) \to \tilde{I}(q_0, q_1).$$

Since u, u', v, v' are all smooth and both $|u|$ and $|v|$ are larger than a positive constant on $[1/N, N]$, we know that $|u'/u|, |v'/v|$ are bounded on $[1/N, N]$. Thus, we infer from (31) and (32) that

$$\int_{1/N}^N f(q_0, q_1) rdr \le \liminf_{n\to\infty} \int_{1/N}^N f(q_{0,n}, q_{1,n}) rdr \le \liminf_{n\to\infty} \tilde{I}(q_{0,n}, q_{1,n}) = \tilde{I}(q_0, q_1).$$

Here we used the non-negativity of $f(q_{0,n}, q_{1,n})$. Letting $N \to \infty$, it follows that

$$\int_0^\infty f(q_0, q_1) rdr \le \tilde{I}(q_0, q_1).$$

If the left hand side is zero, then we must have (up to multiply a nonzero constant)

$$q_1 = u, \quad q_0 = -3v.$$

However, $u/r \notin L^2((0,\infty), rdr)$ due to $\lim_{r\to+\infty} u \ne 0$. This is a contradiction. Thus, $\tilde{I}(q_0, q_1) \ge \int_0^\infty f(q_0, q_1) rdr > 0$.

By noting (27), (29), (30) and the fact that $u, u/r \notin L^2((0,\infty), rdr)$, we can prove (2) and (3) in a similar way. Thus, the null space is given by (12).

REFERENCES

[1] P. Bauman, J. Park and D. Philips, Analysis of nematic liquid crystals with disclination lines, *Arch. Ration. Mech. Anal.*, **205** (2012), 795–826.

[2] P. Biscari and G. G. Peroli, A hierarchy of defects in biaxial nematics, *Commun. Math. Phys*, **186** (1997), 381–392.

[3] G. Canevari, Biaxiality in the asymptotic analysis of a 2-d Landau-de Gennes model for liquid crystals, *ESAIM Control Optim. Calc. Var.*, **21** (2015), 101–137.

[4] P. de Gennes and J. Prost, *The Physics of Liquid Crystals*, 2nd edition, Oxford University Press, Oxford, 1995.

[5] G. Di Fratta, J. M. Robbins, V. Slastikov and A. Zarnescu, Half-integer point defects in the Q-tensor theory of nematic liquid crystals, *Journal of Nonlinear Science*, **26** (2016), 121–140.

[6] J. Ericksen, Liquid crystals with variable degree of orientation, *Arch. Ration. Mech. Anal.*, **113** (1990), 97–120.

[7] D. Golovaty and J. A. Montero, On minimizers of a Landau-de Gennes energy functional on planar domains, *Arch. Ration. Mech. Anal.*, **213** (2014), 447–490.

[8] S. Gustafson and I. M. Sigal, The stability of magnetic vortices, *Commun. Math. Phys.*, **212** (2000), 257–275.

[9] R. Hardt, D. Kinderlehrer and F.-H. Lin, Existence and partial regularity of static liquid crystal configurations, *Commun. Math. Phys.*, **105** (1986), 547–570.

[10] F. Hélein, Minima de la fonctionelle energie libre des cristaux liquides, *C. R. Acad. Sci. Paris*, **305** (1987), 565–568.

[11] Y. Hu, Y. Qu and P. Zhang, On the disclination lines of nematic liquid crystals, *Communications in Computational Physics* , **19** (2016), 354–379.

[12] R. Ignat, L. Nguyen, V. Slastikov and A. Zarnescu, Uniqueness results for an ODE related to a generalized Ginzburg-Landau model for liquid crystals, *SIAM J. Math. Anal.*, **46** (2014), 3390–3425.

[13] R. Ignat, L. Nguyen, V. Slastikov and A. Zarnescu, Stability of the melting hedgehog in the Landau-de Gennes theory of nematic liquid crystals, *Arch. Ration. Mech. Anal.*, **215** (2015), 633–673.

[14] R. Ignat, L. Nguyen, V. Slastikov and A. Zarnescu, Instability of point defects in a two-dimensional nematic liquid crystal model, *Ann. I. H. Poincare-AN*, **33** (2016), 1131–1152.

[15] R. Ignat, L. Nguyen, V. Slastikov and A. Zarnescu, Stability of point defects of degree $\pm 1/2$ in a two-dimensional nematic liquid crystal model, *Calculus of Variations and Partial Differential Equations*, **55** (2016), 33pp.

[16] M. Kleman and O. D. Lavrentovich, Topological point defects in nematic liquid crystals, *Philosophical Magazine*, **86** (2006), 4117–4137.

[17] X. Lamy, Some properties of the nematic radial hedgehog in the Landau-de Gennes theory, *J. Math. Anal. Appl.*, **397** (2013), 586–594.

[18] E. H. Lieb and M. Loss, Symmetry of the Ginzburg-Landau mimimizer in a disc, *Math. Res. Lett.*, **1** (1994), 701–715.

[19] F.-H. Lin and C. Liu, Static and dynamic theories of liquid crystals, *J. Partial Differ. Equ.*, **14** (2001), 289–330.

[20] T.-C. Lin, The stability of the radial solution to the Ginzburg-Landau equation, *Commun. PDE*, **22** (1997), 619–632.

[21] A. Majumdar, The radial-hedgehog solution in Landau-de Gennes' theory for nematic liquid crystals, *Euro. J. Appl. Math.*, **23** (2012), 61–97.

[22] A. Majumdar and A. Zarnescu, Landau-de Gennes theory of nematic liquid crystals: The Oseen-Frank limit and beyond, *Arch. Ration. Mech. Anal.*, **196** (2010), 227–280.

[23] N. D. Mermin, The topological theory of defects in ordered media, *Rev. Modern Phys.*, **51** (1979), 591–648.

[24] P. Mironescu, On the stability of radial solutions of the Ginzburg-Landau equation, *J. Funct. Anal.*, **130** (1995), 334–344.

[25] Manuel de Pino, P. Felmer and M. Kowalczyk, Minimality and nondegeneracy of degree-one Ginzburg-Landau vortex as a Hardy's inequality, *IMRN*, **30** (2004), 1511–1527.

[26] R. Rosso and E. G. Virga, Metastable nematic hedgehogs, *J. Phys. A*, **29** (1996), 4247–4264.

[27] G. Toulouse and M. Kleman, Principles of a classification of defects in ordered media, *Journal de Physique Lettres*, **37** (1976), 149–151.

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