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Well-Posedness for the Dumbbell Model
of Polymeric Fluids

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Abstract: The dumbbell model is a coupled hydrodynamic-kinetic model for poly-
meric fluids in which the configurations of the dumbbells are described by stochastic
differential equations. We prove well-posedness of this model by deriving directly a
priori estimates on the stochastic model. Our results can be used to analyze stochastic
simulation methods such as the ones that are based on Brownian configuration fields.

1. Introduction

The dumbbell model is the simplest model of polymeric fluids that takes into account
the microscopic behavior of the solute polymers [1]. It models the dilute polymers by
dumbbells, each with two beads connected by a spring. The configuration of the spring
then specifies the conformation of the polymer. Denote by $\mathbf{u}$ and p the velocity and
pressure of the fluid, and $\mathbf{Q}$ the configuration of the spring, and hence the dumbbell,
then $\mathbf{Q}$ obeys the following equation:

$$\frac{\partial \mathbf{Q}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{Q} - (\nabla\mathbf{u})^T\mathbf{Q} = -\mathbf{F}(\mathbf{Q}) + \dot{\mathbf{w}}(t). \tag{1}$$

Here $\dot{\mathbf{w}}(t)$ is Gaussian white noise in time.

This equation is the result of the balance between the friction force (caused by the
viscous fluid) on the left hand side and the spring and Brownian force on the right hand
side. For simplicity we will set all physical constants to be 1, and we will write the spring
force in the form $\mathbf{F}(\mathbf{Q}) = \gamma(|Q|^2)\mathbf{Q}$.

In writing down (1) for an individual dumbbell we made the crucial assumption
that the polymer-polymer interaction can be neglected. Thermal noise is then naturally

expressed as white noise in time. In this description, the conformation of the dumbbell is described by a stochastic field $\mathbf{Q}(\mathbf{x}, t)$: for any fixed $\mathbf{x}$ in the flow domain, the ensemble $\{\mathbf{Q}(\mathbf{x}, t)\}$ represents possible dumbbell conformation at $\mathbf{x}$. $\mathbf{Q}$ is called Brownian configuration fields. An alternative description, which was used in earlier stochastic simulation methods such as CONNFFESSIT[10], attempts to keep track of each individual dumbbells, which are then subjected to independent thermal noises. The relation between these two descriptions are not fully understood.

The complete model for the full polymer-polymer system is then given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u} + \nabla \cdot \tau, \quad \nabla \cdot \mathbf{u} = 0, \tag{2}$$

where τ is the polymer contribution to stress, which is expressed via the Kramers expression

$$\tau(\mathbf{x}, t) = \mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}). \tag{3}$$

In contrast to traditional models of complex fluids which express polymer stress τ using empirical constitutive relations, (1)-(3) expresses the polymer stress in terms of the microscopic conformations of the polymers using (3). To this end, a new dynamic equation (1) is added to the model which describes the evolution of the internal degrees of freedom of the polymers. Since the polymeric stress is computed directly from the configuration distribution of the polymers, there is no need to introduce ad hoc constitutive relations.

Equations (1)–(3) is a system of stochastic differential equations in which the dynamics of $(\mathbf{u}, p)$ is deterministic. One way of studying such systems is to use the equivalent Fokker-Planck equation for the $(\mathbf{x}, \mathbf{Q})$ distribution function, denoted by ψ, of the dumbbells

$$\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla)\psi + \nabla_{\mathbf{Q}} \cdot \{(\nabla \mathbf{u})^T \mathbf{Q} \psi - \mathbf{F}(\mathbf{Q})\psi\} = \Delta_{\mathbf{Q}} \psi. \tag{4}$$

In (4), $\mathbf{Q}$ is an independent variable and we use the subscript $\mathbf{Q}$ to denote differentiation with respect to $\mathbf{Q}$. Without the subscript, the differentiation is understood to be in $\mathbf{x}$.

The novelty of Eqs. (2)–(4) is that the macroscopic fluid equation is coupled with the mesoscopic Fokker-Planck equation in kinetic theory. Mathematical study of such systems is still in its infancy. In [16, 17, 11], the local existence and uniqueness has been established.

In the special case when the spring force is linear, $\mathbf{F}(\mathbf{Q}) = H\mathbf{Q}$, we get from (1)-(3) a reduced system of equations for $\mathbf{u}$ and τ,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u} + \nabla \cdot \tau, \quad \nabla \cdot \mathbf{u} = 0, \tag{5}$$

$$\frac{\partial \tau}{\partial t} + (\mathbf{u} \cdot \nabla)\tau - (\nabla \mathbf{u})^T \tau - \tau \nabla \mathbf{u} + \tau - I = 0. \tag{6}$$

In this way, one eliminates $\mathbf{Q}$ as an independent variable. This is the well-known Oldroyd-B model. Its well-posedness has been studied by Saut et al. [5], Lions and Masmoudi [13]. However, their methods do not seem to extend to more general cases when closed systems of equations such as (4)–(5) are not available.

Despite the fact that a purely deterministic analysis based on the Fokker-Planck equation is possible, it is of great interest to treat directly the stochastic system (1)–(3) for several reasons:

1. Equation (1) gives a more direct and intuitive description of the conformation and dynamics of the polymers.
2. One may expect that analysis based on (1) is easier to generalize to more general models of polymers such as liquid crystal polymers, bead and spring models. Some evidence is already provided in [12].
3. There has been a great deal of interest in designing stochastic modelling techniques for polymeric fluids [15]. One of our interest is in the analysis of such stochastic methods. So far this is only done in [7] for one-dimensional shear flows using the specified structure of the shear flow system. The numerical analysis in the general case will depend crucially on proving that (1)-(3) is locally well-posed which is the main purpose of the present paper.

To avoid complications from the boundary of the physical domain, we assume that the physical domain is $D = [0, 1]^d$ with periodic boundary conditions. We will denote by $\mathbf{Q}_0(\mathbf{x}) = \mathbf{Q}(\mathbf{x}, 0)$ the initial condition for the configuration field. We take it to be deterministic. But certainly our results can be extended to the case when it is random.

Our main result is the following:

Theorem 1.1. *Assume that the spring force* $\mathbf{F}$ *and the initial value satisfy the following conditions (A) and (B):*

(A) The function γ *is* C^∞-*smooth from* $[0, +\infty)$ *to* $(0, +\infty)$, *and* $\gamma'(|\mathbf{Q}|^2) \geq 0$, *and the derivative of* $\mathbf{F}$ *satisfies that* $|\nabla_{\mathbf{Q}}^m \mathbf{F}(\mathbf{Q})| \leq C(1 + |\mathbf{Q}|^p)$ *(*$m = 0, 1, 2, 3, 4$*), where* C *is a constant and* p *is a certain non-negative integer.*

(B)

$$\|\mathbf{u}_0\|_{H^4} \leq \ Const., \tag{7}$$

$$|\nabla^m \mathbf{Q}_0| \leq \ Const. \ (m = 0, 1, 2, 3, 4). \tag{8}$$

Then there exists a T^* *such that for* $t \leq T^*$, *(1)-(3) has a unique strong solution* $\mathbf{u}$ *with* $\mathbf{u} \in C^1([0, T^*], C^2(D)) \cap L^2([0, T^*], H^5(D))$.

Our strategy is quite simple. We will treat (1)-(3) as if it is deterministic and follow general strategies for proving local well-posedness of such systems (see [14]). It is well-known that the crucial step in such a procedure is to establish a priori estimates for $\int_0^t \|\nabla \mathbf{u}(\cdot, s)\|_\infty ds$. As usual this is done by controlling sufficiently high Sobolev norms. This is carried out in Sect. 2. As one might expect, the key technical component is to estimate $\mathbf{Q}$. The equation for $\mathbf{Q}$ is a vector transport equation, plus a noise term. Fortunately the noise term depends only on time which disappears after differentiation in $\mathbf{x}$. This makes it possible to control high Sobolev norm.

2. A Priori Estimates

In this section, we derive a priori estimates for smooth solutions of the system

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u} + \nabla \cdot \tau, \quad \nabla \cdot \mathbf{u} = 0, \tag{9}$$

$$\tau = \mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}), \tag{10}$$

$$\frac{\partial \mathbf{Q}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{Q} = (\nabla \mathbf{u})^T \mathbf{Q} - \mathbf{F}(\mathbf{Q}) + \dot{\mathbf{w}}(t). \tag{11}$$

Here $\mathbb{E}$ denotes expectation with respect to the statistics of the Gaussian white noise $\dot{\mathbf{w}}(t)$.

Equations (9), (10), and (11) are supplemented with initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{12}$$
$$\mathbf{Q}(\mathbf{x}, 0) = \mathbf{Q}_0(\mathbf{x}), \tag{13}$$

which are assumed to be smooth. $\{\mathbf{Q}_0\}$ can be random, however.

2.1. A priori estimates for $\mathbf{u}$.

2.1. A priori estimates for $\mathbf{u}$. Here we recall some standard estimates for Navier-Stokes type of equations.

Consider

$$\begin{cases} \mathbf{u}_t + (\mathbf{h} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u} + \nabla \cdot \tau, \ \ \nabla \cdot \mathbf{u} = 0 \\ \nabla \cdot \mathbf{h} = 0 \end{cases}, \tag{14}$$

where $\mathbf{u}, \mathbf{h}$ and f are assumed to be smooth. Then we have

$$\|\mathbf{u}(\cdot, t)\|_{L^2}^2 + \int_0^t \|\nabla \mathbf{u}(\cdot, s)\|_{L^2}^2 ds \leq \|\mathbf{u}_0\|_{L^2}^2 + \int_0^t \|\tau(\cdot, s)\|_{L^2}^2 ds. \tag{15}$$

Lemma 2.1. *Let* $\mathbf{v} = \nabla \mathbf{u}$, $\mathbf{v}_0 = \nabla \mathbf{u}_0$, *then*

$$\|\mathbf{v}(\cdot, t)\|_{L^2}^2 \leq e^{\int_0^t \|\nabla \mathbf{h}(\cdot, s)\|_{L^\infty} ds} \left(\|\mathbf{v}_0\|_{L^2}^2 + \int_0^t \|\nabla \tau(\cdot, s)\|_{L^2}^2 ds \right), \tag{16}$$

$$\int_0^t \|\nabla \mathbf{v}(\cdot, s)\|_{L^2}^2 ds \leq \|\mathbf{v}_0\|_{L^2}^2 + \int_0^t ds \left(\|\nabla \mathbf{h}(\cdot, s)\|_{L^\infty} \|\mathbf{v}(\cdot, s)\|_{L^2}^2 + \|\nabla \tau(\cdot, s)\|_{L^2}^2 \right). \tag{17}$$

Lemma 2.2. *For* $\alpha = 0, 1, 2, 3, 4$,

$$\|\mathbf{u}(\cdot, t)\|_{H^\alpha}^2 \leq e^{\int_0^t \|\mathbf{h}(\cdot, s)\|_{H^4} ds} \left(\|\mathbf{u}_0\|_{H^\alpha}^2 + \int_0^t \|\tau(\cdot, s)\|_{H^\alpha}^2 ds \right), \tag{18}$$

$$\int_0^t \|\mathbf{u}(\cdot, s)\|_{H^{\alpha+1}}^2 ds \leq \|\mathbf{u}_0\|_{H^\alpha}^2 + \int_0^t (\|\mathbf{h}(\cdot, s)\|_{H^4} \|\mathbf{u}(\cdot, s)\|_{H^\alpha}^2 + \|\tau(\cdot, s)\|_{H^\alpha}^2) ds. \tag{19}$$

2.2. A priori estimates for $\mathbf{Q}$. Consider

$$\partial_t \mathbf{Q} + (\mathbf{u} \cdot \nabla)\mathbf{Q} = \kappa \mathbf{Q} - \mathbf{F}(\mathbf{Q}) + \dot{\mathbf{w}}(t), \tag{20}$$

where $\kappa = (\nabla \mathbf{u})^T$, $\mathbf{F}$ is a smooth function. To be precise (20) should be written as

$$d\mathbf{Q} = (-\mathbf{u} \cdot \nabla \mathbf{Q} + \kappa \mathbf{Q} - \mathbf{F}(\mathbf{Q}))dt + d\mathbf{w}. \tag{21}$$

$\mathbf{u}$ is assumed to be a given smooth deterministic velocity field, s.t. $\nabla \cdot \mathbf{u} = 0$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space upon which the Wiener process $\mathbf{w}(\cdot)$ is defined. We will use ω to denote realizations of the Brownian path.

We will always assume that $\mathbf{F}$ satisfies the growth condition (A) in Theorem 1.1.

Denote by $\mathbf{X}(\boldsymbol{\alpha}, t)$ the Eulerian-Lagrangian flow map induced by the velocity field $\mathbf{u}$:

$$\begin{cases} \dfrac{d}{dt}\mathbf{X}(\boldsymbol{\alpha}, t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t) \\ \mathbf{X}(\boldsymbol{\alpha}, 0) = \boldsymbol{\alpha} \end{cases}. \tag{22}$$

Let $\mathbf{Q}$ be the solution of (20). Fix a $\boldsymbol{\alpha} \in R^d$, let $\mathbf{q}(\boldsymbol{\alpha}, t)$ be the solution of

$$d\mathbf{q} = (k(t)\mathbf{q} - \mathbf{F}(\mathbf{q}))dt + d\mathbf{w}, \quad \mathbf{q}(\boldsymbol{\alpha}, 0) = \mathbf{Q}_0(\boldsymbol{\alpha}), \tag{23}$$

where $k(t) = \kappa(\mathbf{X}(\boldsymbol{\alpha}, t), t)$. Then

Lemma 2.3. *For almost all $\omega \in \Omega$,*

$$\mathbf{q}(\boldsymbol{\alpha}, t) = \mathbf{Q}(\mathbf{X}(\boldsymbol{\alpha}, t), t)$$

for all $\boldsymbol{\alpha} \in R^d$.

Proof. Let $\bar{\mathbf{q}}_\alpha(t) = \mathbf{Q}(\mathbf{X}(\boldsymbol{\alpha}, t), t)$. It is easy to see that $\bar{\mathbf{q}}_\alpha(\cdot)$ is a solution of (23) under the $C^1(D)$-smooth condition of $\mathbf{u}$ which will be shown later. Hence Lemma 2.3 follows from the uniqueness results for (23). $\quad\square$

Lemma 2.4. *There exists a unique solution to the SDEs (23) with values $C([0, +\infty), R^d)$.*

Proof. It is a standard procedure to prove the existence and uniqueness of the SDEs before the stopping time $\tau_N = \inf\{t \mid |\mathbf{Q}| > N\}$ for the smoothness of $\mathbf{F}$ and $\mathbf{u}$. The only needed thing is to show that the lifespan of the solution of (23) is $\mathbb{R}^+$ a.s., i.e. $\lim_{N \to +\infty} \tau_N = +\infty$. Because the stretching term $k(t)\mathbf{q}$ is just a linear growth term of $\mathbf{q}$, we only need to show that the equation

$$d\mathbf{q} = -\mathbf{F}(\mathbf{q})dt + d\mathbf{w} \tag{24}$$

will not blow up in finite time. We use Feller's test for explosion to deal with this problem [8, 9].

Consider the equation for $X_t = |\mathbf{q}|^2$ by applying Itô's formula

$$\begin{aligned} dX_t &= 2\mathbf{q} \cdot d\mathbf{q} + ndt \\ &= -2\mathbf{q} \cdot \mathbf{F}(\mathbf{q})dt + ndt + 2\mathbf{q} \cdot d\mathbf{w} \\ &= (n - 2X_t\gamma(X_t))dt + 2\sqrt{X_t}dB_t, \end{aligned} \tag{25}$$

where B_t is a Brownian motion by Paul Lévy characterization. $n = d$ is the spatial dimension. Define the scale function $p(x)$ satisfies

$$2(n - 2\gamma(x)x)p'(x) + 4xp''(x) = 0, \tag{26}$$

and, we obtain

$$p(x) = \int_c^x y^{-\frac{n}{2}} e^{h(y)}dy, \tag{27}$$

where $h'(x) = \gamma(x)$, c is a fixed positive number. Clearly $h(x)$ is a monotonely increasing function of x. The speed measure is defined as

$$m(dx) = \frac{1}{2}x^{\frac{n}{2}-1}e^{-h(x)}dx; \tag{28}$$

thus if $n = 2$, we have

$$v(x) = \int_c^x \Big(p(x) - p(y) \Big) m(dy)$$

$$= \int_c^x \Big(\frac{1}{2} \int_y^x \frac{y}{z} \cdot \frac{1}{y} e^{(h(z)-h(y))} dz \Big) dy$$

$$\geq \frac{1}{2} \int_c^x \ln \frac{x}{y} dy = \frac{x}{2} \int_1^{\frac{c}{x}} \ln z dz. \tag{29}$$

It is clear that $v(+\infty) = +\infty$.

If $n = 3$, we have

$$v(x) = \int_c^x \Big(p(x) - p(y) \Big) m(dy)$$

$$= \int_c^x \Big(\frac{1}{2} \int_y^x \Big(\frac{y}{z} \Big)^{\frac{3}{2}} \cdot \frac{1}{y} e^{(h(z)-h(y))} dz \Big) dy$$

$$\geq \frac{1}{2} \int_c^x \sqrt{y} \big(- 2z^{-\frac{1}{2}} |_y^x \big) dy = \int_c^x \Big(1 - \sqrt{\frac{y}{x}} \Big) dy$$

$$= x \Big(1 - \int_{\frac{c}{x}}^1 z^{\frac{1}{2}} dz \Big) - c. \tag{30}$$

It is clear that $v(+\infty) = +\infty$.

Thus we obtain the existence and uniqueness on $\mathbb{R}^+$. □

Here and in the following we will often use the Frobenius-norm of the vectors or the tensors A which is defined as

$$|A|_F \triangleq \Big(\sum_{i,j,\dots,k} a_{i,j,\dots,k}^2 \Big)^{\frac{1}{2}}. \tag{31}$$

It is not difficult to find that this norm satisfies the common triangle inequality and the product inequality

$$|A * B|_F \leq |A|_F |B|_F, \tag{32}$$

where $*$ may be $\cdot$, : or higher order contraction operators. We will still abbreviate $|\cdot|_F$ as $|\cdot|$ through the paper.

Lemma 2.5. *Define:*

$$\mathcal{Q}_m^{(0)}(t) = sup_{\boldsymbol{\alpha}} \mathbb{E}|\mathbf{q}(\boldsymbol{\alpha}, t)|^m = sup_{\mathbf{x}} \mathbb{E}|\mathbf{Q}|^m, \tag{33}$$

then we have the following recursive inequality

$$\mathcal{Q}_m^{(0)}(t) \leq \mathcal{Q}_m^{(0)}(0) + m \int_0^t \|\nabla \mathbf{u}\|_{L^\infty} \mathcal{Q}_m^{(0)}(s) ds + \frac{1}{2}(mn + m(m-2)) \int_0^t \mathcal{Q}_{m-2}^{(0)}(s) dt. \tag{34}$$

If $m = 2$, then we have

$$\mathcal{Q}_2^{(0)}(t) \leq \mathcal{Q}_2^{(0)}(0) + nt + e^{\int_0^t 2\|\nabla \mathbf{u}\|_{L^\infty} ds} \int_0^t 2(\mathcal{Q}_2^{(0)}(0) + ns) \|\nabla \mathbf{u}\|_{L^\infty} ds, \tag{35}$$

where $n = d$ is the spatial dimension.

Proof. For $|\mathbf{q}|^m$, we use Itô's formula

$$d|\mathbf{q}|^m = m|\mathbf{q}|^{m-2}\mathbf{q} \cdot d\mathbf{q} + \frac{1}{2}(mn + m(m-2))|\mathbf{q}|^{m-2}dt. \tag{36}$$

Paying attention that the term $\mathbf{q} \cdot \mathbf{F}(\mathbf{q}) \geq 0$, we obtain

$$d|\mathbf{q}|^m \leq m|\nabla\mathbf{u}||\mathbf{q}|^m dt + m|\mathbf{q}|^{m-2}\mathbf{q} \cdot d\mathbf{w} + \frac{1}{2}(mn + m(m-2))|\mathbf{q}|^{m-2}dt. \tag{37}$$

Integrating on $[0, t]$, and taking the expectation on both sides, we get

$$\mathcal{Q}_m^{(0)}(t) \leq \mathcal{Q}_m^{(0)}(0) + m \int_0^t \|\nabla\mathbf{u}\|_{L^\infty}\mathcal{Q}_m^{(0)}(s)ds + \frac{1}{2}(mn + m(m-2)) \int_0^t \mathcal{Q}_{m-2}^{(0)}(s)dt. \tag{38}$$

Taking $m = 2$, we have

$$\mathcal{Q}_2^{(0)}(t) \leq \mathcal{Q}_2^{(0)}(0) + nt + e^{\int_0^t 2\|\nabla\mathbf{u}\|_{L^\infty}ds} \int_0^t 2(\mathcal{Q}_2^{(0)}(0) + ns)\|\nabla\mathbf{u}\|_{L^\infty}ds. \tag{39}$$

For general m, we will obtain the recursive estimate for $\mathcal{Q}_m^{(0)}(t)$. If m is odd, we may get the corresponding estimate by using the inequality $|\mathbf{q}| \leq 1 + |\mathbf{q}|^2$. $\quad\square$

Remark 1. For arbitrary m, we have the L^∞-norm estimate of $\mathbb{E}|\mathbf{Q}|^m$.

Lemma 2.6. *Define*

$$\mathcal{Q}_m^{(1)}(t) = \int_D \mathbb{E}|\nabla\mathbf{Q}|^m dx, \tag{40}$$

then we have

$$\mathcal{Q}_m^{(1)}(t) \leq e^{\int_0^t (\|\nabla\mathbf{u}\|_{L^\infty}+c)ds} \left(\mathcal{Q}_m^{(1)}(0) + \int_0^t \mathcal{Q}_m^{(0)}(s)\|\nabla^2\mathbf{u}(\cdot, s)\|_{L^m}^m ds \right). \tag{41}$$

Proof. Define $R \triangleq \nabla\mathbf{Q}$; we have

$$R_t + (\mathbf{u} \cdot \nabla)R + (\nabla\mathbf{u} \cdot \nabla)\mathbf{Q} = \nabla\kappa \cdot \mathbf{Q} + \kappa \cdot R - \nabla\mathbf{F}(\mathbf{Q}). \tag{42}$$

Taking the inner product on both sides with $R|R|^{m-2}$, and noting that

$$(\nabla\mathbf{F}(\mathbf{Q}))_{ij} R_{ij} = 2\gamma'(|\mathbf{Q}|^2)Q_l\partial_i Q_l Q_j\partial_i Q_j + \gamma(|\mathbf{Q}|^2)|R|^2 \geq 0, \tag{43}$$

where the summation convention is applied, thus we get

$$\frac{1}{m}|R|_t^m \leq \|\nabla\mathbf{u}\|_{L^\infty}|R|^m - (\mathbf{u} \cdot \nabla)R * R|R|^{m-2} + |\nabla^2\mathbf{u}||\mathbf{Q}||R|^{m-1}. \tag{44}$$

Here and following we will represent the inner product as $*$. Hence we have

$$\frac{1}{m}\frac{d}{dt}\mathcal{Q}_m^{(1)} \leq \|\nabla\mathbf{u}\|_{L^\infty}\mathcal{Q}_m^{(1)} - \int_D \mathbb{E}((\mathbf{u} \cdot \nabla)R * R|R|^{m-2})dx + \int_D \mathbb{E}(|\nabla^2\mathbf{u}||\mathbf{Q}||R|^{m-1})dx. \tag{45}$$

Notice that

$$\int_D (\mathbf{u} \cdot \nabla) R * R |R|^{m-2} dx = 0; \tag{46}$$

we get

$$\frac{1}{m} \frac{d}{dt} \mathcal{Q}_m^{(1)} \leq \|\nabla u\|_{L^\infty} \mathcal{Q}_m^{(1)} + \int_D |\nabla^2 \mathbf{u}| \mathbb{E}(|\mathbf{Q}||R|^{m-1}). \tag{47}$$

Since

$$\int_D \mathbb{E}(|\nabla^2 \mathbf{u}||\mathbf{Q}||R|^{m-1}) \leq \int_D (\mathbb{E}|R|^m)^{\frac{m-1}{m}} (\mathbb{E}|\nabla^2 \mathbf{u}|^m |\mathbf{Q}|^m)^{\frac{1}{m}} dx$$

$$\leq \frac{m-1}{m} \mathcal{Q}_m^{(1)} + \frac{\mathcal{Q}_m^{(0)}(t)}{m} \int_D |\nabla^2 \mathbf{u}|^m dx, \tag{48}$$

finally we get

$$\mathcal{Q}_m^{(1)}(t) \leq e^{\int_0^t (\|\nabla \mathbf{u}\|_{L^\infty} + c) ds} \left(\mathcal{Q}_m^{(1)}(0) + \int_0^t \mathcal{Q}_m^{(0)}(s) \|\nabla^2 \mathbf{u}(\cdot, s)\|_{L^m}^m ds \right). \tag{49}$$

□

Remark 2. Notice that if $m < +\infty$, we have $\|\nabla^2 \mathbf{u}\|_{L^m} \leq \|\mathbf{u}\|_{H^4} \in L^\infty(0, t)$. The estimate of $\mathcal{Q}_m^{(1)}(t)$ is valid.

Lemma 2.7. *Define*

$$\mathcal{Q}_m^{(2)}(t) = \int_D \mathbb{E}|\nabla^2 \mathbf{Q}|^m dx, \tag{50}$$

then we have

$$\mathcal{Q}_m^{(2)}(t) \leq e^{\int_0^t (\|\nabla \mathbf{u}\|_{L^\infty} + c) ds} \cdot \left(\mathcal{Q}_m^{(2)}(0) + \int_0^t (\|\nabla^2 \mathbf{u}\|_{L^{2m}}^{2m} \right.$$

$$\left. + \mathcal{Q}_m^{(0)}(s) \|\nabla^3 \mathbf{u}\|_{L^m}^m + \mathcal{Q}_{4p}^{(0)}(s) + \mathcal{Q}_{4m}^{(1)}(s)) ds \right). \tag{51}$$

Proof. Define $S \triangleq \nabla R = \nabla^2 \mathbf{Q}$, then

$$S_t + (\mathbf{u} \cdot \nabla) S + 2(\nabla \mathbf{u} \cdot \nabla) R + (\nabla^2 \mathbf{u} \cdot \nabla) \mathbf{Q} = \nabla^2 \kappa \cdot \mathbf{Q} + 2\nabla \kappa \cdot R + \kappa \cdot S - \nabla^2 \mathbf{F}(\mathbf{Q}), \tag{52}$$

$$\nabla^2 \mathbf{F}(\mathbf{Q}) = \nabla_x (\nabla_{\mathbf{Q}} \mathbf{F} \cdot \nabla_x \mathbf{Q}) = \nabla_{\mathbf{Q}}^2 \mathbf{F} \nabla_x \mathbf{Q} \nabla_x \mathbf{Q} + \nabla_{\mathbf{Q}} \mathbf{F} \cdot \nabla_x^2 \mathbf{Q}. \tag{53}$$

Taking the inner product on both sides with $S|S|^{m-2}$, and noting that

$$(\nabla_{\mathbf{Q}} \mathbf{F})_{il} S_{ljk} S_{ijk} = 2\gamma'(|\mathbf{Q}|^2) Q_i Q_l \partial_{lj} Q_k \partial_{ij} Q_k + \gamma(|\mathbf{Q}|^2)|S|^2 \geq 0, \tag{54}$$

where the summation convention is applied, thus we get

$$\frac{1}{m} |S|_t^m \leq \|\nabla \mathbf{u}\|_{L^\infty} |S|^m - (\mathbf{u} \cdot \nabla) S * S|S|^{m-2} + |\nabla^2 \mathbf{u}||\nabla \mathbf{Q}||S|^{m-1}$$

$$+ |\nabla^3 \mathbf{u}||\mathbf{Q}||S|^{m-1} + (1 + |\mathbf{Q}|^p)|\nabla \mathbf{Q}|^2 |S|^{m-1}. \tag{55}$$

Integrating on D and taking the expectation

$$\frac{1}{m}\frac{d}{dt}\mathcal{Q}_m^{(2)} \leq \|\nabla\mathbf{u}\|_{L^\infty}\mathcal{Q}_m^{(2)} - \int_D \mathbb{E}((\mathbf{u}\cdot\nabla)S * S|S|^{m-2})dx$$

$$+ \int_D \mathbb{E}(|\nabla^2\mathbf{u}||\nabla\mathbf{Q}||S|^{m-1})dx + \int_D \mathbb{E}(|\nabla^3\mathbf{u}||\mathbf{Q}||S|^{m-1})dx$$

$$+ \int_D \mathbb{E}\Big((1+|\mathbf{Q}|^p)|\nabla\mathbf{Q}|^2|S|^{m-1}\Big)dx, \tag{56}$$

and using

$$\int_D (\mathbf{u}\cdot\nabla)S * S|S|^{m-2}dx = 0, \tag{57}$$

we get

$$\frac{1}{m}\frac{d}{dt}\mathcal{Q}_m^{(2)} \leq \|\nabla\mathbf{u}\|_{L^\infty}\mathcal{Q}_m^{(2)} + \int_D \mathbb{E}(|\nabla^2\mathbf{u}||\nabla\mathbf{Q}||S|^{m-1})dx + \int_D \mathbb{E}(|\nabla^3\mathbf{u}||\mathbf{Q}||S|^{m-1})dx$$

$$+ \int_D \mathbb{E}\Big((1+|\mathbf{Q}|^p)|\nabla\mathbf{Q}|^2|S|^{m-1}\Big)dx$$

$$= P1 + P2 + P3 + P4, \tag{58}$$

and we have

$$P2 \leq \int_D (\mathbb{E}|S|^m)^{\frac{m-1}{m}}(\mathbb{E}|\nabla^2\mathbf{u}|^m|\nabla\mathbf{Q}|^m)^{\frac{1}{m}}dx$$

$$\leq \frac{m-1}{m}\mathcal{Q}_m^{(2)} + \frac{1}{2m}(\|\nabla^2\mathbf{u}\|_{L^{2m}}^{2m} + \mathcal{Q}_{2m}^{(1)}(t)), \tag{59}$$

$$P3 \leq \int_D (\mathbb{E}|S|^m)^{\frac{m-1}{m}}(\mathbb{E}|\nabla^3\mathbf{u}|^m|\mathbf{Q}|^m)^{\frac{1}{m}}dx$$

$$\leq \frac{m-1}{m}\mathcal{Q}_m^{(2)} + \frac{\mathcal{Q}_m^{(0)}(t)}{m}\|\nabla^3\mathbf{u}\|_{L^m}^m, \tag{60}$$

$$P4 \leq \int_D (\mathbb{E}|S|^m)^{\frac{m-1}{m}}\big(\mathbb{E}((1+|\mathbf{Q}|^{2p})|\nabla\mathbf{Q}|^{2m})\big)^{\frac{1}{m}}dx$$

$$\leq \frac{m-1}{m}\mathcal{Q}_m^{(2)} + \frac{1}{2m}(\mathcal{Q}_{4m}^{(1)}(t) + 2\mathcal{Q}_{2m}^{(1)}(t) + \mathcal{Q}_{4p}^{(0)}(t)). \tag{61}$$

After dropping some lower order terms, we obtain

$$\mathcal{Q}_m^{(2)}(t) \leq e^{\int_0^t(\|\nabla\mathbf{u}\|_{L^\infty}+c)ds} \cdot \Big(\mathcal{Q}_m^{(2)}(0) + \int_0^t (\|\nabla^2\mathbf{u}\|_{L^{2m}}^{2m}$$

$$+\mathcal{Q}_m^{(0)}(s)\|\nabla^3\mathbf{u}\|_{L^m}^m + \mathcal{Q}_{4p}^{(0)}(s) + \mathcal{Q}_{4m}^{(1)}(s))ds\Big). \tag{62}$$

$\square$

Remark 3. Notice that if $m \leq 6$, we have $\|\nabla^2\mathbf{u}\|_{L^{2m}} \leq \|\mathbf{u}\|_{H^4} \in L^\infty(0,t)$, $\|\nabla^3\mathbf{u}\|_{L^m} \leq \|\mathbf{u}\|_{H^4} \in L^\infty(0,t)$. The estimate of $\mathcal{Q}_m^{(2)}(t)$ is valid.

Lemma 2.8. *Define*

$$\mathcal{Q}_m^{(3)}(t) = \int_D \mathbb{E}|\nabla^3 \mathbf{Q}|^m dx, \tag{63}$$

then we have

$$\mathcal{Q}_m^{(3)}(t) \le e^{\int_0^t (\|\nabla \mathbf{u}\|_{L^\infty} + c)ds} \cdot \left(\mathcal{Q}_m^{(3)}(0) + \int_0^t (\|\nabla^2 \mathbf{u}\|_{L^{2m}}^{2m} + \|\nabla^3 \mathbf{u}\|_{L^{2m}}^{2m} \right.$$
$$\left. + \mathcal{Q}_m^{(0)}(s)\|\nabla^4 \mathbf{u}\|_{L^m}^m + \mathcal{Q}_{4mp}^{(0)}(t) + \mathcal{Q}_{6m}^{(1)}(s) + \mathcal{Q}_{2m}^{(2)}(s))ds \right). \tag{64}$$

Proof. Define $T \triangleq \nabla S = \nabla^3 \mathbf{Q}$, then we have

$$T_t + (\mathbf{u} \cdot \nabla)T + 3(\nabla \mathbf{u} \cdot \nabla)S + 3(\nabla^2 \mathbf{u} \cdot \nabla)R + (\nabla^3 \mathbf{u} \cdot \nabla)\mathbf{Q}$$
$$= \nabla^3 \kappa \cdot \mathbf{Q} + 3\nabla^2 \kappa \cdot R + 3\nabla \kappa \cdot S + \kappa \cdot T - \nabla^3 \mathbf{F}(\mathbf{Q}), \tag{65}$$

$$\nabla_x^3 \mathbf{F}(\mathbf{Q}) = \nabla_x (\nabla_{\mathbf{Q}}^2 \mathbf{F} \nabla_x \mathbf{Q} \nabla_x \mathbf{Q} + \nabla_{\mathbf{Q}} \mathbf{F} \cdot \nabla_x^2 \mathbf{Q})$$
$$= \nabla_{\mathbf{Q}}^3 \mathbf{F} \nabla_x \mathbf{Q} \nabla_x \mathbf{Q} \nabla_x \mathbf{Q} + \nabla_{\mathbf{Q}}^2 \mathbf{F} \nabla_x^2 \mathbf{Q} \nabla_x \mathbf{Q} + \nabla_{\mathbf{Q}} \mathbf{F} \nabla_x^3 \mathbf{Q}. \tag{66}$$

Taking the inner product on both sides of the equation for T with $T|T|^{m-2}$, we get

$$\frac{1}{m}|T|_t^m \le \|\nabla \mathbf{u}\|_{L^\infty} |T|^m - (\mathbf{u} \cdot \nabla)T * T|T|^{m-2} - (\nabla_{\mathbf{Q}} \mathbf{F} \cdot T) * T|T|^{m-2}$$
$$+ |\nabla^2 \mathbf{u}||\nabla^2 \mathbf{Q}||T|^{m-1} + |\nabla^3 \mathbf{u}||\nabla \mathbf{Q}||T|^{m-1} + |\nabla^4 \mathbf{u}||\mathbf{Q}||T|^{m-1}$$
$$+ (|\nabla \mathbf{Q}|^3 |T|^{m-1} + |\nabla^2 \mathbf{Q}||\nabla \mathbf{Q}||T|^{m-1})(1 + |\mathbf{Q}|^p). \tag{67}$$

Integrating both sides and using the identity

$$\int_D (\mathbf{u} \cdot \nabla)T * T|T|^{m-2} dx = 0 \tag{68}$$

and the inequality $(\nabla_{\mathbf{Q}} \mathbf{F} \cdot T) * T \ge 0$, we get

$$\frac{1}{m}\frac{d}{dt}\mathcal{Q}_m^{(3)} \le \|\nabla \mathbf{u}\|_{L^\infty} \mathcal{Q}_m^{(3)} + \int_D \mathbb{E}(|\nabla^2 \mathbf{u}||\nabla^2 \mathbf{Q}||T|^{m-1})dx$$
$$+ \int_D \mathbb{E}(|\nabla^3 \mathbf{u}||\nabla \mathbf{Q}||T|^{m-1})dx + \int_D \mathbb{E}(|\nabla^4 \mathbf{u}||\mathbf{Q}||T|^{m-1})dx$$
$$+ \int_D \mathbb{E}((1 + |\mathbf{Q}|^p)|\nabla \mathbf{Q}|^3 |T|^{m-1})dx$$
$$+ \int_D \mathbb{E}((1 + |\mathbf{Q}|^p)|\nabla^2 \mathbf{Q}||\nabla \mathbf{Q}||T|^{m-1})dx$$
$$= P1 + P2 + P3 + P4 + P5 + P6. \tag{69}$$

Note that

$$P2 \le \int_D (\mathbb{E}|T|^m)^{\frac{m-1}{m}} (\mathbb{E}|\nabla^2 \mathbf{u}|^m |\nabla^2 \mathbf{Q}|^m)^{\frac{1}{m}} dx$$
$$\le \frac{m-1}{m}\mathcal{Q}_m^{(3)} + \frac{1}{2m}(\|\nabla^2 \mathbf{u}\|_{L^{2m}}^{2m} + \mathcal{Q}_{2m}^{(2)}(t)), \tag{70}$$

$$P3 \leq \int_D (\mathbb{E}|T|^m)^{\frac{m-1}{m}} (\mathbb{E}|\nabla^3 \mathbf{u}|^m |\nabla \mathbf{Q}|^m)^{\frac{1}{m}} dx$$

$$\leq \frac{m-1}{m} \mathcal{Q}_m^{(3)} + \frac{1}{2m} (\|\nabla^3 \mathbf{u}\|_{L^{2m}}^{2m} + \mathcal{Q}_{2m}^{(1)}(t)), \tag{71}$$

$$P4 \leq \int_D (\mathbb{E}|T|^m)^{\frac{m-1}{m}} (\mathbb{E}|\nabla^4 \mathbf{u}|^m |\mathbf{Q}|^m)^{\frac{1}{m}} dx$$

$$\leq \frac{m-1}{m} \mathcal{Q}_m^{(3)} + \frac{\mathcal{Q}_m^{(0)}(t)}{m} \|\nabla^4 \mathbf{u}\|_{L^m}^m, \tag{72}$$

$$P5 \leq \int_D (\mathbb{E}|T|^m)^{\frac{m-1}{m}} (\mathbb{E}|\nabla \mathbf{Q}|^{3m} (1+|\mathbf{Q}|^{mp}))^{\frac{1}{m}} dx$$

$$\leq \frac{m-1}{m} \mathcal{Q}_m^{(3)} + \frac{1}{2m} (\mathcal{Q}_{6m}^{(1)}(t) + 2\mathcal{Q}_{3m}^{(1)}(t) + \mathcal{Q}_{2mp}^{(0)}(t)), \tag{73}$$

$$P6 \leq \int_D (\mathbb{E}|T|^m)^{\frac{m-1}{m}} (\mathbb{E}|\nabla^2 \mathbf{Q}|^m |\nabla \mathbf{Q}|^m (1+|\mathbf{Q}|^{mp}))^{\frac{1}{m}} dx$$

$$\leq \frac{m-1}{m} \mathcal{Q}_m^{(3)} + \frac{1}{4m} (2\mathcal{Q}_{2m}^{(2)}(t) + 2\mathcal{Q}_{2m}^{(1)}(t) + \mathcal{Q}_{4m}^{(1)}(t) + \mathcal{Q}_{4mp}^{(0)}(t)), \tag{74}$$

where some constants have been omitted. After dropping some lower order terms, we obtain

$$\mathcal{Q}_m^{(3)}(t) \leq e^{\int_0^t (\|\nabla \mathbf{u}\|_{L^\infty} + c) ds} \cdot \left(\mathcal{Q}_m^{(3)}(0) + \int_0^t (\|\nabla^2 \mathbf{u}\|_{L^{2m}}^{2m} + \|\nabla^3 \mathbf{u}\|_{L^{2m}}^{2m} \right.$$

$$\left. + \mathcal{Q}_m^{(0)}(s) \|\nabla^4 \mathbf{u}\|_{L^m}^m + \mathcal{Q}_{6m}^{(1)}(s) + \mathcal{Q}_{4mp}^{(0)}(t) + \mathcal{Q}_{2m}^{(2)}(s)) ds \right). \tag{75}$$

$\square$

Remark 4. Notice that if $m \leq 3$, we have $\|\nabla^2 \mathbf{u}\|_{L^{2m}}$, $\|\nabla^3 \mathbf{u}\|_{L^{2m}} \leq \|\mathbf{u}\|_{H^4} \in L^\infty(0, t)$. If $m = 3$, we have

$$\|\nabla^4 \mathbf{u}\|_{L^3}^3 \leq \|\nabla^4 \mathbf{u}\|_{H^{\frac{1}{2}}}^3 \leq (\|\nabla^4 \mathbf{u}\|_{H^0}^{\frac{1}{2}})^3 (\|\nabla^4 \mathbf{u}\|_{H^1}^{\frac{1}{2}})^3 \leq \|\nabla^4 \mathbf{u}\|_{L^2}^{\frac{3}{2}} \|\mathbf{u}\|_{H^5}^{\frac{3}{2}}, \tag{76}$$

and a simple Hölder inequality shows that $\|\nabla^4 \mathbf{u}\|_{L^3}^3$ belongs to $L^1(0, t)$. The estimate of $\mathcal{Q}_m^{(3)}(t)$ is valid.

Lemma 2.9. *Define*

$$\mathcal{Q}_m^{(4)}(t) = \int_D \mathbb{E}|\nabla^4 \mathbf{Q}|^m dx, \tag{77}$$

then we have

$$\mathcal{Q}_2^{(4)}(t) \leq e^{\int_0^t (\|\nabla \mathbf{u}\|_{L^\infty} + c) ds} \cdot \left(\mathcal{Q}_2^{(4)}(0) + \int_0^t (\|\nabla^2 \mathbf{u}\|_{L^6}^6 + \|\nabla^3 \mathbf{u}\|_{L^4}^4 + \|\nabla^4 \mathbf{u}\|_{L^3}^3 \right.$$

$$\left. + \mathcal{Q}_2^{(0)}(s) \|\nabla^5 \mathbf{u}\|_{L^2}^2 + \mathcal{Q}_{12p}^{(0)}(s) + \mathcal{Q}_{16}^{(1)}(s) + \mathcal{Q}_6^{(2)}(s) + \mathcal{Q}_3^{(3)}(s)) ds \right). \tag{78}$$

Proof. Define $U \triangleq \nabla T = \nabla^4 \mathbf{Q}$, then we have

$$U_t + (\mathbf{u} \cdot \nabla)U + 4(\nabla\mathbf{u} \cdot \nabla)T + 6(\nabla^2\mathbf{u} \cdot \nabla)S + 4(\nabla^3\mathbf{u} \cdot \nabla)R + (\nabla^4\mathbf{u} \cdot \nabla)\mathbf{Q}$$
$$= \nabla^4\kappa \cdot \mathbf{Q} + 4\nabla^3\kappa \cdot R + 6\nabla^2\kappa \cdot S + 4\nabla\kappa \cdot T + \kappa \cdot U - \nabla^4\mathbf{F}(\mathbf{Q}). \tag{79}$$

Straightforward calculation gives $\nabla_x^4 \mathbf{F}(\mathbf{Q}) = \nabla_\mathbf{Q}\mathbf{F} \cdot U + Rem$, where

$$|Rem| \le C(|\nabla^3\mathbf{Q}||\nabla\mathbf{Q}| + |\nabla^2\mathbf{Q}|^2 + |\nabla^2\mathbf{Q}||\nabla\mathbf{Q}|^2 + |\nabla\mathbf{Q}|^4)(1 + |\mathbf{Q}|^p). \tag{80}$$

Hence

$$\frac{1}{2}|U|_t^2 \le \|\nabla\mathbf{u}\|_{L^\infty}|U|^2 - (\mathbf{u} \cdot \nabla)U * U - (\nabla_\mathbf{Q}\mathbf{F} \cdot U) * U + |\nabla^2\mathbf{u}||\nabla^3\mathbf{Q}||U|$$
$$+ |\nabla^3\mathbf{u}||\nabla^2\mathbf{Q}||U| + |\nabla^4\mathbf{u}||\nabla\mathbf{Q}||U| + |\nabla^5\mathbf{u}||\mathbf{Q}||U| + (|\nabla^3\mathbf{Q}||\nabla\mathbf{Q}|$$
$$+ |\nabla^2\mathbf{Q}|^2 + |\nabla^2\mathbf{Q}||\nabla\mathbf{Q}|^2 + |\nabla\mathbf{Q}|^4)(1 + |\mathbf{Q}|^p)|U|. \tag{81}$$

Integrating both sides and using

$$\int (\mathbf{u} \cdot \nabla)U * U \, dx = 0 \tag{82}$$

and the inequality $(\nabla_\mathbf{Q}\mathbf{F} \cdot U) * U \ge 0$, we get

$$\frac{1}{2}\frac{d}{dt}\mathcal{Q}_2^{(4)}(t) \le \|\nabla\mathbf{u}\|_{L^\infty}\mathcal{Q}_2^{(4)}(t) + \int_D \mathbb{E}|\nabla^2\mathbf{u}||\nabla^3\mathbf{Q}||U|dx + \int_D \mathbb{E}|\nabla^3\mathbf{u}||\nabla^2\mathbf{Q}||U|dx$$
$$+ \int_D \mathbb{E}|\nabla^4\mathbf{u}||\nabla\mathbf{Q}||U|dx + \int_D \mathbb{E}|\nabla^5\mathbf{u}||\mathbf{Q}||U|dx$$
$$+ \int_D \mathbb{E}|\nabla^3\mathbf{Q}||\nabla\mathbf{Q}|(1 + |\mathbf{Q}|^p)|U|dx + \int_D \mathbb{E}|\nabla^2\mathbf{Q}|^2(1 + |\mathbf{Q}|^p)|U|dx$$
$$+ \int_D \mathbb{E}|\nabla^2\mathbf{Q}||\nabla\mathbf{Q}|^2(1 + |\mathbf{Q}|^p)|U|dx + \int_D \mathbb{E}|\nabla\mathbf{Q}|^4(1 + |\mathbf{Q}|^p)|U|dx$$
$$\triangleq P1 + P2 + P3 + P4 + P5 + P6 + P7 + P8 + P9, \tag{83}$$

$$P2 \le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^2\mathbf{u}|^2|\nabla^3\mathbf{Q}|^2)dx$$
$$\le \mathcal{Q}_2^{(4)}(t) + \|\nabla^2\mathbf{u}\|_{L^6}^6 + \mathcal{Q}_3^{(3)}(t), \tag{84}$$

$$P3 \le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^3\mathbf{u}|^2|\nabla^2\mathbf{Q}|^2)dx$$
$$\le \mathcal{Q}_2^{(4)}(t) + \|\nabla^3\mathbf{u}\|_{L^4}^4 + \mathcal{Q}_4^{(2)}(t), \tag{85}$$

$$P4 \le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^4\mathbf{u}|^2|\nabla\mathbf{Q}|^2)dx$$
$$\le \mathcal{Q}_2^{(4)}(t) + \|\nabla^4\mathbf{u}\|_{L^3}^3 + \mathcal{Q}_6^{(1)}(t), \tag{86}$$

$$P5 \leq \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^5 \mathbf{u}|^2 |\mathbf{Q}|^2)dx$$
$$\leq \mathcal{Q}_2^{(4)}(t) + \mathcal{Q}_2^{(0)}(t)\|\nabla^5 \mathbf{u}\|_{L^2}^2, \tag{87}$$

$$P6 \leq \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^3 \mathbf{Q}|^2 |\nabla \mathbf{Q}|^2 (1 + |\mathbf{Q}|^{2p}))dx$$
$$\leq \mathcal{Q}_2^{(4)}(t) + \mathcal{Q}_{12p}^{(0)}(t) + \mathcal{Q}_{12}^{(1)}(t) + \mathcal{Q}_6^{(1)}(t) + \mathcal{Q}_3^{(3)}(t), \tag{88}$$

$$P7 \leq \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^2 \mathbf{Q}|^4 (1 + |\mathbf{Q}|^{2p}))dx$$
$$\leq \mathcal{Q}_2^{(4)}(t) + \mathcal{Q}_{6p}^{(0)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_6^{(2)}(t), \tag{89}$$

$$P8 \leq \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^2 \mathbf{Q}|^2 |\nabla \mathbf{Q}|^2 (1 + |\mathbf{Q}|^{2p}))dx$$
$$\leq \mathcal{Q}_2^{(4)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_4^{(1)}(t) + \mathcal{Q}_8^{(1)}(t) + \mathcal{Q}_{8p}^{(0)}(t), \tag{90}$$

$$P9 \leq \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla \mathbf{Q}|^8 (1 + |\mathbf{Q}|^{2p}))dx$$
$$\leq \mathcal{Q}_2^{(4)}(t) + \mathcal{Q}_{16}^{(1)}(t) + \mathcal{Q}_8^{(1)}(t) + \mathcal{Q}_{4p}^{(0)}(t), \tag{91}$$

where some constants are omitted. After dropping some lower order terms, we obtain

$$\mathcal{Q}_2^{(4)}(t) \leq e^{\int_0^t (\|\nabla \mathbf{u}\|_{L^\infty} + c)ds} \cdot \left(\mathcal{Q}_2^{(4)}(0) + \int_0^t (\|\nabla^2 \mathbf{u}\|_{L^6}^6 + \|\nabla^3 \mathbf{u}\|_{L^4}^4 + \|\nabla^4 \mathbf{u}\|_{L^3}^3 \right.$$
$$\left. + \mathcal{Q}_2^{(0)}(s)\|\nabla^5 \mathbf{u}\|_{L^2}^2 + \mathcal{Q}_{12p}^{(0)}(s) + \mathcal{Q}_{16}^{(1)}(s) + \mathcal{Q}_6^{(2)}(s) + \mathcal{Q}_3^{(3)}(s))ds \right). \tag{92}$$

$\square$

Lemma 2.10. *Assume that* $\mathbf{F}$ *satisfies Condition (A) as in Theorem 1.1, then*

$$|\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}| \leq C(1 + |\mathbf{Q}|^{p_1}), \tag{93}$$

$$|\nabla_x (\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})| \leq C|\nabla \mathbf{Q}|(1 + |\mathbf{Q}|^{p_2}), \tag{94}$$

$$|\nabla_x^2 (\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})| \leq C(|\nabla \mathbf{Q}|^2 + |\nabla^2 \mathbf{Q}|)(1 + |\mathbf{Q}|^{p_3}), \tag{95}$$

$$|\nabla_x^3 (\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})| \leq C(|\nabla \mathbf{Q}|^3 + |\nabla^2 \mathbf{Q}||\nabla \mathbf{Q}| + |\nabla^3 \mathbf{Q}|)(1 + |\mathbf{Q}|^{p_4}), \tag{96}$$

$$|\nabla_x^4 (\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})| \leq C(|\nabla^4 \mathbf{Q}| + |\nabla^3 \mathbf{Q}||\nabla \mathbf{Q}| + |\nabla^2 \mathbf{Q}|^2$$
$$+ |\nabla^2 \mathbf{Q}||\nabla \mathbf{Q}|^2 + |\nabla \mathbf{Q}|^4)(1 + |\mathbf{Q}|^{p_5}), \tag{97}$$

where p_1, p_2, p_3, p_4 *and* p_5 *are suitable integers which are greater than* p.

Proof. This follows from a direct calculation. $\square$

Lemma 2.11. *The stress τ has the following estimates:*

$$\|\tau\|_{L^2}^2 \leq C(1 + \mathcal{Q}_{2p_1}^{(0)}(t)) \cdot meas(D), \tag{98}$$

$$\|\nabla\tau\|_{L^2}^2 \leq C(1 + \mathcal{Q}_{2p_2}^{(0)}(t))\mathcal{Q}_2^{(1)}(t), \tag{99}$$

$$\|\nabla^2\tau\|_{L^2}^2 \leq (1 + \mathcal{Q}_{2p_3}^{(0)}(t))(\mathcal{Q}_4^{(1)}(t) + \mathcal{Q}_2^{(2)}(t)), \tag{100}$$

$$\|\nabla^3\tau\|_{L^2}^2 \leq (1 + \mathcal{Q}_{2p_4}^{(0)}(t))(\mathcal{Q}_6^{(1)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_2^{(3)}(t)), \tag{101}$$

$$\|\nabla^4\tau\|_{L^2}^2 \leq (1 + \mathcal{Q}_{2p_5}^{(0)}(t))(\mathcal{Q}_8^{(1)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_3^{(3)}(t) + \mathcal{Q}_2^{(4)}(t)), \tag{102}$$

where some lower order terms have been omitted.

Proof. The estimates of different order derivatives of τ will be done in different steps:

Step 1. Estimate of $\|\tau\|_{L^2}$:

$$\begin{aligned}
\|\tau\|_{L^2}^2 &= \int |\mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})|^2 dx \leq \int (\mathbb{E}|\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}|)^2 dx \\
&\leq C \int \left(\mathbb{E}(1 + |\mathbf{Q}|^{p_1})\right)^2 dx \leq C \int (1 + \mathbb{E}|\mathbf{Q}|^{2p_1}) dx \\
&= C \int (1 + \mathbb{E}|q|^{2p_1}) dx \leq C(1 + \mathcal{Q}_{2p_1}^{(0)}(t)) \cdot meas(D).
\end{aligned} \tag{103}$$

Step 2. Estimate of $\|\nabla\tau\|_{L^2}$:

$$\begin{aligned}
\|\nabla\tau\|_{L^2}^2 &= \int |\nabla\mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})|^2 dx \\
&\leq C \int \mathbb{E}(1 + |\mathbf{Q}|^{2p_2})\mathbb{E}|\nabla\mathbf{Q}|^2 dx \\
&\leq C(1 + \mathcal{Q}_{2p_2}^{(0)}(t))\mathcal{Q}_2^{(1)}(t).
\end{aligned} \tag{104}$$

Step 3. Estimate of $\|\nabla^2\tau\|_{L^2}$:

$$\begin{aligned}
\|\nabla^2\tau\|_{L^2}^2 &= \int |\nabla_x^2\mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})|^2 dx \\
&\leq C \int \mathbb{E}(|\nabla\mathbf{Q}|^4 + |\nabla^2\mathbf{Q}|^2)\mathbb{E}(1 + |\mathbf{Q}|^{2p_3}) dx \\
&\leq C(1 + \mathcal{Q}_{2p_3}^{(0)}(t))(\mathcal{Q}_4^{(1)}(t) + \mathcal{Q}_2^{(2)}(t)).
\end{aligned} \tag{105}$$

Step 4. Estimate of $\|\nabla^3\tau\|_{L^2}$:

$$\begin{aligned}
\|\nabla^3\tau\|_{L^2}^2 &= \int |\nabla_x^3\mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})|^2 dx \\
&\leq C \int \mathbb{E}(|\nabla\mathbf{Q}|^6 + |\nabla^2\mathbf{Q}|^2|\nabla\mathbf{Q}|^2 + |\nabla^3\mathbf{Q}|^2)\mathbb{E}(1 + |\mathbf{Q}|^{2p_4}) dx \\
&\leq C(1 + \mathcal{Q}_{2p_4}^{(0)}(t))(\mathcal{Q}_6^{(1)}(t) + \mathcal{Q}_4^{(1)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_2^{(3)}(t)).
\end{aligned} \tag{106}$$

Step 5. Estimate of $\|\nabla^4 \tau\|_{L^2}$:

$$
\begin{aligned}
\|\nabla^4 \tau\|_{L^2}^2 &= \int |\nabla_x^4 \mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})|^2 dx \\
&\leq C \int \mathbb{E}(|\nabla^4 \mathbf{Q}|^2 + |\nabla^3 \mathbf{Q}|^2 |\nabla \mathbf{Q}|^2 + |\nabla^2 \mathbf{Q}|^4 + |\nabla^2 \mathbf{Q}|^2 |\nabla \mathbf{Q}|^4 \\
&\quad + |\nabla \mathbf{Q}|^8) \mathbb{E}(1 + |\mathbf{Q}|^{2p_5}) dx \\
&\leq C(1 + \mathcal{Q}_{2p_5}^{(0)}(t))(\mathcal{Q}_8^{(1)}(t) + \mathcal{Q}_6^{(1)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_3^{(3)}(t) + \mathcal{Q}_2^{(4)}(t)).
\end{aligned}
\tag{107}
$$

This completes the a priori estimates. □

3. The Local Well-Posedness

To prove local well-posedness, namely Theorem 1.1, we set up a standard iteration scheme. Let $\mathbf{u}^0(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x})$, $\mathbf{Q}^0(\mathbf{x}, t, \omega) = \mathbf{Q}_0(\mathbf{x}, \omega)$. From $\{\mathbf{u}^0, \mathbf{Q}^0\}$, we obtain a sequence $\{\mathbf{u}^n, \mathbf{Q}^n\}$ by solving the following system

$$
\begin{cases}
\partial_t \mathbf{u}^{n+1} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} = \Delta \mathbf{u}^{n+1} + \nabla \cdot \tau^n, \\
\nabla \cdot \mathbf{u}^{n+1} = 0, \\
\tau^n = \mathbb{E}(\mathbf{F}(\mathbf{Q}^n) \otimes \mathbf{Q}^n), \\
\partial_t \mathbf{Q}^{n+1} + (\mathbf{u}^{n+1} \cdot \nabla)\mathbf{Q}^{n+1} = \kappa^{n+1} \mathbf{Q}^{n+1} - \mathbf{F}(\mathbf{Q}^{n+1}) + \dot{\mathbf{w}}(t),
\end{cases}
\tag{108}
$$

where $\kappa^{n+1} = \nabla \mathbf{u}^{n+1}$.

We will prove that for short times, the sequence $\{\mathbf{u}^n\}$ is uniformly bounded in a high enough Sobolev norm, and contractive in the L^2 norm.

Define the norm:

$$
|||\mathbf{u}^n||| = \left(\max_{s \leq T^*} \|\mathbf{u}^n(s)\|_{H^4}^2 + \int_0^{T^*} \|\mathbf{u}^n(s)\|_{H^5}^2 ds \right)^{\frac{1}{2}},
\tag{109}
$$

where T^* is a constant to be determined later.

Step 1. Uniform boundedness of $\mathbf{u}^n$. It follows from Lemma 2.2 that

$$
\begin{aligned}
|||\mathbf{u}^{n+1}|||^2 &\leq e^{\int_0^{T^*} \|\mathbf{u}^n\|_{H^4} ds} (\|\mathbf{u}_0\|_{H^4}^2 + \int_0^{T^*} \|\tau^n\|_{H^4}^2 ds) + \|\mathbf{u}_0\|_{H^4}^2 \\
&\quad + \int_0^{T^*} \{\|\mathbf{u}^n\|_{H^4} [e^{\int_0^{T^*} \|\mathbf{u}^n\|_{H^4} ds} (\|\mathbf{u}_0\|_{H^4}^2 + \int_0^{T^*} \|\tau^n\|_{H^4}^2 ds)] + \|\tau^n\|_{H^4}^2\} ds.
\end{aligned}
\tag{110}
$$

Define $C_0 = \|\mathbf{u}_0\|_{H^4}^2$, assume $|||\mathbf{u}^n||| \leq K$, then

$$
\begin{aligned}
|||\mathbf{u}^{n+1}|||^2 &\leq e^{KT^*} \left(C_0 + \int_0^{T^*} \|\tau^n\|_{H^4}^2 ds \right) + C_0 \\
&\quad + \int_0^{T^*} \left(K[e^{KT^*}(C_0 + \int_0^t \|\tau^n\|_{H^4}^2 ds)] + \|\tau^n\|_{H^4}^2 \right) ds.
\end{aligned}
\tag{111}
$$

We only need to consider $\int_0^t \|\tau^n\|_{H^4}^2 ds$.

From Lemma 2.11, the highest order terms in $\int_0^t \|\tau^n\|_{H^4}^2 ds$ are $\mathcal{Q}_6^{(2)}(t)$, $\mathcal{Q}_3^{(3)}(t)$, $\mathcal{Q}_2^{(4)}(t)$, which corresponds to the highest order spatial derivative terms. Their validity is easily obtained by using

$$\|\nabla^4 \mathbf{u}\|_{L^3}^3 \leq \|\nabla^4 \mathbf{u}\|_{H^{\frac{1}{2}}}^3 \leq (\|\nabla^4 \mathbf{u}\|_{H^0}^{\frac{1}{2}})^3 (\|\nabla^4 \mathbf{u}\|_{H^1}^{\frac{1}{2}})^3 \leq \|\nabla^4 \mathbf{u}\|_{L^2}^{\frac{3}{2}} \|\mathbf{u}\|_{H^5}^{\frac{3}{2}}. \qquad (112)$$

We will simplify the term $\int_0^t \|\tau^n\|_{H^4}^2 ds \leq f(t, K)$, where $f(t, K)$ denotes the right hand side of the last inequality in Lemma 2.10. $f(t, K)$ is a monotonely increasing continuous function of t, and $f(0, K) = 0$, thus

$$|||\mathbf{u}^{n+1}|||^2 \leq e^{KT^*}(C_0 + f(T^*, K)) + C_0 + KT^* e^{KT^*}(C_0 + f(T^*, K)) + Kf(T^*, K). \quad (113)$$

Let $K = 2C_0 + 1$, and choose T^* sufficiently small such that

$$e^{KT^*}(C_0 + f(T^*, K)) + C_0 + KT^* e^{KT^*}(C_0 + f(T^*, K)) + Kf(T^*, K) \leq K^2, \quad (114)$$

then we have

$$|||\mathbf{u}^n||| \leq K. \qquad (115)$$

Step 2. Contraction in the low norm:

Define $\mathbf{v}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}^n$, $\pi^{n+1} = p^{n+1} - p^n$, $\mathbf{R}^{n+1} = \mathbf{Q}^{n+1} - \mathbf{Q}^n$, then

$$\mathbf{v}_t^{n+1} + (\mathbf{u}^n \cdot \nabla)\mathbf{v}^{n+1} + (\mathbf{v}^n \cdot \nabla)\mathbf{u}^n + \nabla\pi^{n+1} = \Delta\mathbf{v}^{n+1} + \nabla \cdot (\tau^n - \tau^{n-1}), \quad (116)$$

$$\nabla \cdot \mathbf{v}^{n+1} = 0,$$

$$\mathbf{R}_t^{n+1} - (\mathbf{u}^{n+1} \cdot \nabla)\mathbf{R}^{n+1} - (\mathbf{v}^{n+1} \cdot \nabla)\mathbf{Q}^n = \kappa^{n+1}\mathbf{R}^{n+1} + \nabla(\mathbf{u}^{n+1} - \mathbf{u}^n)\mathbf{Q}^n$$
$$- (\mathbf{F}(\mathbf{Q}^{n+1}) - \mathbf{F}(\mathbf{Q}^n)). \qquad (117)$$

From the definition of τ we have

$$\|\tau^n - \tau^{n-1}\|_{L^2}^2 \leq \int_D \mathbb{E}|\mathbf{F}(\mathbf{Q}^n)|^2 \mathbb{E}|\mathbf{Q}^n - \mathbf{Q}^{n-1}|^2 dx$$
$$+ \int_D \mathbb{E}|\mathbf{Q}^{n-1} \otimes \nabla_{\mathbf{Q}}\mathbf{F}(\mathbf{Q}^\theta)|^2 \mathbb{E}|\mathbf{Q}^n - \mathbf{Q}^{n-1}|^2 dx$$
$$\leq C(1 + \mathcal{Q}_{2p+2}^{(0)}(t)) \int_D \mathbb{E}|\mathbf{R}^n|^2 dx, \qquad (118)$$

where $\mathbf{Q}^\theta = \theta\mathbf{Q}^n + (1-\theta)\mathbf{Q}^{n-1}$, $\theta \in [0, 1]$. From (116)

$$\frac{d}{dt}\|\mathbf{v}^{n+1}\|_{L^2}^2 \leq \|\mathbf{v}^{n+1}\|_{L^2}^2 - \|\nabla\mathbf{v}^{n+1}\|_{L^2}^2 + \|\tau^n - \tau^{n-1}\|_{L^2}^2$$
$$+ \|\nabla\mathbf{u}^n\|_{L^\infty}(\|\mathbf{v}^n\|_{L^2}^2 + \|\mathbf{v}^{n+1}\|_{L^2}^2)$$
$$\leq \|\mathbf{v}^{n+1}\|_{L^2}^2 - \|\nabla\mathbf{v}^{n+1}\|_{L^2}^2 + C(1 + \mathcal{Q}_{2p+2}^{(0)}(t))\mathbb{E}\|\mathbf{R}^n\|_{L^2}^2$$
$$+ K(\|\mathbf{v}^n\|_{L^2}^2 + \|\mathbf{v}^{n+1}\|_{L^2}^2). \qquad (119)$$

From (117)

$$\frac{d}{dt}\|\mathbf{R}^{n+1}\|_{L^2}^2 \leq \|\nabla\mathbf{u}^{n+1}\|_{L^\infty}\|\mathbf{R}^{n+1}\|_{L^2}^2 + 2\int_D (1+|\mathbf{Q}^{n+1}|^p + |\mathbf{Q}^n|^p)|\mathbf{R}^{n+1}|dx$$

$$+ \int_D (\mathbf{v}^{n+1}\cdot\nabla\mathbf{Q}^n\cdot\mathbf{R}^{n+1})dx + \int_D (\nabla(\mathbf{u}^{n+1}-\mathbf{u}^n)\cdot\mathbf{Q}^n\cdot\mathbf{R}^{n+1})dx$$

$$\triangleq P1 + P2 + P3 + P4, \tag{120}$$

$$P3 \leq \int_D |\mathbf{v}^{n+1}|^2|\nabla\mathbf{Q}^n|^2 dx + \|\mathbf{R}^{n+1}\|_{L^2}^2$$

$$\leq \|\nabla\mathbf{Q}^n\|_{L^\infty}^2 \int_D |\mathbf{v}^{n+1}|^2 dx + \|\mathbf{R}^{n+1}\|_{L^2}^2$$

$$\leq \|\mathbf{Q}^n\|_{H^3}^2 \int_D |\mathbf{v}^{n+1}|^2 dx + \|\mathbf{R}^{n+1}\|_{L^2}^2, \tag{121}$$

$$P4 \leq C_1\int_D |\nabla\mathbf{v}^{n+1}|^2|\mathbf{Q}^n|^2 dx + C_2\|\mathbf{R}^{n+1}\|_{L^2}^2, \tag{122}$$

where C_1, C_2 is chosen such that $C_1\mathcal{Q}_2^{(0)}(T^*) \leq 1$. Thus

$$\frac{d}{dt}\mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2 \leq C(\|\nabla\mathbf{u}^{n+1}\|_{L^\infty} + \sqrt{\mathcal{Q}_{2p}^{(0)}(t)} + 1)\mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2$$

$$+ \mathbb{E}\|\mathbf{Q}^n\|_{H^3}^2 \int_D |\mathbf{v}^{n+1}|^2 dx$$

$$+ C_1\int_D |\nabla\mathbf{v}^{n+1}|^2\mathbb{E}|\mathbf{Q}^n|^2 dx$$

$$\leq C(\|\nabla\mathbf{u}^{n+1}\|_{L^\infty} + \sqrt{\mathcal{Q}_{2p}^{(0)}(T^*)} + 1)\mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2 + C\|\mathbf{v}^{n+1}\|_{L^2}^2$$

$$+ C_1\mathcal{Q}_2^{(0)}(T^*)\int_D |\nabla\mathbf{v}^{n+1}|^2 dx$$

$$\leq C\mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2 + C\|\mathbf{v}^{n+1}\|_{L^2}^2 + \int_D |\nabla\mathbf{v}^{n+1}|^2 dx. \tag{123}$$

Hence we have, for some constant C_*,

$$\frac{d}{dt}(\|\mathbf{v}^{n+1}\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2) \leq C_*(\|\mathbf{v}^{n+1}\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2)$$

$$+ (\|\mathbf{v}^n\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^n\|_{L^2}^2). \tag{124}$$

Gronwall's inequality shows

$$\|\mathbf{v}^{n+1}(t)\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^{n+1}(t)\|_{L^2}^2 \leq e^{C_* t}\int_0^t (\|\mathbf{v}^n\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^n\|_{L^2}^2)ds. \tag{125}$$

Let $\|\mathbf{v}^n\|_C = \max_{s\leq t}(\|\mathbf{v}^n(s)\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^n(s)\|_{L^2}^2)^{\frac{1}{2}}$. If

$$T^* e^{C_* T^*} = \beta < 1, \tag{126}$$

then we have

$$\|\mathbf{v}^{n+1}\|_C^2 \leq \beta \|\mathbf{v}^n\|_C^2. \tag{127}$$

This establishes the contraction property.

The rest of the proof follows from standard arguments, see for example [14].

4. Conclusion

In this paper, we give a proof of well-posedness of the stochastic model (1)-(3) for a dumbbell-solvent system. We believe that our method, which is based on analyzing directly the stochastic model (1) is of interest by itself. The results of this paper have been used in [4] for the numerical analysis of stochastic simulation methods based on Brownian configuration fields (BCF). Through our analysis, we also demonstrate how to handle directly the stochastic system (1)-(3).

The main technical restriction in our analysis is the fact that noise depends only on time. This is fine for dilute polymer solutions since polymer-polymer interaction can be neglected. But for semi-dilute or for concentrated solutions, this kind of technique can only handle systems under the mean field approximation (see [2]). The mean field approximation is typically made for liquid crystal polymer systems. Therefore a natural next step is to extend the results of the present paper for that case [12].

In the special case when $\mathbf{F}$ is linear in $\mathbf{Q}$, our results recover the local well-posedness results of Saut, but not the results of Lions and Masmoudi for the global existence of weak solutions for the Oldroyd model [13].

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