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004

CONVERGENCE OF THE VARIABLE-ELLIPTIC-VORTEX METHOD FOR EULER EQUATIONS*

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Abstract. A general formulation of the variable-elliptic-vortex method for the incompressible Euler equations is derived, and its consistency, stability and convergence are proved. The main feature of this method is that not only the centers of the vortex blobs are transported by the induced velocity field, but also the blobs themselves are rotated and deformed in the elliptic shape according to the Jacobian matrix of the induced velocity field. The variable-elliptic-vortex method provides a more flexible and more reasonable approach to mimic physical flows and allows a smooth transition from vortex blobs to sheets and vice versa. The theoretic analysis indicates that the discretization error using variable blobs is smaller than that using fixed blobs. Several issues on the practical aspects of the method are also addressed.

Key words. vortex method, variable elliptic blobs, convergence

AMS subject classifications. 65M25, 76D05

1. Introduction.

Vortex methods based on Lagrangian formulations are effective methods for the stimulation of incompressible flows (see Chorin [4] and Leonard [12]). The features of these methods are that the interactions of the numerical vortices mimic the physical mechanisms in actual fluid flow, vortex methods are automatically adaptive because the vortex blobs concentrate in the regions of physical interest, and there are no inherent errors with behavior like the numerical viscosity of Eulerian difference methods. The basic idea of the vortex methods for the two-dimensional inviscid case is to approximate the vorticity distribution by a collection of radially symmetric vortex blobs of fixed shape and to let the centers of the blobs be moved by the velocity field that is induced by the approximate vorticity distribution. The convergence of vortex methods was first obtained by Hald [8]; then the results were improved and different proofs were given by Anderson and Greengard [1], Beale and Majda [2], [3], and Raviart [13]. All of the results and some new advances on the initial boundary problem are included in the book *Vortex Method* by Ying and Zhang [17].

However, notice that all the standard numerical vortex blobs mentioned above are assumed to retain a fixed shape for all time, while the actual flow can undergo substantial distortion. The "nonphysical behavior" of the vortex blobs might reduce the accuracy of the vortex methods even though it does not interfere with the convergence of vortex methods. The variable-elliptic-vortex method proposed by the first author of this paper in [15] and [16] can follow the distortion of the actual vortex blobs and allow a smooth transition from vortex blobs to sheets and vice versa. The feature of this method is that not only the centers of the blobs are transported by the induced velocity field, but also the blobs themselves are rotated and deformed in the elliptic shape according to the Jacobian matrix of the induced velocity field.

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Therefore, the variable-elliptic-vortex method provides a more flexible and more reasonable approach to mimic physical flows and also a more accurate method to stimulate flows with strong local shear and, in particular, to stimulate boundary layer flow [15], [18]. Another potential application of the variable-elliptic-vortex method is to capture small-scale structures of turbulence, since the elliptic-vortex blob might be a basic model with which to approach turbulence (see Chorin [5], [6]).

In this paper we first derive a general formulation of the variable-elliptic-vortex method for the two-dimensional Euler equations and then prove its consistency, stability, and convergence. At last we also discuss several issues on the practical aspects of the proposed method. The theoretic analysis indicates that the discretization error and the moment error using variable blobs are smaller than those using fixed blobs, and the convergence theorem shows that the variable-elliptic-vortex method can approximate not only the exact particle trajectories of the fluid but also the Jocobian matrices of the flow map.

Here we would like to mention that the study of this paper is also motivated by the work of Hou [10], in which he proved the convergence of a variable-blob vortex method for the Euler and Navier–Stokes equations under the assumption that the deformations of the vortex blobs are known and the upper and lower bounds on the deformations are assumed.

2. Formulation of the variable-elliptic-vortex method. The incompressible two-dimensional Euler equation can be written in the vorticity form

$$\omega_t + (u \cdot \nabla)\omega = 0, \quad \omega(x,0) = \omega_0(x), \tag{2.1}$$

where u is defined by the Biot–Savart law through ω

$$u(x,t) = \int_{\mathbf{R}^2} K(x-y)\omega(y,t)\,dy \tag{2.2a}$$

and K is the Biot–Savart kernel

$$K(x) = \frac{1}{2\pi|x|^2}(-x_2, x_1). \tag{2.2b}$$

Let $\phi(\alpha,t)$ be a flow map (characteristic line) that is defined by

$$\frac{d\phi(\alpha,t)}{dt} = u(\phi(\alpha,t),t), \quad \phi(\alpha,0) = \alpha, \tag{2.3}$$

where α is the Lagrangian coordinate for the Euler equations. This, along with (2.1) and (2.2), gives

$$\frac{d\omega(\phi(\alpha,t),t)}{dt} = 0, \quad \omega(\phi(\alpha,t),t) = \omega_0(\alpha),$$

and further

$$\frac{d\phi(\alpha,t)}{dt} = u(\phi(\alpha,t),t) = \int_{\mathbf{R}^2} K(\phi(\alpha,t) - x')\omega(x',t)\,dx'. \tag{2.4}$$

Using the transformation $x' = \phi(\alpha',t)$ to (2.4), which satisfies $\det(\nabla\phi(\alpha',t)) = 1$, we obtain

$$\begin{aligned}\frac{d\phi(\alpha,t)}{dt} &= \int K(\phi(\alpha,t) - \phi(\alpha',t))\omega(\phi(\alpha',t),t)\,d\alpha' \\ &= \int K(\phi(\alpha,t) - \phi(\alpha',t))\omega_0(\alpha')\,d\alpha'.\end{aligned}$$

Thus we obtain the equivalent Lagrangian formulation of the Euler equations

$$\frac{d\phi(\alpha,t)}{dt} = \int K\big(\phi(\alpha,t)-\phi(\alpha',t)\big)\omega_0(\alpha')\,d\alpha', \quad \phi(\alpha,0)=\alpha, \tag{2.5}$$

which is the basis for our definition of the variable-vortex method. To solve (2.5), we cover the α-plane by nonoverlapping square meshes with mesh length h and centered at $\alpha_j = jh$ and let

$$k_j = \omega_0(\alpha_j)h^2, \quad \alpha_j = jh = (j_1h, j_2h).$$

$\rho(x) = \rho(|x|)$ is called an mth-order blob function if it satisfies

$$\rho(x) = 0 \quad \text{for} \quad |x| \geq 1,$$
$$\int_{\mathbf{R}^2} \rho(x)\,dx = 1,$$
$$\int_{\mathbf{R}^2} x^\beta \rho(x)\,dx = 0 \quad \forall \beta \in N^2 \quad \text{with} \quad 1 \leq |\beta| \leq m-1,$$

and $\rho_\delta(\alpha)$ is defined by

$$\rho_\delta(\alpha) = \frac{1}{\delta^2}\rho\left(\frac{\alpha}{\delta}\right).$$

Now let us formulate the variable-vortex method. To begin, we approximate the initial vorticity $\omega_0(\alpha')$ by a collection of vortex blobs

$$\overline{\omega}_0(\alpha') = \sum_j k_j \rho_\delta(\alpha'-\alpha_j), \tag{2.6}$$

and then we write (2.5) in the following form:

$$\begin{aligned}\frac{d\phi(\alpha,t)}{dt} &= \int K\big(\phi(\alpha,t)-\phi(\alpha',t)\big)\big(\omega_0(\alpha')-\overline{\omega}_0(\alpha')\big)\,d\alpha' \\ &\quad + \sum_j k_j \int \Big[K\big(\phi(\alpha,t)-\phi(\alpha',t)\big) - K\big(\phi(\alpha,t)-\phi(\alpha_j,t) \\ &\qquad - \nabla\phi(\alpha_j,t)(\alpha'-\alpha_j)\big)\Big]\rho_\delta(\alpha'-\alpha_j)\,d\alpha' \\ &\quad + \sum_j k_j \int K\big(\phi(\alpha,t)-\phi(\alpha_j,t)-\nabla\phi(\alpha_j,t)(\alpha'-\alpha_j)\big)\rho_\delta(\alpha'-\alpha_j)\,d\alpha', \\ \phi(\alpha,0) &= \alpha.\end{aligned} \tag{2.7}$$

Taking the derivative with respect to α in the above equations, we have

$$\begin{aligned}\frac{d\nabla\phi(\alpha,t)}{dt} &= \int \nabla K\big(\phi(\alpha,t)-\phi(\alpha',t)\big)\big(\omega_0(\alpha')-\overline{\omega}_0(\alpha')\big)\,d\alpha' \cdot \nabla\phi(\alpha,t) \\ &\quad + \sum_j k_j \int \Big[\nabla K\big(\phi(\alpha,t)-\phi(\alpha',t)\big) - \nabla K\big(\phi(\alpha,t)-\phi(\alpha_j,t) \\ &\qquad - \nabla\phi(\alpha_j,t)(\alpha'-\alpha_j)\big)\Big]\rho_\delta(\alpha'-\alpha_j)\,d\alpha' \cdot \nabla\phi(\alpha,t) \\ &\quad + \sum_j k_j \int \nabla K\big(\phi(\alpha,t)-\phi(\alpha_j,t)-\nabla\phi(\alpha_j,t)(\alpha'-\alpha_j)\big)\rho_\delta(\alpha'-\alpha_j)\,d\alpha' \cdot \nabla\phi(\alpha,t), \\ \nabla\phi(\alpha,0) &= E,\end{aligned} \tag{2.8}$$

where E is the identity matrix.

If we set $\alpha = \alpha_i$ into (2.7) and (2.8) and drop the first two "truncation error" terms on the right-hand sides of (2.7) and (2.8), then we obtain a differential system that defines the variable-elliptic-vortex solution $\overline{\phi}_i(t)$ and $\overline{\nabla\phi}_i(t)$ for $i \in \mathbf{Z}^2$ as

$$\frac{d\overline{\phi}_i(t)}{dt} = \sum_j k_j \int K\big(\overline{\phi}_i(t) - \overline{\phi}_j(t) - \overline{\nabla\phi}_j(t)(\alpha' - \alpha_j)\big)\rho_\delta(\alpha' - \alpha_j)\, d\alpha', \tag{2.9}$$

$$\frac{d\overline{\nabla\phi}_i(t)}{dt} = \sum_j k_j \int \nabla K\big(\overline{\phi}_i(t) - \overline{\phi}_j(t) - \overline{\nabla\phi}_j(t)(\alpha' - \alpha_j)\big)\rho_\delta(\alpha' - \alpha_j)\, d\alpha' \cdot \overline{\nabla\phi}_i(t) \tag{2.10}$$

with initial conditions

$$\overline{\phi}_i(0) = ih, \quad \overline{\nabla\phi}_i(0) = E. \tag{2.11}$$

Here $\overline{\phi}_i(t)$ is an approximation of $\phi(\alpha_i, t)$ and $\overline{\nabla\phi}_i(t)$, which is a 2×2 matrix, is an approximation of $\nabla\phi(\alpha_i, t)$.

We first prove some properties of the variable-elliptic-vortex solution $\overline{\phi}_i(t)$ and $\overline{\nabla\phi}_i(t)$.

PROPOSITION 1. *If the variable-elliptic-vortex solution $\overline{\phi}_i(t)$ and $\overline{\nabla\phi}_i(t)$ for $i \in \mathbf{Z}^2$ exists, then*

$$\det(\overline{\nabla\phi}_i(t)) = 1. \tag{2.12}$$

Proof. Since $K = \nabla^\perp g(x)$, where $g(x) = \frac{1}{2\pi}\ln|x|$, $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$, we have

$$\operatorname{tr}(\nabla K) = 0, \tag{2.13}$$

where $\operatorname{tr}(a_{ij})_{2\times 2} = a_{11} + a_{22}$. According to the Liouville theorem [14], we find

$$\begin{aligned}
&\det\big(\overline{\nabla\phi}_i(t)\big) \\
&= \det\big(\overline{\nabla\phi}_i(0)\big) \exp \int_0^t \operatorname{tr}\Big(\sum_j k_j \int \nabla K\big(\overline{\phi}_i(t) - \overline{\phi}_j(t) - \overline{\nabla\phi}_j(t)(\alpha' - \alpha_j)\big) \\
&\qquad\qquad \times \rho_\delta(\alpha' - \alpha_j)\, d\alpha'\Big)\, dt \\
&= \det(E) \exp \int_0^t \sum_j k_j \int \operatorname{tr}\Big(\nabla K\big(\overline{\phi}_i(t) - \overline{\phi}_j(t) - \overline{\nabla\phi}_j(t)(\alpha' - \alpha_j)\big)\Big) \\
&\qquad\qquad \times \rho_\delta(\alpha' - \alpha_j)\, d\alpha'\, dt.
\end{aligned}$$

In view of (2.13), we get (2.12). $\square$

Using the transformation

$$x' = \overline{\nabla\phi}_j(t)(\alpha' - \alpha_j)$$

to (2.9) and noting that

$$\det\big(\overline{\nabla\phi}_j(t)\big) = 1,$$

we find that

$$\frac{d\overline{\phi}_i(t)}{dt} = \sum_j k_j \int K\big(\overline{\phi}_i(t) - \overline{\phi}_j(t) - x'\big)\rho_\delta\left(\overline{\nabla\phi}_j(t)^{-1} \cdot x'\right) dx'. \tag{2.14}$$

The support of $\rho_\delta\left(\overline{\nabla\phi}_j(t)^{-1}\cdot x'\right)$ with respect to x' is

$$\Omega_j(t) := \{x' \,|\, x'\overline{\nabla\phi}_j(t)^{-1}\left(\overline{\nabla\phi}_j(t)^{-1}\right)^T x'^T \le \delta^2, \quad x' \in \mathbf{R}^2\} \tag{2.15}$$

and has the following properties.

PROPOSITION 2. *The shape of* $\Omega_j(t)$ *defined by* (2.15) *is an ellipse with conserved area in time*

$$\text{meas}\,(\Omega_j(t)) = \pi\delta^2.$$

Proof. By (2.12) and (2.15) we can easily arrive at the conclusion. □

In comparison with the standard (fixed) vortex method [4], [12], we present the governing equations for the fixed vortex solution $\widehat{\phi}_i(t)$:

$$\frac{d\widehat{\phi}_i(t)}{dt} = \sum_j k_j \int K\left(\widehat{\phi}_i(t) - x'\right)\rho_\delta\left(x' - \widehat{\phi}_j(t)\right) dx'$$

or

$$\frac{d\widehat{\phi}_i(t)}{dt} = \sum_j k_j \int K\left(\widehat{\phi}_i(t) - \widehat{\phi}_j(t) - x'\right)\rho_\delta(x')\, dx'. \tag{2.16}$$

We can see that the shape of the blob function $\rho_\delta(\cdot)$ given by (2.16) is fixed while that of $\rho_\delta(\overline{\nabla\phi}_j(t)^{-1}(\cdot))$ given by (2.14) can be changed in the elliptic shape. Therefore, we call (2.9) and (2.10) the variable-elliptic-vortex method.

Now we define a regularized kernel $K_\delta(z; A)$ by

$$K_\delta(z; A) = \int K(z - A\alpha')\rho_\delta(\alpha')\, d\alpha' \tag{2.17a}$$

or, equivalently,

$$K_\delta(z; A) = \int K(y)\rho_\delta\left(A^{-1}(z - y)\right) dy, \tag{2.17b}$$

where $A = (a_{ij})$ is a 2×2 matrix with the properties

$$|a_{ij}| \le C, \tag{2.18}$$

$$\det A = 1. \tag{2.19}$$

Thus the variable-elliptic-vortex method (2.9)–(2.11) can be written as follows:

$$\frac{d\overline{\phi}_i(t)}{dt} = \sum_j k_j K_\delta\left(\overline{\phi}_i(t) - \overline{\phi}_j(t); \overline{\nabla\phi}_j(t)\right), \quad \overline{\phi}_i(0) = \alpha_i, \tag{2.20}$$

$$\frac{d\overline{\nabla\phi}_i(t)}{dt} = \sum_j k_j \nabla K_\delta\left(\overline{\phi}_i(t) - \overline{\phi}_j(t); \overline{\nabla\phi}_j(t)\right)\cdot \overline{\nabla\phi}_i(t), \quad \overline{\nabla\phi}_i(0) = E. \tag{2.21}$$

The vorticity distribution $\overline{\omega}(x,t)$ for the variable-elliptic-vortex method is defined by the sum of variable blobs, centered at $\overline{\phi}_j(t)$ and of the shape $\rho_\delta(\overline{\nabla\phi}_j(t)^{-1}(\cdot))$,

$$\overline{\omega}(x,t) = \sum_j k_j\rho_\delta\left(\overline{\nabla\phi}_j(t)^{-1}\left(x - \overline{\phi}_j(t)\right)\right), \tag{2.22}$$

and the velocity field $\overline{u}(x,t)$ is defined by

$$\overline{u}(x,t) = \int K(y)\overline{\omega}(x-y,t)\,dy$$

or

$$\overline{u}(x,t) = \sum_j k_j K_\delta\left(x - \overline{\phi}_j(t); \overline{\nabla\phi}_j(t)\right). \tag{2.23}$$

3. Main theorem. We define some discrete norm as follows:

$$\|\nu_i\|_{l_h^p} = \left(h^2 \sum_{j\in\mathbf{Z}^2} |\nu_j|^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$|A| = \max_{ij} |a_{ij}|, \quad A = (a_{ij}).$$

Our main result in this paper is the following theorem.

THEOREM 1. *Assume the solution of* (2.1) *is sufficiently smooth,* ω_0 *has compact support, and, moreover, that* $\rho(x)$ *is an mth-order* ($m \ge 2$) *blob function and* ρ *also has compact support,* $\rho \in C_0^l(\mathbf{R}^2)$. *Let* $h \le \delta^a$ *and* $(a-1)l \ge 1$. *Then the solutions of* (2.20)–(2.23) *satisfy*

$$\left\|\overline{\phi}_i(t) - \phi(\alpha_i,t)\right\|_{l_h^p} \le C\delta^2|\log\delta|, \tag{3.1}$$

$$\left\|\overline{\nabla\phi}_i(t) - \nabla\phi(\alpha_i,t)\right\|_{l_h^p} \le C\delta|\log\delta|, \tag{3.2}$$

$$\left\|\overline{u}(\overline{\phi}_i(t),t) - u(\phi(\alpha_i,t),t)\right\|_{l_h^p} \le C\delta^2|\log\delta|, \tag{3.3}$$

$$\left\|\nabla\overline{u}(\overline{\phi}_i(t),t) - \nabla u(\phi(\alpha_i,t),t)\right\|_{l_h^p} \le C\delta|\log\delta| \tag{3.4}$$

for $0 \le t \le T$, *where* $\phi(\alpha,t)$ *and* $u(x,t)$ *are the corresponding exact solutions of the Euler equations.*

In this paper C and C_1 denote constants that are independent of δ and h but may depend on ρ, ω_0, T, and bounds for a finite number of derivatives of the exact solution ϕ, while C' denotes the same kind of constant but does not depend on T and ϕ. In different places C and C' may stand for different values.

In order to prove the theorem we need a number of technical lemmas.

LEMMA 1. *Suppose the components of a matrix* $A(\theta) = (a_{ij}(\theta))$ *are smooth functions on* $0 \le \theta \le 1$ *and satisfy* (2.18), (2.19). *Then the regularized kernel* $K_\delta(z; A(\theta))$ *has the following properties:*

$$\left|D_A^\gamma D_z^\beta K_\delta(z; A(\theta))\right| \le C|\delta|^{-1-|\beta|-|\gamma|} \quad \textit{for all } z, \tag{3.5}$$

$$\left|D_A^\gamma D_z^\beta K_\delta(z; A(\theta))\right| \le C|z|^{-1-|\beta|-|\gamma|} \quad \textit{for} \quad |z| \ge C_1\delta, \tag{3.6}$$

where $\beta = (\beta_1, \beta_2)$ *and* $\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ *are multi-index with* $|\beta| = \sum_{i=1}^2 \beta_i$ *and* $|\gamma| = \sum_{i=1}^4 \gamma_i$ *and*

$$\left|D_\theta K_\delta(z; A(\theta))\right| \le C|\delta|^{-1}\left|\frac{dA(\theta)}{d\theta}\right| \quad \textit{for all } z, \tag{3.7}$$

$$|D_\theta K_\delta(z;A(\theta))| \le C\delta|z|^{-2}\left|\frac{dA(\theta)}{d\theta}\right| \quad \text{for} \quad |z| \ge C_1\delta, \tag{3.8}$$

$$|D_A^\gamma D_z^\beta D_\theta K_\delta(z;A(\theta))| \le C|\delta|^{-2}\left|\frac{dA(\theta)}{d\theta}\right| \quad \text{for all } z \text{ and } |\gamma|+|\beta|=1, \tag{3.9}$$

$$|D_A^\gamma D_z^\beta D_\theta K_\delta(z;A(\theta))| \le C|z|^{-2}\left|\frac{dA(\theta)}{d\theta}\right| \quad \text{for } |z| \ge C_1\delta \text{ and } |\gamma|+|\beta|=1, \tag{3.10}$$

where $\frac{dA(\theta)}{d\theta} = (\frac{da_{ij}(\theta)}{d\theta})$ *and* D_A^γ *is a* $|\gamma|$*-order derivative operator with respect to the components of* A.

Proof. In virtue of the boundedness of $A(\theta)$ and $A^{-1}(\theta)$, we can derive (3.5) and (3.6) by using a similar argument as given in the proof of Lemma 2 in [2], and therefore we omit the proof here.

If $A(\theta) = (a_{ij}(\theta))$, then we have

$$A^{-1}(\theta) = (a_{ij}^*(\theta)) = \begin{pmatrix} a_{22}(\theta) & -a_{12}(\theta) \\ -a_{21}(\theta) & a_{11}(\theta) \end{pmatrix}$$

and

$$D_\theta\rho_\delta\left(A^{-1}(\theta)(z-y)\right) = D\rho_\delta\left(A^{-1}(\theta)(z-y)\right)\frac{dA^{-1}(\theta)}{d\theta}(z-y),$$

where

$$\frac{dA^{-1}(\theta)}{d\theta} = \left(\frac{da_{ij}^*(\theta)}{d\theta}\right).$$

For $|z| \ge C_1\delta$, (3.8) will imply (3.7), and thus we only need to prove (3.7) for $|z| \le C_1\delta$. We write

$$D_\theta K_\delta(z;A(\theta)) = \int K(y)D_\theta\rho_\delta\left(A^{-1}(\theta)(z-y)\right)dy = I_1 + I_2 \tag{3.11}$$

where I_1 is the integral over $\{|y| < 2C_1\delta\}$ and I_2 over $\{|y| > 2C_1\delta\}$. For any $\sigma > 0$, we can easily estimate K in $L^1(|y| < \sigma)$. Since $|K(y)|$ is a constant times $|y|^{-1}$, we have

$$\int_{|y|<\sigma} |K(y)|dy = C\int_0^\sigma r^{-1}r\,dr = C\sigma \tag{3.12}$$

where C is a universal constant. Also we know

$$\begin{aligned} |D_\theta\rho_\delta(A^{-1}(\theta)(z-y))| &\le C\left|(z-y)D\rho_\delta(A^{-1}(\theta)(z-y))\right|\left|\frac{dA(\theta)}{d\theta}\right| \\ &= C\left|\delta^{-2}\frac{z-y}{\delta}D\rho\left(\frac{A^{-1}(\theta)(z-y)}{\delta}\right)\right|\left|\frac{dA(\theta)}{d\theta}\right| \\ &\le C\delta^{-2}\left|\frac{dA(\theta)}{d\theta}\right|, \end{aligned}$$

where the last inequality follows from $|xD\rho(x)| \le C$ for any $x \in \mathbf{R}^2$. Using these two facts, we have

$$|I_1| \le C(C_1\delta)\delta^{-2}\left|\frac{dA(\theta)}{d\theta}\right| = C\delta^{-1}\left|\frac{dA(\theta)}{d\theta}\right|.$$

Now we estimate I_2. On the set $\{|y| > 2C_1\delta\}$, we have $|y-z| \geq |y| - C_1\delta \geq C_1\delta$. From this and $|D^\alpha \rho(x)| \leq C|x|^{-3}$, it follows that

$$\begin{aligned}|D_\theta \rho_\delta(A^{-1}(\theta)(z-y))| &\leq C|(z-y)D\rho_\delta(A^{-1}(\theta)(z-y))|\left|\frac{dA(\theta)}{d\theta}\right| \\ &\leq C|z-y|^{-2}\left|\frac{dA(\theta)}{d\theta}\right| \leq C(|y|-C_1\delta)^{-2}\left|\frac{dA(\theta)}{d\theta}\right|.\end{aligned}$$

Using the above inequality and the relation $K(y) \sim |y|^{-1}$ we have

$$\begin{aligned}|I_2| &\leq C\int_{2C_1\delta}^{\infty} r^{-1}(r-C_1\delta)^{-2} r\,dr \left|\frac{dA(\theta)}{d\theta}\right| \\ &\leq C\int_{2C_1\delta}^{\infty} r^{-2}dr\left|\frac{dA(\theta)}{d\theta}\right| \leq C\delta^{-1}\left|\frac{dA(\theta)}{d\theta}\right|.\end{aligned}$$

This completes the proof of (3.7).

We now prove (3.8). In order to estimate (3.11) we denote by I_3 and I_4 the integral (3.11) with K replaced by ψK and $(1-\psi)K$:

$$\begin{aligned}&D_\theta K_\delta(z; A(\theta)) \\ &\quad = \int \psi K(y) D_\theta \rho_\delta(A^{-1}(\theta)(z-y))\,dy + \int (1-\psi)K(y)D_\theta\rho_\delta(A^{-1}(\theta)(z-y))\,dy \\ &\quad = I_3 + I_4,\end{aligned}$$

where $\psi(y) = \psi_0(|y|/|z|)$ and $\psi_0(r)$ is a smooth function, which satisfies: $\psi_0(r) = 0$ for $r \leq \frac{1}{4}$, $\psi_0(r) = 1$ for $r \geq \frac{1}{2}$, and $0 \leq \psi_0 \leq 1$. In the first term, the singularity at $y = 0$ has been removed. Using the variable substitution $y' = A^{-1}(\theta)(z-y)$ to I_3 and noting $\det A(\theta) = 1$, we can write

$$\begin{aligned}I_3 &= \pm\int D_\theta\Big[\psi(z-A(\theta)y)K(z-A(\theta)y)\Big] y\rho_\delta(y)\,dy \\ &= \pm\int D\Big[\psi(z-A(\theta)y)K(z-A(\theta)y)\Big] D_\theta A(\theta) y \rho_\delta(y)\,dy \\ &= \pm\int D\big[\psi(y)K(y)\big] D_\theta A(\theta)A^{-1}(\theta)(z-y)\rho_\delta(A^{-1}(\theta)(z-y))\,dy.\end{aligned} \tag{3.13}$$

Since $|D\psi| \leq C|z|^{-1}$, $|D^\gamma K(y)| \leq C|y|^{-1-|\gamma|}$ and, on the support of ψ, $|y| \geq |z|/4$, we have

$$|D(\psi(y)K(y))| \leq C|z|^{-2}.$$

On the other hand, we have

$$\left|\int (A^{-1}(\theta)(z-y))\rho_\delta(A^{-1}(\theta)(z-y))dy\right| \leq C\delta\int |z-y||\rho(A^{-1}(\theta)(z-y))|dy \leq C\delta.$$

Substituting the above two inequalities into (3.13) we conclude that

$$|I_3| \leq C\delta|z|^{-2}\left|\frac{dA(\theta)}{d\theta}\right|. \tag{3.14}$$

It remains to show that for $|z| \geq c_1\delta$, the above inequality holds for I_4. In doing so it is enough to establish for $|z| \geq c_1\delta$,

$$\left|\int_{|y|<|z|/2} (1-\psi)K(y)D_\theta\rho_\delta\big(A^{-1}(\theta)(z-y)\big)\,dy\right| \leq C\delta|z|^{-2}\left|\frac{dA(\theta)}{d\theta}\right|. \tag{3.15}$$

By using the facts that $|z-y| \geq |z|/2 \geq C_1\delta/2$ for $|z| \geq c_1\delta$ and $|D^l\rho(x)| \leq C_N|x|^{-N}$ for any integer N, we obtain

$$\begin{aligned}|D_\theta\rho_\delta\big(A^{-1}(\theta)(z-y)\big)| &\leq C\delta^{-3}\left|\frac{dA(\theta)}{d\theta}\right||z|\delta^N|z|^{-N}\\ &\leq C\delta^{N-3}|z|^{-N+1}\left|\frac{dA(\theta)}{d\theta}\right|.\end{aligned}$$

Using this inequality with $N=4$, we can bound the left-hand side of (3.15) by

$$C\delta|z|^{-3}\left|\frac{dA(\theta)}{d\theta}\right|\int_{|y|<|z|/2}|K(y)|\,dy \leq C\delta|z|^{-2}\left|\frac{dA(\theta)}{d\theta}\right|.$$

Thus, we have proved (3.15). Combining (3.14) and (3.15) yields (3.8). The proof of (3.9) and (3.10) is the same. Therefore Lemma 1 is completed. □

LEMMA 2. *Let $A(\theta)$ satisfy* (2.18) *and* (2.19), *and define*

$$M_{ij}^{(l,m)} = \max_{|y|\leq C_0\delta}\ \max_{m+|\beta|+|\gamma|=l}\left\{\left|D_\theta^m D_A^\gamma D_y^\beta K_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t)+y;A(\theta)\big)\right|\right\}. \tag{3.16}$$

Then

$$\sum_j M_{ij}^{(l,0)}h^2 \leq \begin{cases} C|\log\delta| & \textit{if } l=1,\\ C\delta^{-1} & \textit{if } l=2;\end{cases}$$

$$\sum_j M_{ij}^{(1,1)}h^2 \leq C\delta|\log\delta|\left|\frac{dA(\theta)}{d\theta}\right|;$$

$$\sum_j M_{ij}^{(2,1)}h^2 \leq C|\log\delta|\left|\frac{dA(\theta)}{d\theta}\right|.$$

Proof. Using estimates from Lemma 1 and arguing exactly as in the proof of Lemma 3.2 in [2], we could prove Lemma 2. □

LEMMA 3. *For $|\beta|+|\gamma|=1$, we have*

$$\left\|\int D_A^\gamma D_z^\beta K_\delta\big(z-y;A(\theta)\big)g(y)\,dy\right\|_{L^p} \leq C\|g\|_{L^p}. \tag{3.17}$$

The proof of this lemma can be found in [10] and therefore is omitted.

4. Consistency. We define

$$u_h\big(\phi(\alpha_i,t),t\big) = \sum_j k_jK_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t);\nabla\phi(\alpha_j,t)\big).$$

Then we obtain the following consistency error estimates.

LEMMA 4 (consistency). *Under the assumption of Theorem* 1, *we have*

$$\big|u_h\big(\phi(\alpha_i,t),t\big)-u\big(\phi(\alpha_i,t),t\big)\big| \leq C\delta^2|\log\delta|, \tag{4.1}$$

$$\big|\nabla u_h\big(\phi(\alpha_i,t),t\big)-\nabla u\big(\phi(\alpha_i,t),t\big)\big| \leq C\delta|\log\delta| \tag{4.2}$$

for $0 \le t \le T$.

Proof. We decompose the consistency error into three parts as follows:

$$
\begin{aligned}
&\left|u_h(\phi(\alpha_i,t),t) - u(\phi(\alpha_i,t),t)\right| \\
&\quad = \Big|\sum_j k_j K_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \nabla\phi(\alpha_j,t)\big) - u\big(\phi(\alpha_i,t),t\big)\Big| \\
&\quad \le \Big|\sum_j k_j \Big[K_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \nabla\phi(\alpha_j,t)\big) - \mathcal{K}_\delta(\alpha_i,\alpha_j,t)\Big]\Big| \\
&\qquad + \Big|\sum_j k_j \mathcal{K}_\delta(\alpha_i,\alpha_j,t) - \int \mathcal{K}_\delta(\alpha_i,\beta,t)\omega_0(\beta)d\beta\Big| \\
&\qquad + \Big|\int \mathcal{K}_\delta(\alpha_i,\beta,t)\omega_0(\beta)d\beta - \int K(\phi(\alpha_i,t) - \phi(\beta,t))\omega_0(\beta)d\beta\Big| \\
&\quad = \text{variable-vortex error} + \text{discretization error} + \text{moment error},
\end{aligned}
\tag{4.3}
$$

where

$$
\mathcal{K}_\delta(\alpha_i,\beta,t) = \int K\big(\phi(\alpha_i,t) - \phi(\alpha',t)\big)\rho_\delta(\alpha'-\beta)d\alpha' \tag{4.4a}
$$

or, equivalently,

$$
\mathcal{K}_\delta(\alpha_i,\beta,t) = \int K(y)\rho_\delta\Big(\phi^{-1}(\phi(\alpha_i,t) - y) - \beta\Big)\,dy. \tag{4.4b}
$$

In order to estimate the moment error, we write

$$
\begin{aligned}
\int_{\mathbf{R}^2} \mathcal{K}_\delta(\alpha_i,\beta,t)\omega_0(\beta)d\beta &= \int_{\mathbf{R}^2}\Big(\int K\big(\phi(\alpha_i,t) - \phi(\alpha',t)\big)\rho_\delta(\alpha'-\beta)d\alpha'\Big)\,\omega_0(\beta)d\beta \\
&= \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} K\big(\phi(\alpha_i,t) - \phi(\alpha',t)\big)\rho_\delta(\alpha'-\beta)\omega_0(\beta)d\alpha' d\beta \\
&= \int_{\mathbf{R}^2} K\big(\phi(\alpha_i,t) - \phi(\alpha',t)\big)\Big(\int_{\mathbf{R}^2}\rho_\delta(\alpha'-\beta)\omega_0(\beta)d\beta\Big)\,d\alpha'.
\end{aligned}
$$

By the assumption that ρ is an mth-order blob function, we can show that (see [13])

$$
\Big|\int_{\mathbf{R}^2}\rho_\delta(\alpha'-\beta)\omega_0(\beta)d\beta - \omega_0(\alpha')\Big| \le C'\|\nabla^m_{\alpha'}\omega_0(\alpha')\|_{L^\infty}\delta^m.
$$

In virtue of compact support of ω_0, we have

$$
\begin{aligned}
&\Big|\int_{\mathbf{R}^2} K\big(\phi(\alpha_i,t) - \phi(\alpha',t)\big)\Big(\int\rho_\delta(\alpha'-\beta)\omega_0(\beta)d\beta - \omega_0(\alpha')\Big)\,d\alpha'\Big| \\
&\quad \le \max_{\alpha'}\Big|\int\rho_\delta(\alpha'-\beta)\omega_0(\beta)d\beta - \omega_0(\alpha')\Big| \int_{|\alpha'|\le R+1}\big|K\big(\phi(\alpha_i,t) - \phi(\alpha',t)\big)\big|d\alpha' \\
&\quad \le C'\|\nabla^m_{\alpha'}\omega_0(\alpha')\|_{L^\infty}\delta^m
\end{aligned}
$$

where R is chosen such that $\omega_0(x) = 0$ for $x > R$. Thus we have

$$
|\text{moment error}| \le C'\delta^m \le C'\delta^2 \quad (m \ge 2), \tag{4.5}
$$

where C' only depends on the smoothness of $\omega_0(\alpha')$.

For the discretization error, we know from Theorem 3.1 of Raviart [13] that

$$\left|\sum_j k_j \mathcal{K}_\delta(\alpha_i,\alpha_j,t) - \int \mathcal{K}_\delta(\alpha_i,\beta,t)\omega_0(\beta)d\beta\right| \le C'h^l \|g(\alpha_i,\cdot,t)\|_{L^1},$$

where

$$g(\alpha_i,\beta,t) = \mathcal{K}_\delta(\alpha_i,\beta,t)\omega_0(\beta);$$

and then we have

$$\begin{aligned} |\text{discretization error}| &\le C'\|\nabla^l_{\alpha'}\omega_0(\alpha')\|_{L^1} h^l \delta^{-(l-1)} \\ &\le C'\|\nabla^l_{\alpha'}\omega_0(\alpha')\|_{L^1} (h/\delta)^l \delta \qquad (4.6)\\ &\le C'\delta^2 \qquad (l(a-1)\ge 1), \end{aligned}$$

where C' only depends on $\omega_0(\alpha')$ and the blob function ρ.

Now we turn to the variable-vortex error given by (4.3). This error can split into two parts:

$$\left|\sum_j k_j \Big[K_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t);\nabla\phi(\alpha_j,t)\big) - \mathcal{K}_\delta(\alpha_i,\alpha_j,t)\Big]\right| = |I_5 + I_6|,$$

where I_5 is the sum over $\{j \,|\, |\phi(\alpha_i,t)-\phi(\alpha_j,t)| \le C_1\delta\}$, I_6 over $\{j \,|\, |\phi(\alpha_i,t) - \phi(\alpha_j,t)| > C_1\delta\}$, and the constant C_1 will be defined later.

In view of (2.17b) and (4.4b) I_5 can be expressed as

$$\begin{aligned} I_5 &= \sum_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)|\le C_1\delta} k_j \int K(y)\Big[\rho_\delta\big(\nabla\phi(\alpha_j,t)^{-1}(\phi(\alpha_i,t)-\phi(\alpha_j,t)-y)\big) \\ &\qquad\qquad - \rho_\delta\big(\phi^{-1}(\phi(\alpha_i,t)-y)-\alpha_j\big)\Big]\,dy \\ &= \sum_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)|\le C_1\delta} k_j \int K(y)\Big[\rho_\delta\big(\nabla\phi(\alpha_j,t)^{-1}(\phi(\alpha_i,t)-\phi(\alpha_j,t)-y)\big) \\ &\qquad\qquad - \rho_\delta\big(\nabla\phi(\alpha_j^*,t)^{-1}(\phi(\alpha_i,t)-\phi(\alpha_j,t)-y)\big)\Big]\,dy \\ &= \sum_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)|\le C_1\delta} k_j \int K(y)\Big[D\rho_\delta\big(\nabla\phi(\alpha_j^{**},t)^{-1}(\phi(\alpha_i,t)-\phi(\alpha_j,t)-y)\big) \\ &\qquad\qquad \times D_\alpha\left(\nabla\phi(\alpha_j^{**},t)^{-1}\right)(\alpha_j-\alpha_j^*)\big(\phi(\alpha_i,t)-\phi(\alpha_j,t)-y\big)\Big]\,dy, \end{aligned}$$

where $\phi(\alpha_j^*,t)$ lies on the segment connecting $\phi(\alpha_i,t)-y$ and $\phi(\alpha_j,t)$ and $\phi(\alpha_j^{**},t)$ lies on the segment connecting $\phi(\alpha_j^*,t)$ and $\phi(\alpha_j,t)$. Since $\phi(\alpha_j^*,t)$ is located between $\phi(\alpha_i,t)-y$ and $\phi(\alpha_j,t)$ and in the above integral $|\phi(\alpha_i,t)-\phi(\alpha_j,t)-y| \le C\delta$, we conclude that

$$\left|D_\alpha\left(\nabla\phi(\alpha_j^{**},t)^{-1}\right)(\alpha_j^*-\alpha_j)\right| \le C\delta$$

and

$$|\phi(\alpha_i,t)-\phi(\alpha_j,t)-y|\left|D\rho_\delta\big(\nabla\phi(\alpha_j^{**},t)^{-1}(\phi(\alpha_i,t)-\phi(\alpha_j,t)-y)\big)\right| \le C\delta^{-2}.$$

Thus we have

$$|I_5| \le C\delta^{-1} \sum_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)|\le C_1\delta} |k_j| \int_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)-y|\le C_2\delta} |K(y)|\,dy.$$

It follows from (3.12) that

$$
\begin{aligned}
|I_5| &\le C \sum_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)|\le C_1\delta} |\omega_0(\alpha_j)|h^2 \\
&\le C \int_{|\phi(\alpha_i,t)-y|\le C_1\delta} \left|\omega_0\left(\phi^{-1}(y,t)\right)\right| dy \le C\delta^2.
\end{aligned} \tag{4.7}
$$

Similarly, from (2.17a) and (4.4a) I_6 can be expressed as

$$
\begin{aligned}
I_6 = &\sum_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)|\ge C_1\delta} k_j \int \Big[K\big(\phi(\alpha_i,t)-\phi(\alpha_j,t)-\nabla\phi(\beta_j^*,t)(\alpha'-\alpha_j)\big) \\
&\qquad - K\big(\phi(\alpha_i,t)-\phi(\alpha_j,t)-\nabla\phi(\alpha_j,t)(\alpha'-\alpha_j)\big)\Big]\rho_\delta(\alpha'-\alpha_j)\,d\alpha' \\
= &\sum_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)|\ge C_1\delta} k_j \int DK\big(\phi(\alpha_i,t)-\phi(\alpha_j,t)-\nabla\phi(\beta_j^{**},t)(\alpha'-\alpha_j)\big) \\
&\qquad \times D_\alpha\nabla\phi(\beta_j^{**},t)(\beta_j^*-\alpha_j)(\alpha'-\alpha_j)\rho_\delta(\alpha'-\alpha_j)\,d\alpha',
\end{aligned}
$$

where β_j^* lies on the segment connecting α_j and α' and β_j^{**} lies on the segment connecting β_j^* and α_j. Since β_j^* is located between α_j and α' and in the above integral $|\alpha'-\alpha_j|\le\delta$, we have

$$
\left|D_\alpha\nabla\phi(\beta_j^{**},t)(\beta_j^*-\alpha_j)\right| \le C\delta.
$$

Thus we have

$$
\begin{aligned}
|I_6| &\le C\delta \sum_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)|\ge C_1\delta} |k_j| \int \left|DK\big(\phi(\alpha_i,t)-\phi(\alpha_j,t)-\nabla\phi(\beta_j^{**},t)(\alpha'-\alpha_j)\big)\right| \\
&\qquad \times |(\alpha'-\alpha_j)\rho_\delta(\alpha'-\alpha_j)|\,d\alpha' \\
&\le C\delta \sum_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)|\ge C_1\delta} |k_j| \int \frac{|(\alpha'-\alpha_j)\rho_\delta(\alpha'-\alpha_j)|}{|\phi(\alpha_i,t)-\phi(\alpha_j,t)-\nabla\phi(\beta_j^{**},t)(\alpha'-\alpha_j)|^2}\,d\alpha'.
\end{aligned}
$$

If C_1 is chosen such that $C_1 > \|\nabla\phi\|_{L^\infty}$, then we have

$$
\begin{aligned}
|I_6| &\le C\delta^2 \sum_{|\phi(\alpha_i,t)-\phi(\alpha_j,t)|\ge C_1\delta} \frac{|\omega_0(\alpha_j)|h^2}{(|\phi(\alpha_i,t)-\phi(\alpha_j,t)|-\|\nabla\phi\|_{L^\infty}\delta)^2} \\
&\le C\delta^2 \int_{|\phi(\alpha_i,t)-y|\ge C_1\delta} \frac{|\omega_0(\phi^{-1}(y,t))|}{(|\phi(\alpha_i,t)-y|-\|\nabla\phi\|_{L^\infty}\delta)^2}\,dy \\
&\le C\delta^2|\log\delta|.
\end{aligned} \tag{4.8}
$$

Combining (4.7) and (4.8) we obtain the estimate for the variable-vortex error

$$
|\text{variable-vortex error}| \le C\delta^2|\log\delta|,
$$

and hence from this and (4.5)–(4.6) we verify (4.1).

The proof of (4.2) is similar to (4.1) and therefore is omitted. □

Remark. We recall that the moment error and discretization error of the variable-elliptic-vortex method are uniformly bounded by

$$
\|\nabla_{\alpha'}^m\omega_0(\alpha')\|_{L^\infty}\delta^m + \|\nabla_{\alpha'}^l\omega_0(\alpha')\|_{L^1}(h/\delta)^l\delta,
$$

which do not depend on the exact solution ϕ. Therefore they produce smaller discretization errors than the fixed-vortex blob method.

5. Stability and convergence. In this section we will prove a stability lemma for the variable-vortex method and then prove the main result, Theorem 1. The following lemma plays a center role in proving the stability lemma.

LEMMA 5. *Let $U = (u_{ij})$ and $V = (v_{ij})$ be 2×2 matrices and satisfy*

$$\begin{aligned} &\det U = 1, \quad \det V = 1, \\ &|u_{ij}| \le C, \quad |v_{ij}| \le C, \\ &|u_{ij} - v_{ij}| \le C\delta^s \quad \text{for some } 0 < s < 1. \end{aligned} \tag{A.1}$$

Then there exists a matrix function $A(\theta) = (a_{ij}(\theta))$ defined on $0 \le \theta \le 1$ such that

$$\begin{aligned} &\det A(\theta) = 1, \quad |a_{ij}(\theta)| \le C, \\ &A(0) = U, \quad A(1) = V, \\ &\left|\frac{dA(\theta)}{d\theta}\right| \le C|U - V| \quad \text{for } 0 \le \theta \le 1 \end{aligned} \tag{A.2}$$

provided that δ is small enough.

Proof. We construct $A(\theta)$ as follows. In virtue of $\det U = 1$, there exists a $u_{i_0 j_0}$ such that

$$|u_{i_0 j_0}| \ge \frac{1}{2}.$$

In particular, we suppose $u_{12} \ge \frac{1}{2}$, and then we have $v_{12} \ge \frac{1}{4}$ provided that δ is small enough. We define

$$A(\theta) = \begin{pmatrix} (1-\theta)u_{11} + \theta v_{11} & (1-\theta)u_{12} + \theta v_{12} \\ a_{21}(\theta) & (1-\theta)u_{22} + \theta v_{22} \end{pmatrix},$$

where

$$a_{21}(\theta) = \frac{((1-\theta)u_{11} + \theta v_{11})((1-\theta)u_{22} + \theta v_{22}) - 1}{(1-\theta)u_{12} + \theta v_{12}}.$$

It is easy to verify that $A(\theta)$ satisfies (A.2). □

LEMMA 6 (stability). *Assume*

$$\max_{0 \le t \le T_*} \max_i \left|\overline{\phi}_i(t) - \phi(\alpha_i, t)\right| \le \delta \tag{5.1}$$

and

$$\max_{0 \le t \le T_*} \max_i \left|\overline{\nabla\phi}_i(t) - \nabla\phi(\alpha_i, t)\right| \le \delta^s \quad \text{for some } 0 < s < 1/2, \tag{5.2}$$

where T_* is some constant satisfying $0 < T_* \le T$. Then

$$\begin{aligned} &\left\|\overline{u}(\overline{\phi}_i(t), t) - u_h(\phi(\alpha_i, t), t)\right\|_{l_h^p} \\ &\qquad \le C\left\|\overline{\phi}_i(t) - \phi(\alpha_i, t)\right\|_{l_h^p} + C\delta\left\|\overline{\nabla\phi}_i(t) - \nabla\phi(\alpha_i, t)\right\|_{l_h^p} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} &\left\|\nabla\overline{u}(\overline{\phi}_i(t), t) - \nabla u_h(\phi(\alpha_i, t), t)\right\|_{l_h^p} \\ &\qquad \le \frac{C}{\delta}\left\|\overline{\phi}_i(t) - \phi(\alpha_i, t)\right\|_{l_h^p} + C\left\|\overline{\nabla\phi}_i(t) - \nabla\phi(\alpha_i, t)\right\|_{l_h^p} \end{aligned} \tag{5.4}$$

uniformly for $t \in [0, T_*]$, where C is independent of T_* but depends on T.

Proof. We first divide the following stability error into three terms:

$$
\begin{aligned}
&\overline{u}(\overline{\phi}_i(t),t) - u_h(\phi(\alpha_i,t),t)\\
&\quad= \sum_j k_j \Big[K_\delta\big(\overline{\phi}_i(t)-\overline{\phi}_j(t);\overline{\nabla\phi}_j(t)\big) - K_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t);\nabla\phi(\alpha_j,t)\big)\Big]\\
&\quad= \sum_j k_j \Big[K_\delta\big(\overline{\phi}_i(t)-\overline{\phi}_j(t);\overline{\nabla\phi}_j(t)\big) - K_\delta\big(\phi(\alpha_i,t)-\overline{\phi}_j(t);\overline{\nabla\phi}_j(t)\big)\Big]\\
&\qquad+ \sum_j k_j \Big[K_\delta\big(\phi(\alpha_i,t)-\overline{\phi}_j(t);\overline{\nabla\phi}_j(t)\big) - K_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t);\overline{\nabla\phi}_j(t)\big)\Big]\\
&\qquad+ \sum_j k_j \Big[K_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t);\overline{\nabla\phi}_j(t)\big) - K_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t);\nabla\phi(\alpha_j,t)\big)\Big]\\
&\quad= v_i^{(1)} + v_i^{(2)} + v_i^{(3)}.
\end{aligned}
$$

By using the mean value theorem we have

$$v_i^{(2)} = \sum_j DK_\delta\left(\phi(\alpha_i,t)-\phi(\alpha_j,t)+y_{ij};\overline{\nabla\phi}_j(t)\right)e_jk_j,$$

where $e_j = \phi(\alpha_j,t)-\overline{\phi}_j(t)$ and $|y_{ij}| \le |\phi(\alpha_j,t)-\overline{\phi}_j(t)| \le \delta$. Furthermore, we write

$$v_i^{(2)} = \sum_j DK_\delta\left(\phi(\alpha_i,t)-\phi(\alpha_j,t);\overline{\nabla\phi}_j(t)\right)e_jk_j + r_i^{(1)},$$

where

$$
\begin{aligned}
r_i^{(1)} = \sum_j \Big[&DK_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t)+y_{ij};\overline{\nabla\phi}_j(t)\big)\\
&- DK_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t);\overline{\nabla\phi}_j(t)\big)\Big]e_jk_j.
\end{aligned}
$$

Using the mean value theorem again yields

$$
\begin{aligned}
|r_i^{(1)}| &\le \sum_{|\beta|=2}\sum_j \left|D^\beta K_\delta\left(\phi(\alpha_i,t)-\phi(\alpha_j,t)+y'_{ij};\overline{\nabla\phi}_j(t)\right)y_{ij}\right||e_jk_j|\\
&\le \sum_j M_{ij}^{(2,0)}\delta|e_j\omega_0(\alpha_j)|h^2,
\end{aligned}
$$

where $|y'_{ij}| \le |y_{ij}| \le \delta$. Let $M_{ij}^{(2,0)}$ denote

$$M_{ij}^{(2,0)} = \max_{|y|\le\delta}\max_{|\beta|=2}\left\{\left|D_y^\beta K_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t)+y;\overline{\nabla\phi}_j(t)\big)\right|\right\}. \tag{5.5}$$

Then by Young's inequality [7] we obtain

$$\|r_i^{(1)}\|_{l_h^p} \le \delta\max\left\{\sum_j M_{ij}^{(2,0)}h^2, \sum_i M_{ij}^{(2,0)}h^2\right\}\|e_i\omega_0(\alpha_i)\|_{l_h^p}.$$

(2.12) and (5.2) imply that $\overline{\nabla\phi}_j(t)$ satisfies (2.18) and (2.19); thus, applying Lemma 2 to (5.5) gives

$$\sum_i M_{ij}^{(2,0)}h^2 \le \frac{C}{\delta}, \quad \sum_j M_{ij}^{(2,0)}h^2 \le \frac{C}{\delta}.$$

Therefore

$$\|r_i^{(1)}\|_{l_h^p} \le C\|e_i\omega_0(\alpha_i)\|_{l_h^p} \le C\|e_i\|_{l_h^p}. \tag{5.6}$$

Now we write $v_i^{(2)}$ further into

$$v_i^{(2)} = \sum_j DK_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \nabla\phi(\alpha_j,t)\big)e_jk_j + r_i^{(2)} + r_i^{(1)}, \tag{5.7}$$

where

$$r_i^{(2)} = \sum_j \Big[DK_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \overline{\nabla\phi}_j(t)\big) - DK_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \nabla\phi(\alpha_j,t)\big)\Big]e_jk_j.$$

Following Lemma 5 we can define a matrix function $A_j(\theta)$ on $(0 \le \theta \le 1)$ such that $A_j(0) = \nabla\phi(\alpha_j,t)$ and $A_j(1) = \overline{\nabla\phi}_j(t)$, and then by the mean value theorem we can write

$$r_i^{(2)} = \sum_j D_\theta DK_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); A_j(\theta_0)\big)e_jk_j,$$

where $0 < \theta_0 < 1$. Thus we obtain

$$|r_i^{(2)}| \le \sum_j M_{ij}^{(2,1)}|e_j\omega_0(\alpha_j)|h^2,$$

where

$$M_{ij}^{(2,1)} = \max_{|\beta|=1}\Big\{\big|D_\theta D^\beta K_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); A_j(\theta_0)\big)\big|\Big\}.$$

Lemma 5 and assumption (5.2) imply that $A_j(\theta)$ satisfies (2.18) and (2.19); thus, by Lemma 2 we have

$$\sum_j M_{ij}^{(2,1)}h^2 \le C|\log\delta|\left|\frac{dA_j(\theta_0)}{d\theta}\right| \le C|\log\delta|\,\big|\overline{\nabla\phi}_j(t) - \nabla\phi(\alpha_j,t)\big| \le C\delta^s|\log\delta| \le C.$$

The symmetry of M_{ij} with respect to i and j gives

$$\sum_i M_{ij}^{(2,1)}h^2 \le C.$$

Therefore

$$\|r_i^{(2)}\|_{l_h^p} \le \max\Big\{\sum_i M_{ij}^{(2,1)}h^2, \sum_j M_{ij}^{(2,1)}h^2\Big\}\|e_j\omega_0(\alpha_j)\|_{l_h^p} \le C\|e_j\|_{l_h^p}. \tag{5.8}$$

In order to complete the estimate on $v_i^{(2)}$, we need to bound

$$\Big\|\sum_j DK_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \nabla\phi(\alpha_j,t)\big)e_j\omega_0(\alpha_j)h^2\Big\|_{l_h^p}.$$

In fact, this is a discrete counterpart of the kind of integration given in the left-hand side of (3.17). Therefore we can use the inequality (3.17) and a similar argument as given in the proof of the stability lemma by Beale and Majda [2] to show that

$$\left\| \sum_j DK_\delta(\phi(\alpha_i,t)-\phi(\alpha_j,t);\nabla\phi(\alpha_j,t))e_j\omega_0(\alpha_j)h^2 \right\|_{l_h^p} \leq C\|e_i\omega_0(\alpha_i)\|_{l_h^p} \leq C\|e_i\|_{l_h^p}. \tag{5.9}$$

Combining (5.6)–(5.9), we conclude that

$$\|v_i^{(2)}\|_{l_h^p} \leq C\|e_i\|_{l_h^p}.$$

Now we turn to $v_i^{(1)}$. Using the mean value theorem yields

$$\begin{aligned} v_i^{(1)} &= \sum_j k_j \left[K_\delta\left(\overline{\phi}_i(t)-\overline{\phi}_j(t);\overline{\nabla\phi}_j(t)\right) - K_\delta\left(\phi(\alpha_i,t)-\overline{\phi}_j(t);\overline{\nabla\phi}_j(t)\right)\right] \\ &= \sum_j DK_\delta\left(\phi(\alpha_i,t)-\overline{\phi}_j(t)+y_{ij};\overline{\nabla\phi}_j(t)\right)e_jk_j \\ &= \sum_j DK_\delta\left(\phi(\alpha_i,t)-\phi(\alpha_j,t)+y'_{ij};\overline{\nabla\phi}_j(t)\right)e_jk_j, \end{aligned}$$

where $|y_{ij}| \leq |\phi(\alpha_i,t)-\overline{\phi}_j(t)|$ and $y'_{ij} = y_{ij}+\phi(\alpha_j,t)-\overline{\phi}_j(t)$. The assumption (5.1) implies that

$$|y'_{ij}| \leq 2\delta. \tag{5.10}$$

Using a similar argument as above we can show

$$v_i^{(1)} = \sum_j DK_\delta(\phi(\alpha_i,t)-\phi(\alpha_j,t);\nabla\phi(\alpha_j,t))e_j\omega_0(\alpha_j)h^2 + s_i^{(1)} + s_2^{(2)}, \tag{5.11}$$

where

$$\begin{aligned} s_i^{(1)} &= \sum_j \Big[DK_\delta\left(\phi(\alpha_i,t)-\phi(\alpha_j,t)+y'_{ij};\overline{\nabla\phi}_j(t)\right) \\ &\qquad - DK_\delta\left(\phi(\alpha_i,t)-\phi(\alpha_j,t);\overline{\nabla\phi}_j(t)\right)\Big]e_jk_j \\ &= \sum_{|\beta|=2}\sum_j D^\beta K_\delta\left(\phi(\alpha_i,t)-\phi(\alpha_j,t)+y''_{ij};\overline{\nabla\phi}_j(t)\right)y'_{ij}e_jk_j \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} s_i^{(2)} &= \sum_j \Big[DK_\delta\left(\phi(\alpha_i,t)-\phi(\alpha_j,t);\overline{\nabla\phi}_j(t)\right) \\ &\qquad - DK_\delta\big(\phi(\alpha_i,t)-\phi(\alpha_j,t);\nabla\phi(\alpha_j,t)\big)\Big]e_jk_j. \end{aligned} \tag{5.13}$$

It follows from (5.10) and (5.12) that

$$|s_i^{(1)}| \leq C\delta\sum_j M_{ij}^{(2,0)}|e_jk_j|,$$

where $M_{ij}^{(2,0)}$ is defined by (3.16) with $c_0 = 2$ and $A(\theta) = \overline{\nabla\phi}_j(t)$. By Lemma 2 and Young's inequality we deduce that

$$\|s_i^{(1)}\|_{l_h^p} \leq C\delta\max\left\{\sum_j M_{ij}^{(2,0)}h^2, \sum_i M_{ij}^{(2,0)}h^2\right\}\|e_i\omega_0(\alpha_i)\|_{l_h^p} \leq C\|e_i\|_{l_h^p}.$$

Following Lemma 5 we can write (5.13) as follows

$$s_i^{(2)} = \sum_j D_\theta DK_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); A_j(\theta_0)\big)e_jk_j,$$

where $0 < \theta_0 < 1$, $A_j(0) = \nabla\phi(\alpha_j,t)$, and $A_j(1) = \overline{\nabla\phi}_j(t)$. Similarly, we can obtain

$$|s_i^{(2)}| \le \sum_j M_{ij}^{(2,1)}|e_jk_j|;$$

and then

$$\|s_i^{(2)}\|_{l_h^p} \le C\max\Big\{\sum_j M_{ij}^{(2,1)}h^2, \sum_i M_{ij}^{(2,1)}h^2\Big\}\|e_i\omega_0(\alpha_i)\|_{l_h^p},$$

where $M_{ij}^{(2,1)}$ is defined by (3.16) with $A(\theta) = A_j(\theta)$. Lemma 2 and Lemma 5 show that

$$\begin{aligned}\|s_i^{(2)}\|_{l_h^p} &\le C|\log\delta|\left|\frac{dA_j(\theta)}{d\theta}\right|\|e_i\|_{l_h^p} \le C|\log\delta|\,\big|\overline{\nabla\phi}_j(t) - \nabla\phi(\alpha_j,t)\big|\,\|e_i\|_{l_h^p}\\ &\le C\delta^s|\log\delta|\|e_i\|_{l_h^p} \le C\|e_i\|_{l_h^p}.\end{aligned}$$

In order to finish the estimate on $v_i^{(1)}$ we need to bound the first term on the right-hand side of (5.11) as follows:

$$\Big\|\sum_j DK_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \nabla\phi(\alpha_j,t)\big)e_j\omega_0(\alpha_j)h^2\Big\|_{l_h^p} \le C\|e_i\|_{l_h^p}. \tag{5.14}$$

In fact, this is a discrete counterpart of the inequality in (3.17). Therefore we can use the inequality (3.17) and the same argument as given by Beale and Majda in [2] to show that (5.14) is valid. Thus, it follows from (5.11) that

$$\|v_i^{(1)}\|_{l_h^p} \le C\|e_i\|_{l_h^p}.$$

Finally we estimate $v_i^{(3)}$. By the mean value theorem and Lemma 5 we can define a matrix function $A_j(\theta)$ such that $A_j(0) = \nabla\phi(\alpha_j,t)$, $A_j(1) = \overline{\nabla\phi}_j(t)$, and for some $0 < \theta_0 < 1$

$$\begin{aligned}v_i^{(3)} &= \sum_j k_j\Big[K_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \overline{\nabla\phi}_j(t)\big) - K_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \nabla\phi(\alpha_j,t)\big)\Big]\\ &= \sum_j D_\theta K_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); A_j(\theta_0)\big)k_j.\end{aligned}$$

By the chain rule we have

$$D_\theta K_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); A(\theta_0)\big) = D_A K_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); A_j(\theta_0)\big)\frac{dA_j(\theta_0)}{d\theta};$$

and then we write

$$v_i^{(3)} = \sum_j k_j D_A K_\delta\big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \nabla\phi(\alpha_j,t)\big)\frac{dA_j(\theta_0)}{d\theta} + r_i^{(3)}, \tag{5.15}$$

where

$$r_i^{(3)} = \sum_j k_j \Big[D_A K_\delta \big(\phi(\alpha_i,t) - \phi(\alpha_j,t); A_j(\theta_0)\big) - D_A K_\delta \big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \nabla\phi(\alpha_j,t)\big)\Big] \frac{dA_j(\theta_0)}{d\theta}.$$

Applying the mean value theorem and Lemma 5 again we have

$$r_i^{(3)} = \sum_j k_j D_\theta D_A K_\delta \left(\phi(\alpha_i,t) - \phi(\alpha_j,t); \tilde{A}_j(\tilde{\theta}_0)\right) \frac{dA_j(\theta_0)}{d\theta},$$

where $\tilde{A}_j(0) = \nabla\phi(\alpha_j,t)$, $\tilde{A}_j(1) = A_j(\theta_0)$, and $0 < \tilde{\theta}_0 < 1$. We denote

$$M_{ij}^{(2,1)} = \max_{|\gamma|=1} \left\{ \left| D_\theta D_A^\gamma K_\delta \left(\phi(\alpha_i,t) - \phi(\alpha_j,t); \tilde{A}_j(\tilde{\theta}_0)\right)\right| \right\},$$

and then by Lemma 2 we have

$$\begin{aligned} \sum_j M_{ij}^{(2,1)} h^2 &\le C\delta |\log\delta| \left| \frac{d\tilde{A}_j(\tilde{\theta}_0)}{d\theta} \right| \le C\delta|\log\delta| \left|\tilde{A}_j(1) - \tilde{A}_j(0)\right| \\ &= C\delta|\log\delta| \left|A_j(\theta_0) - \nabla\phi(\alpha_j,t)\right| \le C\delta|\log\delta| \left|\overline{\nabla\phi}_j(t) - \nabla\phi(\alpha_j,t)\right| \\ &\le C\delta^{1+s}|\log\delta| \le C\delta. \end{aligned}$$

Substituting the above inequality into $r_i^{(3)}$ yields

$$\begin{aligned} \|r_i^{(3)}\|_{l_h^p} &\le C \left\| \sum_j M_{ij}^{(2,1)} \frac{dA(\theta_0)}{d\theta} k_j \right\|_{l_h^p} \\ &\le C \max\left\{ \sum_j M_{ij}^{(2,1)} h^2, \sum_i M_{ij}^{(2,1)} h^2 \right\} \left\| \frac{dA_i(\theta_0)}{d\theta} \omega_0(\alpha_i) \right\|_{l_h^p} \\ &\le C\delta \left\| \frac{dA_i(\theta_0)}{d\theta} \omega_0(\alpha_i) \right\|_{l_h^p} \le C\delta \left\| \frac{dA_i(\theta_0)}{d\theta} \right\|_{l_h^p} \\ &\le C\delta \left\| \overline{\nabla\phi}_i(t) - \nabla\phi(\alpha_i,t) \right\|_{l_h^p} = C\delta \|E_i\|_{l_h^p}, \end{aligned} \tag{5.16}$$

where $E_j = \overline{\nabla\phi}_j(t) - \nabla\phi(\alpha_j,t)$.

To complete the estimate on $v_i^{(3)}$, we need to bound

$$\begin{aligned} I &= \left\| \sum_j k_j D_A K_\delta \big(\phi(\alpha_i,t) - \phi(\alpha_j,t); \nabla\phi(\alpha_j,t)\big) \frac{dA_j(\theta_0)}{d\theta} \right\|_{l_h^p} \\ &= \left\| \sum_j \int DK\big(\phi(\alpha_i,t) - \phi(\alpha_j,t) - \overline{\nabla\phi}_j(t)(\alpha' - \alpha_j)\big) \right. \\ &\qquad \left. \times (\alpha' - \alpha_j)\rho_\delta(\alpha' - \alpha_j)\, d\alpha' \frac{dA_j(\theta_0)}{d\theta} k_j \right\|_{l_h^p}. \end{aligned}$$

By the Calderon–Zygmund inequality we arrive at

$$\begin{aligned} I &\le C\|(\alpha' - \alpha_j)\rho_\delta(\alpha' - \alpha_j)\|_{L^1} \left\| \frac{dA_i(\theta_0)}{d\theta} \omega_0(\alpha_i) \right\|_{l_h^p} \\ &\le C\delta \left\| \frac{dA_i(\theta_0)}{d\theta} \omega_0(\alpha_i) \right\|_{l_h^p} \le C\delta \left\| \frac{dA_i(\theta_0)}{d\theta} \right\|_{l_h^p} \le C\delta \|E_i\|_{l_h^p}. \end{aligned} \tag{5.17}$$

Then combining (5.15)–(5.17), we have

$$\|v_i^{(3)}\|_{l_h^p} \le C\delta \|E_j\|_{l_h^p}.$$

This completes the proof of (5.3). Similarly, we can prove (5.4), and the details of the proof are omitted here. □

Proof of Theorem 1. Let $e_i(t) = \overline{\phi}_i(t) - \phi(\alpha_i, t)$ and $E_i(t) = \overline{\nabla\phi}_i(t) - \nabla\phi(\alpha_i, t)$. Then we have

$$de_i(t) = \left[\overline{u}\left(\overline{\phi}_i(t), t\right) - u(\phi(\alpha_i, t), t)\right] dt,$$
$$dE_i(t) = \left[\nabla\overline{u}\left(\overline{\phi}_i(t), t\right) \overline{\nabla\phi}_i(t) - \nabla u(\phi(\alpha_i, t), t) \nabla\phi(\alpha_i, t)\right] dt.$$

The consistency and stability lemmas imply that

$$\left\| \frac{de_i(t)}{dt} \right\|_{l_h^p} \le C\left[\|e_i(t)\|_{l_h^p} + \delta\|E_i(t)\|_{l_h^p} + \delta^2|\log\delta|\right], \qquad e_i(0) = 0,$$

and

$$\left\| \frac{dE_i(t)}{dt} \right\|_{l_h^p} \le C\left[\frac{1}{\delta}\|e_i(t)\|_{l_h^p} + \|E_i(t)\|_{l_h^p} + \delta|\log\delta|\right], \qquad E_i(0) = 0.$$

Thus we have

$$\text{(5.18)} \qquad \begin{aligned} &\frac{d(\|e_i(t)\|_{l_h^p} + \delta\|E_i(t)\|_{l_h^p})}{dt} \le C\left[(\|e_i(t)\|_{l_h^p} + \delta\|E_i(t)\|_{l_h^p}) + \delta^2|\log\delta|\right], \\ &\|e_i(0)\|_{l_h^p} + \delta\|E_i(0)\|_{l_h^p} = 0. \end{aligned}$$

Applying the Gronwall inequality to (5.18), we obtain

$$\text{(5.19)} \qquad \|e_i(t)\|_{l_h^p} + \delta\|E_i(t)\|_{l_h^p} \le C\delta^2|\log\delta|$$

for $0 \le t \le T_*$, where C is independent of T_*. Thus we have

$$\max_i |e_i(t)| \le h^{-\frac{2}{p}}\|e_i(t)\|_{l_h^p} \le Ch^{-\frac{2}{p}}\delta^2|\log\delta| \le \frac{\delta}{2}$$

and

$$\max_i |E_i(t)| \le h^{-\frac{2}{p}}\|E_i(t)\|_{l_h^p} \le Ch^{-\frac{2}{p}}\delta|\log\delta| \le \frac{1}{2}\delta^s, \quad 0 < s < \frac{1}{2},$$

for $t < T_*$. By choosing p, m, l large enough and h small enough, on account of $h = \delta^a$ with $a > 1$, we can see that $\|e_i(t)\|_{l_h^\infty}$ hardly reach δ and $\|E_i(t)\|_{l_h^\infty}$ hardly reach δ^s $(0 < s < 1/2)$. Hence we conclude that $T_* = T$ and (5.19) holds for $0 \le t \le T$. Thus (3.1) and (3.2) have been proved.

The convergence of discrete velocity follows from

$$\begin{aligned} \left\|\overline{u}\left(\overline{\phi}_i(t), t\right) - u(\phi(\alpha_i, t), t)\right\|_{l_h^p} &\le \left\|\frac{de_i(t)}{dt}\right\|_{l_h^p} \\ &\le C\left[\left(\|e_i(t)\|_{l_h^p} + \delta\|E_i(t)\|_{l_h^p}\right) + \delta^2|\log\delta|\right] \\ &\le C\delta^2|\log\delta|. \end{aligned}$$

Finally, we prove (3.4). From (2.3), (2.20), and (2.22) it follows that

$$\frac{d\nabla\phi(\alpha_i,t)}{dt} = \nabla u(\phi(\alpha_i,t),t)\cdot\nabla\phi(\alpha_i,t),$$
$$\frac{d\overline{\nabla\phi}_i(t)}{dt} = \nabla\overline{u}\left(\overline{\phi}_i(t),t\right)\cdot\overline{\nabla\phi}_i(t);$$

and hence

$$\nabla\overline{u}\left(\overline{\phi}_i(t),t\right)-\nabla u(\phi(\alpha_i,t),t) = \frac{dE_i}{dt}\nabla\phi(\alpha_i,t)^{-1}+\frac{d\overline{\nabla\phi}_i(t)}{dt}\left(\nabla\phi(\alpha_i,t)^{-1}-\overline{\nabla\phi}_i(t)^{-1}\right).$$

By using (5.18), (3.2), and the above inequality we conclude (3.4). This completes the proof of Theorem 1. □

Remark. The error constants C in the stability lemma depend on T and ϕ; hence the error constants C in Theorem 1 also depend on them even though the constants C' in moment error and discretization error are independent of them.

6. Conclusion. We have presented a general formulation of the variable-elliptic-vortex method and proved its consistency, stability, and convergence. Now we turn to the practical aspects of the proposed method and make the following comments.

1. The extension from circular blobs to elliptic blobs can yield a more efficient vortex representation and therefore reduce the number of vortex blobs in calculations. This is justified by numerical experiments with the fixed-elliptic-vortex method [15] and with the variable-elliptic-vortex method [18], where the elliptic blobs are used to mimic the flow over a flat-plate at different Reynolds numbers.

2. If ρ is chosen to be the step function

$$\rho(x)=\begin{cases}1/\pi & \text{if } |x|\le 1,\\ 0 & \text{if } |x|>1,\end{cases}\tag{6.1}$$

then the integral $K_\delta(z;A)$ ($\nabla K_\delta(z;A)$ as well) has an explicit closed-form expression (see [15], [16]), which is important in designing an effective algorithm for (2.20) and (2.21). A full discretization scheme is designed in [16].

Note that the blob function (6.1) is nonsmooth, which does not comply with the assumption on ρ stated in Theorem 1, but the convergence theorem may still follow from the theoretical analysis given in this paper and the techniques used in [9].

3. In order to make the initial vorticity approximation (2.6) more accurate we may let each blob function have its own shape, i.e., $\overline{\omega}_0(\alpha')$ is defined by

$$\overline{\omega}_0(\alpha')=\sum_j k_j\rho_\delta^{(j)}(\alpha'-\alpha_j),\tag{6.2a}$$

where

$$\rho_\delta^{(j)}(\alpha)=\rho_\delta\left(B_j^{-1}\alpha\right)\tag{6.2b}$$

with suitably chosen 2×2 matrices B_j satisfying $\det B_j=1$. It is easy to see that the support of $\rho_\delta^{(j)}(\cdot)$ is an ellipse. We can simply replace $\rho_\delta(\alpha'-\alpha_j)$ in (2.9) and (2.10) by $\rho_\delta^{(j)}(\alpha'-\alpha_j)$ to adapt the method, and all the theoretical results in this paper are also valid for the adapted method.

4. This method can easily be used to approximate a high Reynolds number flow by incorporating a random-walk algorithm to mimic the viscosity effect and a vortex-generating algorithm to maintain the no-slip boundary condition (see [4] and [15]).

5. Kida [11] solved the motion of an elliptic vortex of uniform vorticity in a uniform shear flow exactly and showed that when the strain is very strong, the vortex is always elongated infinitely in the direction of the strain. This may also happen to the numerical elliptic-vortex blobs. In order to preserve numerical stability and accuracy, Zhu [18] suggested taking the following steps: stop deforming a blob if its minor axis is smaller than a given small number and split a blob if its major axis is larger than a given large number.

Here we would like to point out that Zhu [18] has studied the practical aspect of this concept in depth. More practical applications and numerical calculations are needed to verify the capabilities and limitations of the variable-vortex method. We will report this matter elsewhere.

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