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ON THE APPROXIMATION OF QUASIPERIODIC FUNCTIONS WITH DIOPHANTINE FREQUENCIES BY PERIODIC FUNCTIONS*

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Abstract. We present an analysis of the approximation error for a d-dimensional quasiperiodic function f with Diophantine frequencies, approximated by a periodic function with the fundamental domain $[0,L_1)\times[0,L_2)\times\cdots\times[0,L_d)$. When f has a certain regularity, its global behavior can be described by a finite number of Fourier components and has a polynomial decay at infinity. The dominant part of the periodic approximation error is bounded by $O(\max_{1\le j\le d} L_j^{-s_j})$, where L_j belongs to the best simultaneous approximation sequence and s_j is the number of different irrational elements in the jth dimension component of Fourier frequencies, respectively. Meanwhile, we discuss the approximation rate. Finally, these analytical results are verified by some examples.

Key words. quasiperiodic functions, Diophantine frequency, periodic approximation method, rational approximation error

MSC codes. 42A75, 65T40, 68W40, 74S25

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1. Introduction.

Quasiperiodic systems, as a natural extension of periodic systems, are widely found in nature, materials science, and physical systems, such as many-body problems, incommensurate structures, quasicrystals, polycrystalline materials, and quasiperiodic quantum systems [1, 2, 3, 4, 5, 6, 7]. We consider a d-dimensional quasiperiodic function $f(\boldsymbol{x})$, and aim to approximate it using a periodic function $f_p(\boldsymbol{x})$ in a finite domain. This subject is basic and important in the field of approximation theory. Meanwhile, this is also the core idea of the periodic approximation method (PAM), which is widely used to study quasiperiodic systems [8, 9, 10]. Therefore, studying the approximation error of PAM not only expands the intension of the approximation theory, but also establishes a basic theory for the application of PAM. However, it is surprising that there is still a lack of rigorous and systematic theoretical analysis on this approximation problem. In this work, we analyze the approximation error of multidimensional quasiperiodic functions with Diophantine frequencies when approximated by periodic functions.

Assume that $f(\boldsymbol{x})$ has the Fourier series [11]

$$f(\boldsymbol{x})=\sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} a_{\boldsymbol{\lambda}} e^{i2\pi\boldsymbol{\lambda}\cdot\boldsymbol{x}},\ \boldsymbol{x}\in\mathbb{R}^d, \tag{1.1}$$

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where

$$a_{\boldsymbol{\lambda}} = \lim_{T\to+\infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} f(\boldsymbol{x}) e^{-i2\pi\boldsymbol{\lambda}\cdot\boldsymbol{x}}\, d\boldsymbol{x} = ⨍ f(\boldsymbol{x}) e^{-i2\pi\boldsymbol{\lambda}\cdot\boldsymbol{x}}\, d\boldsymbol{x} \tag{1.2}$$

is the Fourier coefficient and $\boldsymbol{\Lambda} = \{\boldsymbol{\lambda} : \boldsymbol{\lambda} = \boldsymbol{P}\boldsymbol{k},\ \boldsymbol{k} \in \mathbb{Z}^n\} \subset \mathbb{R}^d$ is the Fourier exponent set (also called the Fourier frequency set). $\boldsymbol{P} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n) \in \mathbb{R}^{d\times n}$, $n \geq d$, is the projection matrix where $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$ are $\mathbb{Q}$-linearly independent.

Let F be the n-dimensional periodic parent function of f such that $f(\boldsymbol{x}) = F(\boldsymbol{P}^T\boldsymbol{x})$. The convergence of the Fourier series of its parent function can be determined by the convergence of the Fourier series of the quasiperiodic function, and vice versa [12]. Therefore, for the quasiperiodic function f given in (1.1), F can be expanded as

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \hat{F}_{\boldsymbol{k}} e^{i2\pi\boldsymbol{k}\cdot\boldsymbol{z}},\ \boldsymbol{z} \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n,$$

with Fourier coefficient

$$\hat{F}_{\boldsymbol{k}} = \frac{1}{|\mathbb{T}^n|} \int_{\mathbb{T}^n} e^{-i2\pi\boldsymbol{k}\cdot\boldsymbol{z}} F(\boldsymbol{z})\, d\boldsymbol{z}. \tag{1.3}$$

For more properties of parent function F, refer to [12].

Based on the Birkhoff's ergodic theorem [13], we have the following useful result.

THEOREM 1.1 (see [12]). *For a given quasiperiodic function*

$$f(\boldsymbol{x}) = F(\boldsymbol{p}_1 \cdot \boldsymbol{x}, \ldots, \boldsymbol{p}_n \cdot \boldsymbol{x}),\quad \boldsymbol{x} \in \mathbb{R}^d,$$

where $F(\boldsymbol{z})$ is its parent function defined on $\mathbb{T}^n$, and $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ are $\mathbb{Q}$-linearly independent, we have

$$a_{\boldsymbol{\lambda}} = \hat{F}_{\boldsymbol{k}},$$

where $\boldsymbol{\lambda} = \boldsymbol{P}\boldsymbol{k}$, $\boldsymbol{k} \in \mathbb{Z}^n$. $a_{\boldsymbol{\lambda}}$, and $\hat{F}_{\boldsymbol{k}}$ are defined in (1.2) *and* (1.3)*, respectively.*

According to Theorem 1.1, Fourier coefficients $\hat{F}_{\boldsymbol{k}}$ of F and Fourier coefficients $a_{\boldsymbol{\lambda}}$ of f have the same decay behavior. Consequently, if the parent function F has certain regularity, it is reasonable to define the set of fundamental Fourier exponents that globally describe the behavior of f. For a given positive integer $N \in \mathbb{Z}^+$, denote

$$K_N^n = \{\boldsymbol{k} = (k_j)_{j=1}^n \in \mathbb{Z}^n : -N \leq k_j < N\},$$

and the fundamental Fourier exponents set of f can be defined as

$$\boldsymbol{\Lambda}_N^d = \{\boldsymbol{\lambda} = \boldsymbol{P}\boldsymbol{k} : \boldsymbol{k} \in K_N^n\} \subset \boldsymbol{\Lambda}.$$

Obviously, the order of the set $\boldsymbol{\Lambda}_N^d$ is $\#(\boldsymbol{\Lambda}_N^d) = D = (2N)^n$. Let $\mathrm{QP}(\mathbb{R}^d)$ represent the space of all d-dimensional quasiperiodic functions. From $\boldsymbol{\Lambda}_N^d$, we can obtain a finite dimensional linear subspace of $\mathrm{QP}(\mathbb{R}^d)$,

$$S_N = \operatorname{span}\left\{ e^{i2\pi\boldsymbol{\lambda}\cdot\boldsymbol{x}}, \boldsymbol{x} \in \mathbb{R}^d, \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_N^d \right\}.$$

Denote the projection operator $\mathcal{P}_N : \mathrm{QP}(\mathbb{R}^d) \mapsto S_N$. Then we can split the quasiperiodic function f into two parts:

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{\lambda}_\ell \in \boldsymbol{\Lambda}_N^d} a_\ell e^{i2\pi\boldsymbol{\lambda}_\ell\cdot\boldsymbol{x}} + \sum_{\boldsymbol{\lambda}_\ell \in \boldsymbol{\Lambda}\setminus\boldsymbol{\Lambda}_N^d} a_\ell e^{i2\pi\boldsymbol{\lambda}_\ell\cdot\boldsymbol{x}} = \mathcal{P}_N f + (f - \mathcal{P}_N f).$$

From the viewpoint of $\boldsymbol{x}$-space, a periodic function is used to approximate $\mathcal{P}_N f$. Concretely, for some vectors $\boldsymbol{x} = (x_j)_{j=1}^d$, $\boldsymbol{y} = (y_j)_{j=1}^d$, and $\boldsymbol{z} = (z_j)_{j=1}^d$ with $z_j \neq 0$, we define Hadamard product $\boldsymbol{x} \circ \boldsymbol{y} = (x_j y_j)_{j=1}^d$ and $\boldsymbol{x}/\boldsymbol{z} = (x_j/z_j)_{j=1}^d$, $|\boldsymbol{z}| = z_1 z_2 \cdots z_d$. Then, for a given positive integer vector $\boldsymbol{L} = (L_j)_{j=1}^d$, we rewrite $\mathcal{P}_N f$ as

$$\mathcal{P}_N f(\boldsymbol{x}) = \sum_{\ell=1}^{D} a_\ell e^{i2\pi \boldsymbol{v}_\ell \cdot \boldsymbol{x}/\boldsymbol{L}},$$

where $\boldsymbol{v}_\ell = \boldsymbol{L} \circ \boldsymbol{\lambda}_\ell$ with $\boldsymbol{\lambda}_\ell \in \boldsymbol{\Lambda}_N^d$. Using a periodic function

$$f_p(\boldsymbol{x}) = \sum_{\ell=1}^{D} b_\ell e^{i2\pi \boldsymbol{h}_\ell \cdot \boldsymbol{x}/\boldsymbol{L}}, \ \ \boldsymbol{x} \in \mathbb{T}^d = \mathbb{R}^d/(\boldsymbol{L} \circ \mathbb{Z}^d), \tag{1.4}$$

where $\boldsymbol{h}_\ell \in \mathbb{Z}^d$ and b_ℓ is the Fourier coefficient, we approximate $\mathcal{P}_N f$ in $\Omega = [0, L_1) \times [0, L_2) \times \cdots \times [0, L_d)$.

In related work, Gomez, Mondelo, and Simo applied the PAM to recover frequencies and amplitudes of a one-dimensional quasiperiodic function from regular sampling data [14]. Correspondingly, a special error analysis was also provided for their considered one-dimensional cases [15]. The analysis in high-dimensional cases meets many difficulties compared to one-dimensional ones. The main challenge is that irrational frequencies in Fourier exponents may exist across different dimensions. In this paper, we are devoted to giving a theoretical analysis of the periodic approximation problem for arbitrary-dimensional quasiperiodic function.

From the viewpoint of reciprocal space, the periodic approximation problem involves using the integer vector $\boldsymbol{h}_\ell$ to approximate the irrational vector $\boldsymbol{v}_\ell$, which is related to the Diophantine approximation theory. For any vector $\boldsymbol{x} = (x_j)_{j=1}^d \in \mathbb{R}^d$, let $[\boldsymbol{x}] = ([x_j])_{j=1}^d$ denote the integer vector whose element $[x_j]$ is the distance between x_j and its nearest integer and $\|\boldsymbol{x}\|_{\ell^\infty} = \max_{1 \le j \le d} |x_j|$. In subsection 2.2, we will demonstrate that f_p presents a good approximation to f when $\boldsymbol{h}_\ell = [\boldsymbol{v}_\ell]$. Correspondingly, we can obtain the Diophantine inequality

$$\|\boldsymbol{h}_\ell - \boldsymbol{v}_\ell\|_{\ell^\infty} = \max_{1 \le j \le d} |v_{\ell,j} - h_{\ell,j}| < 1/2.$$

Clearly, the approximation analysis of Fourier frequencies is directly related to Diophantine theory. The following Dirichlet's theorem by Dirichlet is a fundamental result in this field.

THEOREM 1.2 (Dirichlet's theorem on simultaneous approximation [16]). *Suppose that $\alpha_1, \ldots, \alpha_s$ are s real numbers. Then there are infinitely integer points $(q, p_1, \ldots, p_s)$ with $q \neq 0$ such that*

$$\max_{1 \le j \le s} |\alpha_j q - p_j| < C_s q^{-1/s},$$

where $C_s = s/(s+1)$.

Denote $Y_D^d = \{\boldsymbol{v}_1, \ \boldsymbol{v}_2, \ldots, \boldsymbol{v}_D\}$. According to Theorem 1.2, there exists an increasing sequence $\{q_{j,1}, \ q_{j,2}, \ldots\}$ such that

$$E_{L_j} = \sum_{\ell=1}^{D} |v_{\ell,j} - h_{\ell,j}| = \sum_{\ell=1}^{D} |L_j \lambda_{\ell,j} - [L_j \lambda_{\ell,j}]| \le D C_{s_j} L_j^{-1/s_j},$$

where $L_j \in \{q_{j,1}, \ q_{j,2}, \ldots\}$ and s_j is the number of different irrational elements in the jth dimension of Y_D^d, $j = 1, 2, \ldots, d$.

DEFINITION 1.1. *For $j=1,2,\dots,d$, the j column increasing subsequence $\mathcal{T}_j(Y_D^d)=\{t_{j,1},\, t_{j,2},\dots\}\subset\{q_{j,1},\, q_{j,2},\dots\}$ is the best simultaneous approximation sequence of Y_D^d by taking $t_{j,1}=q_{j,1}$ and $t_{j,k}=\operatorname{argmin}_{E_{t_{j,k}}<E_{t_{j,k-1}}}\{q_{j,\ell}\}_{\ell=\ell_k}^{\infty}$ with $t_{j,k-1}=q_{j,\ell_k}$.*

Assume that the first ζ Fourier exponents of Y_D^d belong to $\mathbb{Q}^d$, and the rest belong to $\mathbb{R}^d\backslash\mathbb{Q}^d$. In fact, $\boldsymbol{v}_j=\boldsymbol{h}_j$ $(j=1,2,\dots,\zeta)$. Denote $Y_\zeta^d=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_\zeta\}$. Without loss of generality, we always have $Y_\zeta^d\neq\emptyset$. Otherwise, we can obtain a new vector $\boldsymbol{v}_1\in\mathbb{Q}_*^d=\mathbb{Q}^d\backslash\{\boldsymbol{0}\}$ through dividing $v_{j\ell}$ by $v_{1\ell}$, where $j=1,2,\dots,D$, $\ell=1,2,\dots,d$, and $v_{1\ell}\neq 0$.

To analyze the approximation error, we introduce the definition of Diophantine number.

DEFINITION 1.2 (see [17]). *A real number α is said to be a Diophantine number if for any $\tau>0$ and there exists a constant $C_a>0$ such that*

$$\left|\alpha-\frac{p}{q}\right|\geq\frac{C_a}{q^{2+\tau}}$$

for every rational number p/q.

From the definition of Diophantine number, we give the Diophantine condition on the fundamental Fourier exponents set $\boldsymbol{\Lambda}_N^d$.

Assumption 1.3. Assume that irrational elements $\lambda_{\ell,j}$ in the Fourier exponents $\boldsymbol{\lambda}_\ell=\boldsymbol{P}\boldsymbol{k}\in Y_D^d\backslash Y_\zeta^d$ are Diophantine numbers. In particular, $\boldsymbol{\lambda}_\ell$ satisfy the Diophantine condition when $p=0$, $q=\|\boldsymbol{k}\|_{\ell^\infty}$, and

$$|\lambda_{\ell,j}|>\frac{C_a}{\|\boldsymbol{k}\|_{\ell^\infty}^{2+\tau}},\ \ \tau>0.$$

Remark 1.4. An irrational number is either a Diophantine number or a Liouville number. The set of Liouville numbers has a Hausdorff dimension of zero [17]. Specifically, all algebraic numbers in $\mathbb{R}\backslash\mathbb{Q}$ are Diophantine numbers [18].

Remark 1.5. Since $\boldsymbol{v}=\boldsymbol{L}\circ\boldsymbol{\lambda}$ and all rational elements λ_j in $\boldsymbol{\lambda}$ can be transformed into integers by an appropriate choice of $\boldsymbol{L}$, our analysis focuses on distinguishing between integers and irrational numbers in $\boldsymbol{v}$.

Main results. The approximation error between the quasiperiodic function and the approximated periodic function is measured by the infinity norm $\|f_p-f\|_\infty=\sup_{\boldsymbol{x}\in\Omega}|f_p(\boldsymbol{x})-f(\boldsymbol{x})|$. According to the triangle inequality, the approximation error becomes

$$\|f_p-f\|_\infty\leq\|f_p-\mathcal{P}_Nf\|_\infty+\|\mathcal{P}_Nf-f\|_\infty. \tag{1.5}$$

The right terms of inequality (1.5) are the rational approximation error and the truncation error, respectively. The truncation error is related to the regularity of the quasiperiodic function f. If f is α-order derivative, the truncation error can be bounded by $O(N^{\kappa-\alpha})$ where $\alpha>\kappa>d/2$. The rational approximation error is estimated as $O(\max_{1\leq j\leq d}L_j^{-s_j})$ where s_j is the number of different irrational elements in the jth dimension of Y_D^d and $L_j\in\mathcal{T}_j(Y_D^d)$. The detailed analysis of the main results will be presented in section 2.

2. Analysis.

Let $\boldsymbol{a}=(a_1,a_2,\dots,a_n)^T\in\mathbb{R}^n$ and $\boldsymbol{A}=(a_{ij})\in\mathbb{R}^{d\times n}$, and denote that $\|\boldsymbol{a}\|=\sum_{j=1}^n|a_j|$ and $\|\boldsymbol{A}\|_1=\max_{1\leq j\leq n}\sum_{i=1}^d|a_{ij}|$. Denote $A\lesssim B$ as an estimate of the form $A < cB$ for a positive constant c. For any positive integer α, the space $H^{\alpha}_{QP}(\mathbb{R}^d)$ comprises all quasiperiodic functions with partial derivatives order $\alpha \geq 1$ with respect to the inner product $(\cdot,\cdot)_{\alpha}$,

$$(f_1, f_2)_{\alpha} = ⨍ f_1 \bar{f}_2 d\boldsymbol{x} + \sum_{\|\boldsymbol{m}\|=\alpha} ⨍ \partial^{\boldsymbol{m}}_{\boldsymbol{x}} f_1 \cdot \overline{\partial^{\boldsymbol{m}}_{\boldsymbol{x}} f_2} d\boldsymbol{x},$$

and $\partial^{\boldsymbol{m}}_{\boldsymbol{x}} = \partial^{m_1}_{x_1} \cdots \partial^{m_d}_{x_d}$. Then, we can define the norm $\|f\|^2_{\alpha} = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} (1+\|\boldsymbol{\lambda_k}\|^2)^{\alpha} |a_{\boldsymbol{\lambda_k}}|^2$ and seminorm $|f|^2_{\alpha} = \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \|\boldsymbol{\lambda_k}\|^{2\alpha} |a_{\boldsymbol{\lambda_k}}|^2$.

Assume that the quasiperiodic function $f \in H^{\alpha}_{QP}(\mathbb{R}^d)$; the estimate of $\|\mathcal{P}_N f - f\|_{\infty}$ has been given in [12], i.e.,

$$\|\mathcal{P}_N f - f\|_{\infty} \lesssim N^{\kappa-\alpha} |f|_{\alpha},$$

where $\alpha > \kappa > d/2$. The truncation error becomes negligible when f exhibits sufficient regularity. Hence, the periodic approximation error is mainly dominated by the rational approximation error.

Denote $b_{max} = \max_{1\leq \ell \leq D}\{|b_{\ell}|\}$. Next, we estimate the rational approximation error $\|f_p - \mathcal{P}_N f\|_{\infty}$. According to the definition of f_p and $\mathcal{P}_N f$, we have

$$\begin{aligned}
\|f_p - \mathcal{P}_N f\|_{\infty} &= \sup_{\boldsymbol{x}\in\Omega} |\mathcal{P}_N f(\boldsymbol{x}) - f_p(\boldsymbol{x})| \\
&\leq \left| \sum_{\ell=1}^{D} a_{\ell} e^{i2\pi \boldsymbol{v}_{\ell}\cdot \boldsymbol{x}/\boldsymbol{L}} - \sum_{\ell=1}^{D} b_{\ell} e^{i2\pi \boldsymbol{h}_{\ell}\cdot \boldsymbol{x}/\boldsymbol{L}} \right| \\
&= \left| \sum_{\ell=1}^{D} (a_{\ell} - b_{\ell}) e^{i2\pi \boldsymbol{v}_{\ell}\cdot \boldsymbol{x}/\boldsymbol{L}} - \sum_{\ell=1}^{D} b_{\ell} (e^{i2\pi \boldsymbol{h}_{\ell}\cdot \boldsymbol{x}/\boldsymbol{L}} - e^{i2\pi \boldsymbol{v}_{\ell}\cdot \boldsymbol{x}/\boldsymbol{L}}) \right| \\
&\leq \sum_{\ell=1}^{D} |a_{\ell} - b_{\ell}| \cdot \left| e^{i2\pi \boldsymbol{v}_{\ell}\cdot \boldsymbol{x}/\boldsymbol{L}} \right| + \sum_{\ell=1}^{D} |b_{\ell}| \cdot \left| e^{i2\pi \boldsymbol{v}_{\ell}\cdot \boldsymbol{x}/\boldsymbol{L}} \right| \cdot \left| e^{i2\pi (\boldsymbol{h}_{\ell} - \boldsymbol{v}_{\ell})\cdot \boldsymbol{x}/\boldsymbol{L}} - 1 \right| \\
&= \sum_{\ell=1}^{D} |a_{\ell} - b_{\ell}| + \sum_{\ell=1}^{D} |b_{\ell}| \cdot \left| 2\sin[\pi(\boldsymbol{h}_{\ell} - \boldsymbol{v}_{\ell})\cdot \boldsymbol{x}/\boldsymbol{L}] \right| \\
&< \sum_{\ell=1}^{D} |a_{\ell} - b_{\ell}| + 2\pi b_{max} \sum_{\ell=1}^{D} \sum_{j=1}^{d} |h_{\ell,j} - v_{\ell,j}|. \qquad (2.1)
\end{aligned}$$

In fact, $\|\boldsymbol{h}_{\ell} - \boldsymbol{v}_{\ell}\|$ can be arbitrarily small (see subsection 2.2). Hence, the last inequality in (2.1) is reasonable when $|h_{\ell,j} - v_{\ell,j}| \leq 1/2d$, $j = 1, 2, \dots, d$.

Denote the quasiperiodic and periodic Fourier coefficient vectors

$$\boldsymbol{y} = (a_1, a_2, \dots, a_D)^T, \ \boldsymbol{y}_p = (b_1, b_2, \dots, b_D)^T,$$

the difference is $\Delta\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{y}_p$, and

$$\Delta\boldsymbol{V} = (\boldsymbol{h}_1 - \boldsymbol{v}_1, \boldsymbol{h}_2 - \boldsymbol{v}_2, \dots, \boldsymbol{h}_D - \boldsymbol{v}_D) \in \mathbb{R}^{d\times D}.$$

Define $\|\Delta\boldsymbol{V}\|_e = \sum_{\ell=1}^{D} \sum_{j=1}^{d} |h_{\ell,j} - v_{\ell,j}|$. Then, the inequality (2.1) can be reduced to

$$\|f_p - \mathcal{P}_N f\|_{\infty} < \|\Delta\boldsymbol{y}\| + 2\pi b_{max} \|\Delta\boldsymbol{V}\|_e, \qquad (2.2)$$

where $\|\Delta\boldsymbol{V}\|_e$ is the Diophantine approximation error and $\|\Delta\boldsymbol{y}\|$ is the frequency approximation error. Then we will estimate $\|\Delta\boldsymbol{y}\|$ and $\|\Delta\boldsymbol{V}\|_e$ in subsections 2.1 and 2.2, respectively.

2.1. Error estimation $\|\Delta \boldsymbol{y}\|$. In this subsection, we will estimate the upper bound of $\|\Delta \boldsymbol{y}\|$ with the help of the discrete Fourier transform (DFT). The windowed DFT with G_j discretization nodes in the jth dimension is

$$F^{\eta}_{f,\boldsymbol{L},\boldsymbol{G}}(\boldsymbol{\beta}) = \frac{1}{|\boldsymbol{G}|}\sum_{\boldsymbol{j}\in K^d_{\boldsymbol{G}}} H^{\eta}_{\boldsymbol{G}}(\boldsymbol{j}) f(\boldsymbol{j}\circ \boldsymbol{L}/\boldsymbol{G}) e^{-i2\pi\boldsymbol{\beta}\cdot\boldsymbol{j}/\boldsymbol{G}},$$

where $\boldsymbol{G}=(G_\ell)^d_{\ell=1}$, $K^d_{\boldsymbol{G}} = \{\boldsymbol{j}=(j_\ell)^d_{\ell=1}\in\mathbb{Z}^d : -G_\ell/2 \le j_\ell \le G_\ell/2-1\}$, $\boldsymbol{\beta}\in\mathbb{Z}^d$, and

$$H^{\eta}_{\boldsymbol{G}}(\boldsymbol{j}) = \begin{cases} \displaystyle\prod_{\ell=1}^{d} \frac{\eta!}{(2\eta-1)!!}\left(1-\cos\frac{2\pi j_\ell}{G_\ell}\right)^{\eta}, & \boldsymbol{j}\in K^d_{\boldsymbol{G}},\\ 0 & \text{otherwise.}\end{cases}$$

Note that one can always choose $\boldsymbol{G}$ such that $G_\ell \gg L_\ell$. Concretely, we require that $\mathcal{P}_N f$ and f_p are equal through the DFT with η-order Hanning windowed function

$$F^{\eta}_{\mathcal{P}_N f,\boldsymbol{L},\boldsymbol{G}}(\boldsymbol{h}_s) = F^{\eta}_{f_p,\boldsymbol{L},\boldsymbol{G}}(\boldsymbol{h}_s),\ \ s=1,2,\dots,D. \tag{2.3}$$

This is equivalent to

$$\frac{1}{|\boldsymbol{G}|}\sum_{\boldsymbol{j}\in K^d_{\boldsymbol{G}}} H^{\eta}_{\boldsymbol{G}}(\boldsymbol{j})\mathcal{P}_N f(\boldsymbol{j}\circ\boldsymbol{L}/\boldsymbol{G})e^{-i2\pi\boldsymbol{h}_s\cdot\boldsymbol{j}/\boldsymbol{G}} = \frac{1}{|\boldsymbol{G}|}\sum_{\boldsymbol{j}\in K^d_{\boldsymbol{G}}} H^{\eta}_{\boldsymbol{G}}(\boldsymbol{j}) f_p(\boldsymbol{j}\circ\boldsymbol{L}/\boldsymbol{G})e^{-i2\pi\boldsymbol{h}_s\cdot\boldsymbol{j}/\boldsymbol{G}},$$

where $\boldsymbol{h}_s \in X^d_D = \{\boldsymbol{h}_1,\, \boldsymbol{h}_2,\dots,\boldsymbol{h}_D\} \subseteq K^d_{\boldsymbol{G}}$. The matrix form of (2.3) is

$$M\boldsymbol{y} = M_p\boldsymbol{y}_p, \tag{2.4}$$

where

(2.5)

$$M = (u_{st})\in\mathbb{C}^{D\times D},\ u_{st} = F^{\eta}_{e^{i2\pi\boldsymbol{v}_t\cdot\boldsymbol{x}/\boldsymbol{L}},\boldsymbol{L},\boldsymbol{G}}(\boldsymbol{h}_s) = \frac{1}{|\boldsymbol{G}|}\sum_{\boldsymbol{j}\in K^d_{\boldsymbol{G}}} H^{\eta}_{\boldsymbol{G}}(\boldsymbol{j}) e^{i2\pi(\boldsymbol{v}_t-\boldsymbol{h}_s)\cdot\boldsymbol{j}/\boldsymbol{G}},$$

$$M_p = (u^p_{st})\in\mathbb{C}^{D\times D},\ u^p_{st} = F^{\eta}_{e^{i2\pi\boldsymbol{h}_t\cdot\boldsymbol{x}/\boldsymbol{L}},\boldsymbol{L},\boldsymbol{G}}(\boldsymbol{h}_s) = \frac{1}{|\boldsymbol{G}|}\sum_{\boldsymbol{j}\in K^d_{\boldsymbol{G}}} H^{\eta}_{\boldsymbol{G}}(\boldsymbol{j}) e^{i2\pi(\boldsymbol{h}_t-\boldsymbol{h}_s)\cdot\boldsymbol{j}/\boldsymbol{G}}.$$

From (2.4), we can obtain the Fourier coefficient vector $\boldsymbol{y}_p$ if $\boldsymbol{y}$ is known, and vice versa.

Subsection 2.1.2 demonstrates that M is invertible if the Fourier exponent $\boldsymbol{v}_t$ satisfies Assumption 1.3, and $\boldsymbol{L}$, $\boldsymbol{G}$ satisfy Assumption 2.2. As a result, by applying the norm property to the linear system (2.4), we can obtain

$$\|\Delta\boldsymbol{y}\| \le b_{max}\|M^{-1}\|_1\|\|M_p - M\|_e.$$

In the following, we give the upper bound estimates of $\|M_p - M\|_e$ and $\|M^{-1}\|_1$ in subsections 2.1.1 and 2.1.2, respectively.

2.1.1. The bound of $\|M_p - M\|_e$. The difference between u_{st} in M and u^p_{st} in M_p can be estimated as

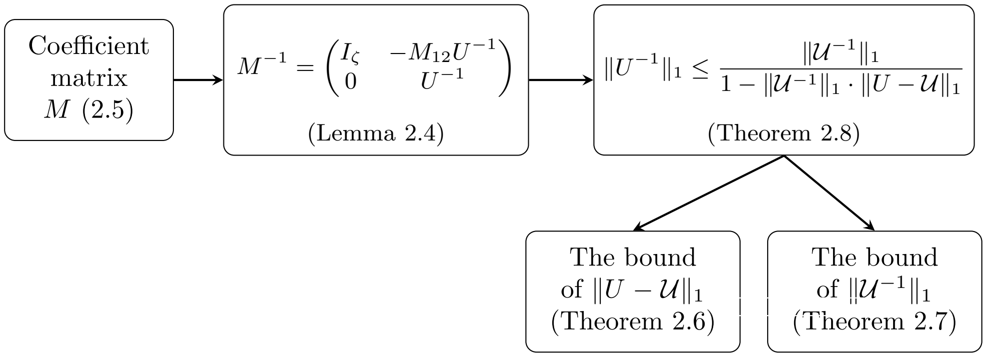


FIG. 1. *An overview of the upper bound proof of* $\|M^{-1}\|_1$.

$$\begin{aligned}
|u_{st} - u_{st}^p| &= \left| \frac{1}{|\boldsymbol{G}|} \sum_{\boldsymbol{j} \in K_{\boldsymbol{G}}^d} H_{\boldsymbol{G}}^\eta(\boldsymbol{j}) e^{i2\pi(\boldsymbol{v}_t - \boldsymbol{h}_s)\cdot \boldsymbol{j}/\boldsymbol{G}} - \frac{1}{|\boldsymbol{G}|} \sum_{\boldsymbol{j} \in K_{\boldsymbol{G}}^d} H_{\boldsymbol{G}}^\eta(\boldsymbol{j}) e^{i2\pi(\boldsymbol{h}_t - \boldsymbol{h}_s)\cdot \boldsymbol{j}/\boldsymbol{G}} \right| \\
&\le \frac{1}{|\boldsymbol{G}|} \sum_{\boldsymbol{j} \in K_{\boldsymbol{G}}^d} H_{\boldsymbol{G}}^\eta(\boldsymbol{j}) \cdot \left| e^{i2\pi(\boldsymbol{h}_t - \boldsymbol{h}_s)\cdot \boldsymbol{j}/\boldsymbol{G}} \right| \cdot \left| e^{i2\pi(\boldsymbol{v}_t - \boldsymbol{h}_t)\cdot \boldsymbol{j}/\boldsymbol{G}} - 1 \right| \\
&= \frac{1}{|\boldsymbol{G}|} \sum_{\boldsymbol{j} \in K_{\boldsymbol{G}}^d} H_{\boldsymbol{G}}^\eta(\boldsymbol{j}) \cdot \left| 2\sin[\pi(\boldsymbol{v}_t - \boldsymbol{h}_t)\cdot \boldsymbol{j}/\boldsymbol{G}] \right| \\
&\le 2\pi \|\boldsymbol{v}_t - \boldsymbol{h}_t\|,
\end{aligned}$$

where the last inequality is reasonable with $|h_{t,j} - v_{t,j}| \le 1/2d$, $j = 1, 2, \ldots, d$. Consequently, we have

$$\|M_p - M\|_e = \sum_{t=1}^{D} \sum_{s=1}^{D} |u_{st}^p - u_{st}| < 2\pi D \|\Delta \boldsymbol{V}\|_e.$$

Obviously, the Diophantine approximation error $\|\Delta \boldsymbol{V}\|_e$ controls the bound of $\|M_p - M\|_e$.

2.1.2. The bound of $\|M^{-1}\|_1$. This subsection proves that $\|M^{-1}\|_1$ is bounded and we give its upper bound. Figure 1 illustrates the flowchart of the upper bound proof of $\|M^{-1}\|_1$. For the purpose of error analysis, we introduce some required notations. Let $\mathbb{Z}_*^d = \mathbb{Z}^d \backslash \{\mathbf{0}\}$ for $d \in \mathbb{Z}^+$. Denote that $I_0(\boldsymbol{\ell})$ and $I_{in}(\boldsymbol{v})$ are index sets of zero and integer entries of $\boldsymbol{\ell} \in \mathbb{R}^d$, respectively, i.e.,

$$\begin{aligned}
I_0(\boldsymbol{\ell}) &= \{j : \ell_j = 0,\ 1 \le j \le d\}, \\
I_{in}(\boldsymbol{\ell}) &= \{j : \ell_j \in \mathbb{Z},\ 1 \le j \le d\}.
\end{aligned}$$

For $0 \le r \le d$, denote

$$J_r = \{\boldsymbol{\ell} \in \mathbb{Z}^d : \# I_0(\boldsymbol{\ell}) = r\} \subset \mathbb{Z}^d. \tag{2.6}$$

Obviously, $\cup_{r=0}^d J_r = \mathbb{Z}^d$.

Continuous normalized windowed Fourier transform. The continuous normalized windowed Fourier transform (NWFT) with Hanning window function of order $\eta \in \mathbb{Z}^+$ is

$$\phi_{f,\boldsymbol{L}}^\eta(\boldsymbol{w}) = \frac{1}{|\Omega|} \int_\Omega H_{\boldsymbol{L}}^\eta(\boldsymbol{x}) f(\boldsymbol{x}) e^{-i2\pi \boldsymbol{w}\cdot \boldsymbol{x}}\, d\boldsymbol{x}, \tag{2.7}$$

where $\boldsymbol{w}=(w_j)_{j=1}^d\in\mathbb{R}^d$ and

$$H_{\boldsymbol{L}}^{\eta}(\boldsymbol{x})=\begin{cases}\prod_{j=1}^{d}\frac{\eta!}{(2\eta-1)!!}\left(1-\cos\frac{2\pi x_j}{L_j}\right)^{\eta}, & \boldsymbol{x}=(x_j)_{j=1}^d\in\Omega,\\ 0 & \text{otherwise.}\end{cases}$$

Then $0\le H_{\boldsymbol{L}}^{\eta}(\boldsymbol{x})\le(2\eta+1)^d$. The Hanning window function has the normalization property

$$\frac{1}{|\Omega|}\int_{\Omega}H_{\boldsymbol{L}}^{\eta}(\boldsymbol{x})\,d\boldsymbol{x}=1. \tag{2.8}$$

From (2.7) and (2.8), for a vector $\boldsymbol{v}=(v_j)_{j=1}^d\in\mathbb{R}^d$, we obtain

$$\phi^{\eta}_{e^{i2\pi v_\ell x_\ell},L_\ell}(w_\ell)=1,\ \ \ell\in I_0(\boldsymbol{v}-\boldsymbol{w}).$$

For a given $L_\ell\in\mathbb{Z}^+$, if $|(v_\ell-w_\ell)L_\ell+j_1|>0$ with $-\eta\le j_1\le\eta$, we have

$$\phi^{\eta}_{e^{i2\pi\boldsymbol{v}\cdot\boldsymbol{x}},L_\ell}(\boldsymbol{w})=\prod_{\substack{\ell=1\\ \ell\notin I_0(\boldsymbol{v}-\boldsymbol{w})}}^{d}\frac{(-1)^{\eta}(\eta!)^2[e^{i2\pi(v_\ell-w_\ell)L_\ell}-1]}{i2\pi\Pi_{j_1=-\eta}^{\eta}[(v_\ell-w_\ell)L_\ell+j_1]}. \tag{2.9}$$

We can also give the coefficient matrix $\mathcal{M}=(\mathfrak{u}_{st})\in\mathbb{C}^{D\times D}$ in the NWFT where

$$\mathfrak{u}_{st}=\phi^{\eta}_{e^{i2\pi\boldsymbol{v}_t\cdot\boldsymbol{x}/\boldsymbol{L}},\boldsymbol{L}}(\boldsymbol{h}_s/\boldsymbol{L}).$$

The relation between DFT and NWFT is given in Lemma 2.1.

Lemma 2.1 (relation between the DFT and the NWFT). *When $\eta\ge1$, we have*

$$F^{\eta}_{f,\boldsymbol{L},\boldsymbol{G}}(\boldsymbol{k})=\phi^{\eta}_{f,\boldsymbol{L}}(\boldsymbol{k}/\boldsymbol{L})+\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d_*}\phi^{\eta}_{f,\boldsymbol{L}}\left(\frac{\boldsymbol{k}+\boldsymbol{\ell}\circ\boldsymbol{G}}{\boldsymbol{L}}\right). \tag{2.10}$$

The proof of Lemma 2.1 is similar to the one-dimensional case presented in [19].

Rewriting the form of $\boldsymbol{M}$. From the definition of ϕ, we know that

$$\mathfrak{u}_{st}=\begin{cases}1,\ s=t,\\ 0,\ s\neq t,\end{cases}\qquad \text{for } 1\le s\le D,\ 1\le t\le\zeta. \tag{2.11}$$

According to the properties of Y_D^d, we rewrite M and $\mathcal{M}$ as block matrices

$$M=\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & U\end{pmatrix},\ \ \mathcal{M}=\begin{pmatrix}I_\zeta & \mathcal{M}_{12}\\ 0 & \mathcal{U}\end{pmatrix}, \tag{2.12}$$

where $M_{11}\in\mathbb{C}^{\zeta\times\zeta}$ and $U\in\mathbb{C}^{(D-\zeta)\times(D-\zeta)}$.

From the relation between DFT and NWFT, we have the following proposition.

Proposition 2.2. *Under Assumption* 1.3, $\boldsymbol{L}$ *and* $\boldsymbol{G}$ *satisfy*

$$\frac{L_jC_a}{(2N)^{2+\tau}}-\frac{1}{2}-\eta>0,\ \ G_j-2L_j\|\boldsymbol{P}\|_1N-\frac{1}{2}>\eta,\ \ j=1,2,\dots,d. \tag{2.13}$$

Then, we have $M_{11}=I_\zeta$, $M_{21}=\boldsymbol{0}$.

Proof. For all $\boldsymbol{\ell}\in\mathbb{Z}_*^d$, we first assume that inequalities

$$\begin{cases}|v_{s,j}-h_{t,j}+j_1|>0, & j\notin I_0(\boldsymbol{v}_s-\boldsymbol{h}_t),\\ |v_{s,j}-h_{t,j}-\ell_jG_j+j_1|>0, & j\notin I_0(\boldsymbol{\ell}),\end{cases} \tag{2.14}$$

hold for $|j_1|<\eta$. From Lemma 2.1 and $v_{s,j}-h_{s,j}-\ell_jG_j=-\ell_jG_j\in\mathbb{Z}_*$ with $1\le s\le\zeta$, we can obtain

$$\begin{aligned}|u_{ss}-\mathfrak{u}_{ss}|&=|F^{\eta}_{e^{i2\pi\boldsymbol{v}_s\cdot\boldsymbol{x}/\boldsymbol{L}},\boldsymbol{L},\boldsymbol{G}}(\boldsymbol{h}_s)-\phi^{\eta}_{e^{i2\pi\boldsymbol{v}_s\cdot\boldsymbol{x}/\boldsymbol{L}},\boldsymbol{L}}(\boldsymbol{h}_s/\boldsymbol{L})|\\ &\le\sum_{\boldsymbol{\ell}\in\mathbb{Z}_*^d}\left|\phi^{\eta}_{e^{i2\pi\boldsymbol{v}_s\cdot\boldsymbol{x}/\boldsymbol{L}},\boldsymbol{L}}\left(\frac{\boldsymbol{h}_s+\boldsymbol{\ell}\circ\boldsymbol{G}}{\boldsymbol{L}}\right)\right|\\ &=\sum_{r=0}^{d-1}\sum_{\boldsymbol{\ell}\in J_r}\left|\prod_{j=1,j\notin I_0(\boldsymbol{\ell})}^{d}\frac{(-1)^\eta(\eta!)^2[e^{i2\pi(v_{s,j}-h_{s,j}-\ell_jG_j)}-1]}{i2\pi\Pi_{j_1=-\eta}^{\eta}(v_{s,j}-h_{s,j}-\ell_jG_j+j_1)}\right|=0.\end{aligned}$$

Similarly, based on inequalities (2.14), since $v_{t,j}-h_{s,j}\in\mathbb{Z}_*$ and $v_{t,j}-h_{s,j}-\ell_jG_j\in\mathbb{Z}_*$ with $1\le t\le\zeta$, we have

$$\begin{aligned}|u_{st}-\mathfrak{u}_{st}|\le&\sum_{r=0}^{d-1}\sum_{\boldsymbol{\ell}\in J_r}\left\{\left(\prod_{j\in I_0(\boldsymbol{\ell})\setminus I_0(\boldsymbol{v}_t-\boldsymbol{v}_s)}\frac{(\eta!)^2|e^{i2\pi(v_{t,j}-h_{s,j})}-1|}{2\pi\Pi_{j_1=-\eta}^{\eta}|v_{t,j}-h_{s,j}+j_1|}\right)\right.\\ &\left.\cdot\left(\prod_{j=1,j\notin I_0(\boldsymbol{\ell})}^{d}\frac{(\eta!)^2|e^{i2\pi(v_{t,j}-h_{s,j}-\ell_jG_j)}-1|}{2\pi\Pi_{j_1=-\eta}^{\eta}|v_{t,j}-h_{s,j}-\ell_jG_j+j_1|}\right)\right\}\\ =&\,0.\end{aligned}$$

Therefore, $u_{st}=\mathfrak{u}_{st}$ where $1\le s,t\le\zeta$. Combining with (2.11), it follows that

$$u_{st}=\begin{cases}1, & s=t,\\ 0, & s\ne t,\end{cases}\qquad\text{for } 1\le s\le D,\ 1\le t\le\zeta.$$

This means that $M_{11}=I_\zeta$ and $M_{21}=\mathbf{0}$.

Next, we show that inequalities (2.14) are true when $\boldsymbol{L}$ and $\boldsymbol{G}$ satisfy conditions (2.13).

(i) We prove that the inequality $|v_{s,j}-h_{t,j}+j_1|>0$ holds when $j\notin I_0(\boldsymbol{v}_s-\boldsymbol{h}_t)$ and $|j_1|\le\eta$. For $j\notin I_0(\boldsymbol{v}_s-\boldsymbol{v}_t)$, due to Assumption 1.3 and $|j_1|\le\eta$, we have

$$|v_{s,j}-h_{t,j}+j_1|\ge\min_{j\notin I_0(\boldsymbol{v}_s-\boldsymbol{v}_t)}(|v_{s,j}-h_{t,j}|-|j_1|)>\frac{L_jC_a}{(2N)^{2+\tau}}-\frac{1}{2}-\eta>0. \tag{2.15}$$

For $j\in I_0(\boldsymbol{v}_s-\boldsymbol{v}_t)$, then $|v_{s,j}-h_{t,j}+j_1|=|v_{t,j}-h_{t,j}+j_1|$. From the Diophantine inequality $|v_{t,j}-h_{t,j}|<1/2$, we have $|v_{t,j}-h_{t,j}+j_1|>0$ when $v_{t,j}\ne h_{t,j}$. Therefore, $|v_{s,j}-h_{t,j}+j_1|>0$ is true for $j\notin I_0(\boldsymbol{v}_s-\boldsymbol{h}_t)$.

(ii) We prove that $|v_{t,j}-h_{s,j}-\ell_jG_j+j_1|>0$ is true when $j\notin I_0(\boldsymbol{\ell})$. For $j\notin I_0(\boldsymbol{v}_s-\boldsymbol{v}_t)$, since $|j_1|\le\eta$ and $j\notin I_0(\boldsymbol{\ell})$, then

$$|v_{s,j}-h_{t,j}-\ell_jG_j+j_1|\ge|\ell_j|G_j-|v_{s,j}-h_{t,j}|-|j_1|.$$

Since

$$|v_{s,j}-h_{t,j}|\le|v_{s,j}-v_{t,j}|+|v_{t,j}-h_{t,j}|<2L_j\|\boldsymbol{P}\|_1N+1/2,$$

we can obtain

$$|v_{s,j} - h_{t,j} - \ell_j G_j + j_1| \geq |\ell_j| G_j - 2L_j \|\boldsymbol{P}\|_1 N - 1/2 - \eta.$$

Moreover, we have

$$|v_{s,j} - h_{t,j} - \ell_j G_j| > 0,$$

that is, $j \notin I_0(\boldsymbol{v}_s - \boldsymbol{h}_t - \boldsymbol{\ell} \circ \boldsymbol{G})$.

For $j \in I_0(\boldsymbol{v}_s - \boldsymbol{v}_t)$ and $j \notin I_0(\boldsymbol{\ell})$, we obtain

$$|v_{s,j} - h_{s,j} - \ell_j G_j + j_1| \geq G_j - \|\boldsymbol{v}_s - \boldsymbol{h}_s\|_{\ell^\infty} - \eta > G_j - \frac{1}{2} - \eta > 0,$$

and this also means that $j \notin I_0(\boldsymbol{v}_s - \boldsymbol{h}_s - \boldsymbol{\ell} \circ \boldsymbol{G})$.

Therefore, $|v_{t,j} - h_{s,j} - \ell_j G_j + j_1| > 0$ holds for $j \notin I_0(\boldsymbol{\ell})$. ∎

Remark 2.1. Similarly to Proposition 2.2, it is easy to prove that M_p is the identity matrix when conditions (2.13) are true.

Applying Proposition 2.2, M becomes

$$M = \begin{pmatrix} I_\zeta & M_{12} \\ 0 & U \end{pmatrix}.$$

Analyze the bound of $\|M^{-1}\|_1$. Lemma 2.4 will show that U is nonsingular, then

$$M^{-1} = \begin{pmatrix} I_\zeta & -M_{12}U^{-1} \\ 0 & U^{-1} \end{pmatrix}.$$

Moreover, we can obtain

$$\|M^{-1}\|_1 \leq \max\{1, (1 + \|M_{12}\|_1)\|U^{-1}\|_1\}.$$

The upper bound of $\|M^{-1}\|_1$ can be obtained by estimating bounds of $\|U^{-1}\|_1$ and $\|M_{12}\|_1$.

Subproblem 1: The bound of $\|U^{-1}\|_1$. Before giving the bound of $\|U^{-1}\|_1$, we introduce some necessary lemmas and symbols.

LEMMA 2.3 (Chapter 5.8 in [20]). *Assume that $E = E_1 + E_2$. If E_1 is invertible with $\|E_1^{-1}\| \cdot \|E_2\| < 1$, then E is invertible and*

$$\|E^{-1}\| \leq \frac{\|E_1^{-1}\|}{1 - \|E_1^{-1}\| \cdot \|E_2\|}. \tag{2.16}$$

Set $E_1 = \mathcal{U}$, $E_2 = U - \mathcal{U}$ in Lemma 2.3. Lemma 2.4 provides the sufficient condition such that $\mathcal{U}$ is nonsingular and $\|\mathcal{U}^{-1}\|_1 \cdot \|U - \mathcal{U}\|_1 < 1$. The proof of the upper bound of $\|U^{-1}\|_1$ is split into two parts: upper bounds of $\|U - \mathcal{U}\|_1$ and $\|\mathcal{U}^{-1}\|_1$ (see Theorem 2.6 and (2.31), respectively).

For any $\boldsymbol{v}_s, \boldsymbol{v}_t \in Y_D^d \setminus Y_\zeta^d$, denote

$$r_s = \#I_{in}(\boldsymbol{v}_s), \ \alpha_{st} = \#I_0(\boldsymbol{v}_s - \boldsymbol{v}_t), \ s \neq t, \tag{2.17}$$

and

$$d_m = \min_{\zeta+1 \leq s \leq D}\{r_s\}, \ d_M = \max_{\zeta+1 \leq s,t \leq D}\{\alpha_{st}\}, \tag{2.18}$$

with $0 \le d_m \le d-1$ and $0 < d_M \le d-1$. Denote $L_{min} = \min_{1\le j\le d} L_j$, $L_{max} = \max_{1\le j\le d} L_j$, and $G_{min} = \min_{1\le j\le d} G_j$. In the following analysis, we assume that $\boldsymbol{L}$ and $\boldsymbol{G}$ satisfy Assumption 2.2.

Assumption 2.2. For given positive numbers ϵ, ϵ_r $(0 \le r \le d-1)$, assume that $\boldsymbol{L}$ and $\boldsymbol{G}$ satisfy

$$L_{min} > \frac{(2N)^{2+\tau}}{C_a} \cdot \left\{ \left(\eta + \frac{1}{2}\right) + \max\left\{1, \left[\frac{(\eta!)^2}{\pi}\right]^{\frac{1}{2\eta+1}}, \left[\frac{\pi^{d_M}}{\epsilon}\right]^{\frac{1}{(2\eta+1)(d-d_M)}} \right\} \right\},$$

$$G_{min} > \max_{0\le r\le d-1} \left\{ 2L_{max}\|\boldsymbol{P}\|_1 N + \left(\eta + \frac{1}{2}\right) + \left[\frac{1}{\eta} \left(\frac{C_d^r \cdot \sum_{\beta=0}^{d-r} (2\eta)^\beta C_{d-r}^\beta}{\epsilon_r} \right)^{\frac{1}{d-r}} \right]^{\frac{1}{2\eta+1}} \right\},$$

where $\eta \ge 1$, $C_d^r = \frac{d!}{r!(d-r)!}$ and d_M is defined by (2.18).

LEMMA 2.4. *Under Assumption* 1.3, *for given positive numbers* ϵ *and* ϵ_r $(0 \le r \le d-1)$ *such that*

$$(D-\zeta) \sum_{r=0}^{d-1} \frac{(\eta!)^{-2r}}{\pi^{-r}} \epsilon_r + (D-\zeta-1)\epsilon < \frac{3^d \pi^d}{5^d} \left(\frac{1}{2} + \eta\right)^{-2\eta d}, \tag{2.19}$$

and $\boldsymbol{L}, \boldsymbol{G}$ *satisfy Assumption* 2.2, *then matrices* U *and* $\mathcal{U}$ *defined in* (2.12) *are nonsingular. Moreover, the inequality* $\|\mathcal{U}^{-1}\|_1 \cdot \|U - \mathcal{U}\|_1 < 1$ *holds.*

The proof of Lemma 2.4 will be presented at the end at this subsection.

For convenience, given a vector $\boldsymbol{\ell} = (\ell_j)_{j=1}^d \in \mathbb{Z}_*^d$, denote

$$A^0(v_{s,j}, h_{t,j}) = \frac{(\eta!)^2 |e^{i2\pi(v_{s,j} - h_{t,j})} - 1|}{2\pi \Pi_{j_1=-\eta}^{\eta} |(v_{s,j} - h_{t,j}) + j_1|}, \qquad j \in I_0(\boldsymbol{\ell}) \cap I_0^c(\boldsymbol{v}_s - \boldsymbol{h}_t),$$

$$A^1(v_{s,j}, h_{t,j}, \ell_j) = \frac{(\eta!)^2 |e^{i2\pi(v_{s,j} - h_{t,j} - \ell_j G_j)} - 1|}{2\pi \Pi_{j_1=-\eta}^{\eta} |(v_{s,j} - h_{t,j} - \ell_j G_j) + j_1|}, \ j \in I_0^c(\boldsymbol{\ell}) \cap I_0^c(\boldsymbol{v}_s - \boldsymbol{h}_t - \boldsymbol{\ell} \circ \boldsymbol{G}).$$

Obviously, $A^1(v_{s,j}, h_{t,j}, \ell_j) = A^0(v_{s,j}, h_{t,j})$ for $j \in I_0(\boldsymbol{\ell})$. Next, we prove that $A^0(v_{s,j}, h_{t,j})$ and $A^1(v_{s,j}, h_{t,j}, \ell_j)$ are well-defined and bounded under Assumptions 1.3 and 2.2.

PROPOSITION 2.5. *Under Assumptions* 1.3 *and* 2.2, *for a given vector* $\boldsymbol{\ell} = (\ell_j)_{j=1}^d \in \mathbb{Z}_*^d$, *the following conclusions hold:*

(1) $A^0(v_{s,j}, h_{t,j})$ *and* $A^1(v_{s,j}, h_{t,j}, \ell_j)$ *are well-defined. Moreover, for* $j \notin I_0(\boldsymbol{\ell})$, *we have* $j \notin I_0(\boldsymbol{v}_s - \boldsymbol{h}_t - \boldsymbol{\ell} \circ \boldsymbol{G})$.

(2) *If* $I_{in}(\boldsymbol{v}_s) = \emptyset$, *then* $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) = \emptyset$,

$$A^0(v_{s,j}, h_{t,j}) < 1, \ 1 \le j \le d, \tag{2.20}$$

and

$$A^1(v_{s,j}, h_{t,j}, \ell_j) \le \frac{(\eta!)^2}{\pi(|\ell_j| G_j - \|\boldsymbol{v}_s - \boldsymbol{h}_t\|_{\ell^\infty} - \eta)^{2\eta+1}}, \ j \notin I_0(\boldsymbol{\ell}). \tag{2.21}$$

In particular, when $s \ne t$ *and* $j \notin I_0(\boldsymbol{v}_s - \boldsymbol{v}_t)$, *it follows that*

$$A^0(v_{s,j}, h_{t,j}) < \frac{(\eta!)^2}{\pi(\frac{L_{min} C_a}{(2N)^{2+\tau}} - \frac{1}{2} - \eta)^{(2\eta+1)}} \le 1. \tag{2.22}$$

(3) *If $I_{in}(\boldsymbol{v}_s) \neq \emptyset$, that is $\# I_{in}(\boldsymbol{v}_s) = r_s > 0$, then, $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \neq \emptyset$.*

(3.1) *When $s = t$, we have*

$$A^0(v_{s,j}, h_{s,j}) < 1, \; j \notin I_0(\boldsymbol{v}_s - \boldsymbol{h}_s).$$

When $s \neq t$, if $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \cap I_0^c(\boldsymbol{v}_s - \boldsymbol{v}_t) \neq \emptyset$, we have

$$A^0(v_{s,j}, h_{t,j}) = 0, \;\; j \in I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \cap I_0^c(\boldsymbol{v}_s - \boldsymbol{v}_t). \tag{2.23}$$

Otherwise, $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \cap I_0^c(\boldsymbol{v}_s - \boldsymbol{v}_t) = \emptyset$, then $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \subset I_0(\boldsymbol{v}_s - \boldsymbol{v}_t)$ and

$$\begin{cases} A^0(v_{s,j}, h_{t,j}) < 1, \;\; j \notin I_0(\boldsymbol{v}_s - \boldsymbol{v}_t), \\ A^0(v_{s,j}, h_{t,j}) = A^0(v_{s,j}, h_{s,j}) < 1, \;\; j \in I_0(\boldsymbol{v}_s - \boldsymbol{v}_t) \setminus I_0(\boldsymbol{v}_s - \boldsymbol{h}_s). \end{cases} \tag{2.24}$$

(3.2) *If $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \cap I_0^c(\boldsymbol{\ell}) \neq \emptyset$, we have*

$$A^1(v_{s,j}, h_{t,j}, \ell_j) = 0, \; j \in I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \cap I_0^c(\boldsymbol{\ell}). \tag{2.25}$$

Otherwise, $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \cap I_0^c(\boldsymbol{\ell}) = \emptyset$, we have $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \subset I_0(\boldsymbol{\ell})$, and

$$A^1(v_{s,j}, h_{t,j}, \ell_j) \leq \frac{(\eta!)^2}{\pi(|\ell_j| G_j - \|\boldsymbol{v}_s - \boldsymbol{h}_t\|_{\ell^\infty} - \eta)^{2\eta+1}}, \; j \notin I_0(\boldsymbol{\ell}). \tag{2.26}$$

Proof. (1) From Assumption 2.2, the inequality (2.13) holds. Then, according to the proof of Proposition 2.2, the conclusion is easy to prove.

(2) $I_{in}(\boldsymbol{v}_s) = \emptyset$ $(\zeta + 1 \leq s \leq D)$ implies $I_0(\boldsymbol{v}_s - \boldsymbol{h}_t) = \emptyset$. When $s = t$,

$$\begin{aligned} A^0(v_{s,j}, h_{s,j}) &= \frac{(\eta!)^2 |e^{i2\pi(v_{s,j} - h_{s,j})} - 1|}{2\pi \Pi_{j_1=-\eta}^{\eta} |(v_{s,j} - h_{s,j}) + j_1|} \\ &= \frac{(\eta!)^2 \cdot 2\sin[\pi(v_{s,j} - h_{s,j})]}{2\pi |v_{s,j} - h_{s,j}| \cdot \Pi_{j_1=1}^{\eta} |(v_{s,j} - h_{s,j})^2 - j_1^2|} \\ &< \frac{\sin[\pi(v_{s,j} - h_{s,j})]}{\pi |v_{s,j} - h_{s,j}|} \cdot \frac{1}{\Pi_{j_1=1}^{\eta} [1 - \frac{(v_{s,j} - h_{s,j})^2}{j_1^2}]}. \end{aligned}$$

From the Weierstrass factorization formula

$$\sin(\pi z) = \pi z \prod_{\ell=1}^{\infty} \left(1 - \frac{z^2}{\ell^2}\right), \;\; z \in \mathbb{C},$$

the function

$$F_1(x) = \prod_{j_1=1}^{\eta} \left(1 - \frac{x^2}{j_1^2}\right)$$

monotonically decreases with respect to η when $0 < x < 1/2$. Therefore,

$$A^0(v_{s,j}, h_{s,j}) < 1.$$

When $s \neq t$, we also show that inequality (2.22) holds. For $j \notin I_0(\boldsymbol{v}_s - \boldsymbol{v}_t)$, applying the inequality (2.15), we can obtain

$$A^0(v_{s,j}, h_{t,j}) = \frac{(\eta!)^2 |e^{i2\pi(v_{s,j}-h_{t,j})} - 1|}{2\pi \Pi_{j_1=-\eta}^{\eta} |v_{s,j} - h_{t,j} + j_1|} \leq \frac{(\eta!)^2}{\pi \Pi_{j_1=-\eta}^{\eta} |v_{s,j} - h_{t,j} + j_1|}$$
$$< \frac{(\eta!)^2}{\pi} \cdot \frac{1}{\left(\frac{L_{min}C_a}{(2N)^{2+\tau}} - \frac{1}{2} - \eta\right)^{2\eta+1}} \leq 1.$$

For $j \in I_0(\boldsymbol{v}_s - \boldsymbol{v}_t)$, it follows that

$$A^0(v_{s,j}, h_{t,j}) = A^0(v_{s,j}, h_{s,j}) < 1.$$

Meanwhile, from the definition of $A^1(v_{s,j}, h_{t,j}, \ell_j)$ and

$$|v_{s,j} - h_{t,j} - \ell_j G_j + j_1| \geq |\ell_j| G_{min} - \|\boldsymbol{v}_s - \boldsymbol{h}_t\|_{\ell^\infty} - \eta,$$

then inequality (2.21) holds.

(3) $I_{in}(\boldsymbol{v}_s) \neq \emptyset$ $(\zeta + 1 \leq s \leq D)$ implies $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \neq \emptyset$.

(3.1) For $s = t$, similar to the above analysis in (2), we have

$$A^0(v_{s,j}, h_{s,j}) < 1,\ j \notin I_0(\boldsymbol{v}_s - \boldsymbol{h}_s).$$

For $s \neq t$, we consider two cases. If $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \cap I_0^c(\boldsymbol{v}_s - \boldsymbol{v}_t) \neq \emptyset$, there exists j such that $v_{s,j}$ is an integer and $v_{s,j} - h_{t,j} \in \mathbb{Z}_*$. This means $A^0(v_{s,j}, h_{t,j}) = 0$. Otherwise, $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \cap I_0^c(\boldsymbol{v}_s - \boldsymbol{v}_t) = \emptyset$ implies $I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \subset I_0(\boldsymbol{v}_s - \boldsymbol{v}_t)$. When $j \notin I_0(\boldsymbol{v}_s - \boldsymbol{v}_t)$, we have $v_{s,j} - h_{t,j} \notin \mathbb{Z}$ and $A^0(v_{s,j}, h_{t,j}) < 1$.

(3.2) Similar to the proof of the conclusion (2), (2.25) and (2.26) can be proved. ∎

Next, we estimate the bound of $\|U - \mathcal{U}\|_1$ by the relation between DFT and NWFT. Denote

$$g_0(t_0) = \frac{1}{\left(\frac{L_{min}C_a}{(2N)^{2+\tau}} - \frac{1}{2} - \eta\right)^{t_0}},\ \ g_1(t_1, t_2) = \frac{1}{(t_1 G_{min} - \|\boldsymbol{v}_s - \boldsymbol{h}_t\|_{\ell^\infty} - \eta)^{t_2}}.$$

Inequalities (2.22) and (2.26) are rewritten as

$$A^0(v_{s,j}, h_{t,j}) < \frac{(\eta!)^2}{\pi} g_0(2\eta + 1) \leq 1,\ \ A^1(v_{s,j}, h_{t,j}, \ell_j) \leq \frac{(\eta!)^2}{\pi} g_1(|\ell_j|, 2\eta + 1).$$

Denote

$$g_2 = \left(\frac{1}{\eta(G_{min} - \frac{1}{2} - \eta)^{2\eta+1}}\right)^{d-r},\ \ g_3 = \left(\frac{1}{\eta(G_{min} - 2L_{max}\|\boldsymbol{P}\|_1 N - \frac{1}{2} - \eta)^{2\eta+1}}\right)^{d-r}.$$

THEOREM 2.6. *Under Assumption* 1.3, *then*

$$\|U - \mathcal{U}\|_1 < \sum_{r=d_m}^{d-1} \frac{(\eta!)^{2(d-r)}}{\pi^{d-r}} C_{d-d_m}^{r-d_m} \cdot \sum_{\beta=0}^{d-r} (2\eta)^\beta C_{d-r}^\beta [g_2 + (D - \zeta - 1) g_3],$$

where d_m *is defined in* (2.18).

Proof. Let's prove this theorem for $I_{in}(\boldsymbol{v})=\emptyset$ and $I_{in}(\boldsymbol{v})\neq\emptyset$, respectively.

(1) When $I_{in}(\boldsymbol{v})=\emptyset$, from Assumption 2.2, we have

$$\frac{L_{min}C_a}{(2N)^{2+\tau}}-\frac{1}{2}-\eta\geq\left(\frac{(\eta!)^2}{\pi}\right)^{\frac{1}{2\eta+1}},\ G_{min}-2L_{max}\|\boldsymbol{P}\|_1N-1/2>\eta.$$

For $s,t=\zeta+1,\zeta+2,\dots,D$, from the definition of J_r in (2.6) and Lemma 2.1, we have

$$\begin{aligned}
|u_{st}-\mathfrak{u}_{st}| &= |F^{\eta}_{e^{i2\pi\boldsymbol{v}_t\cdot\boldsymbol{x}/\boldsymbol{L}},\boldsymbol{L},\boldsymbol{G}}(\boldsymbol{h}_s)-\phi^{\eta}_{e^{i2\pi\boldsymbol{v}_t\cdot\boldsymbol{x}/\boldsymbol{L}},\boldsymbol{L}}(\boldsymbol{h}_s/\boldsymbol{L})| \qquad (2.27)\\
&\leq\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d_*}\left|\phi^{\eta}_{e^{i2\pi\boldsymbol{v}_t\cdot\boldsymbol{x}/\boldsymbol{L}},\boldsymbol{L}}\left(\frac{\boldsymbol{h}_s+\boldsymbol{\ell}\circ\boldsymbol{G}}{\boldsymbol{L}}\right)\right|\\
&=\sum_{r=0}^{d-1}\sum_{\boldsymbol{\ell}\in J_r}\left\{\left(\prod_{j\in I_0(\boldsymbol{\ell})}\frac{(\eta!)^2|e^{i2\pi(v_{t,j}-h_{s,j})}-1|}{2\pi\Pi_{j_1=-\eta}^{\eta}|v_{t,j}-h_{s,j}+j_1|}\right)\right. \qquad (2.28)\\
&\quad\left.\cdot\left(\prod_{j=1,j\notin I_0(\boldsymbol{\ell})}^{d}\frac{(\eta!)^2|e^{i2\pi(v_{t,j}-h_{s,j}-\ell_jG_j)}-1|}{2\pi\Pi_{j_1=-\eta}^{\eta}|v_{t,j}-h_{s,j}-\ell_jG_j+j_1|}\right)\right\}\\
&=\sum_{r=0}^{d-1}\sum_{\boldsymbol{\ell}\in J_r}\left\{\prod_{j\in I_0(\boldsymbol{\ell})}A^0(v_{t,j},h_{s,j})\cdot\prod_{j=1,j\notin I_0(\boldsymbol{\ell})}^{d}A^1(v_{t,j},h_{s,j},\ell_j)\right\}. \qquad (2.29)
\end{aligned}$$

In the following, we will analyze the bound of (2.29) for $s\neq t$ and $s=t$, respectively.

When $s\neq t$, according to

$$\|\boldsymbol{v}_s-\boldsymbol{h}_t\|_{\ell^\infty}\leq\|\boldsymbol{v}_s-\boldsymbol{v}_t\|_{\ell^\infty}+\|\boldsymbol{v}_t-\boldsymbol{h}_t\|_{\ell^\infty}<2L_{max}\|\boldsymbol{P}\|_1N+1/2,$$

and from conclusion (2) of Proposition 2.5, we have

$$\begin{aligned}
&\sum_{\boldsymbol{\ell}\in J_r}\left\{\prod_{j\in I_0(\boldsymbol{\ell})}A^0(v_{t,j},h_{s,j})\cdot\prod_{j=1,j\notin I_0(\boldsymbol{\ell})}^{d}A^1(v_{t,j},h_{s,j},\ell_j)\right\}\\
&<\sum_{\boldsymbol{\ell}\in J_r}\prod_{j=1,j\notin I_0(\boldsymbol{\ell})}^{d}\frac{(\eta!)^2}{\pi}g_1(|\ell_j|,2\eta+1)\\
&=\left(\frac{2(\eta!)^2}{\pi}\right)^{d-r}C^r_d\cdot\left\{\sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}^d_*,\boldsymbol{\ell}>0,\\ \ell_1=\cdots=\ell_r=0}}\prod_{j=r+1}^{d}g_1(\ell_j,2\eta+1)\right\}\\
&=\left(\frac{2(\eta!)^2}{\pi}\right)^{d-r}C^r_d\cdot\left\{\sum_{\beta=0}^{d-r}\left[C^{\beta}_{d-r}g_1(1,(2\eta+1)\beta)\sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}^d_*\\ \ell_{r+1}=\cdots=\ell_{r+\beta}=1\\ \ell_j\geq2,j=r+\beta+1,\cdots,d}}\prod_{j=r+\beta+1}^{d}g_1(\ell_j,2\eta+1)\right]\right\}
\end{aligned}$$

$$\leq \left(\frac{2(\eta!)^2}{\pi}\right)^{d-r} C_d^r \cdot \left\{\sum_{\beta=0}^{d-r}\left[C_{d-r}^{\beta} g_1(1,(2\eta+1)\beta) \prod_{j=r+\beta+1}^{d} \int_1^{\infty} g_1(y_j, 2\eta+1)\, dy_j\right]\right\}$$

$$= \left(\frac{2(\eta!)^2}{\pi}\right)^{d-r} C_d^r \cdot \left\{\sum_{\beta=0}^{d-r}\left[C_{d-r}^{\beta} g_1(1,(2\eta+1)\beta)\right.\right.$$

$$\left.\left.\times \left(\frac{1}{2\eta G_{min}(G_{min} - \|\boldsymbol{v}_t - \boldsymbol{h}_s\|_{\ell^\infty} - \eta)^{2\eta}}\right)^{d-r-\beta}\right]\right\}$$

$$\leq \left(\frac{2(\eta!)^2}{\pi}\right)^{d-r} C_d^r \cdot \left\{\sum_{\beta=0}^{d-r}\left[C_{d-r}^{\beta} g_1(1,(2\eta+1)\beta)\right.\right.$$

$$\left.\left.\times \left(\frac{1}{2\eta (G_{min} - \|\boldsymbol{v}_t - \boldsymbol{h}_s\|_{\ell^\infty} - \eta)^{2\eta+1}}\right)^{d-r-\beta}\right]\right\}$$

$$< \frac{(\eta!)^{2(d-r)}}{\pi^{d-r}} C_d^r \cdot \sum_{\beta=0}^{d-r} (2\eta)^{\beta} C_{d-r}^{\beta} \left[\frac{1}{\eta(G_{min} - 2L_{max}\|\boldsymbol{P}\|_1 N - \frac{1}{2} - \eta)^{2\eta+1}}\right]^{d-r}.$$

Therefore,

$$|u_{st} - \mathfrak{u}_{st}| < \sum_{r=0}^{d-1} \frac{(\eta!)^{2(d-r)}}{\pi^{d-r}} C_d^r \cdot \sum_{\beta=0}^{d-r} (2\eta)^{\beta} C_{d-r}^{\beta} g_3.$$

Similarly, when $s = t$,

$$|u_{ss} - \mathfrak{u}_{ss}| < \sum_{r=0}^{d-1} \frac{(\eta!)^{2(d-r)}}{\pi^{d-r}} C_d^r \cdot \sum_{\beta=0}^{d-r} (2\eta)^{\beta} C_{d-r}^{\beta} g_2. \tag{2.30}$$

It follows that

$$\|U - \mathcal{U}\|_1 = \max_{\zeta+1 \leq t \leq D} \sum_{s=\zeta+1}^{D} |u_{st} - \mathfrak{u}_{st}|$$

$$< \sum_{r=0}^{d-1} \frac{(\eta!)^{2(d-r)}}{\pi^{d-r}} C_d^r \cdot \sum_{\beta=0}^{d-r} (2\eta)^{\beta} C_{d-r}^{\beta} [g_2 + (D - \zeta - 1) g_3].$$

(2) When $I_{in}(\boldsymbol{v}) = I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \neq \emptyset$, we can obtain a tighter bound of $\|U - \mathcal{U}\|_1$. We consider the diagonal case of $s = t$ and the nondiagonal one of $s \neq t$ separately.

(i) Let's consider the case of $s = t$. For $r_s \leq r \leq d-1$, we give an index set $\{j_1^*, j_2^*, \ldots, j_r^*\}$ such that

$$I_0(\boldsymbol{v}_s - \boldsymbol{h}_s) \subset \{j_1^*, j_2^*, \ldots, j_r^*\} \subset \{1, 2, \ldots, d\},$$

and denote

$$J_r^* = \{\boldsymbol{\ell} \in \mathbb{Z}^d : \ell_{j_1^*} = \ell_{j_2^*} = \cdots = \ell_{j_r^*} = 0\}.$$

From (2.9), we have

$$\left|\frac{1}{L_j}\int_0^{L_j} \frac{\eta!}{(2\eta-1)!!}\left(1 - \cos\frac{2\pi x_j}{L_j}\right)^{\eta} e^{i2\pi(v_{s,j} - h_{s,j})x_j/L_j}\, dx_j\right|$$

$$= \begin{cases} 1, & j \in I_0(\boldsymbol{v}_s - \boldsymbol{h}_s), \\ A^0(v_{s,j}, h_{s,j}), & j \notin I_0(\boldsymbol{v}_s - \boldsymbol{h}_s). \end{cases}$$

According to conclusion (3) of Proposition 2.5, we only need to consider the case $I_0(\boldsymbol{v}_s-\boldsymbol{h}_s)\cap I_0^c(\boldsymbol{\ell})=\emptyset$ for $\boldsymbol{\ell}\in\mathbb{Z}_*^d$ since

$$A^1(v_{s,j},h_{s,j},\ell_j)=0,\ j\in I_0(\boldsymbol{v}_s-\boldsymbol{h}_s)\cap I_0^c(\boldsymbol{\ell}).$$

Therefore, $I_0(\boldsymbol{v}_s-\boldsymbol{h}_s)\subseteq I_0(\boldsymbol{\ell})$ and $\#I_0(\boldsymbol{v}_s-\boldsymbol{h}_s)\le\#I_0(\boldsymbol{\ell})$. From inequalities (2.24) and (2.26), we can obtain

$$\begin{aligned}
|u_{ss}-\mathfrak{u}_{ss}| &\le \sum_{r=r_s}^{d-1}\sum_{\boldsymbol{\ell}\in J_r}\left\{\prod_{j\in I_0(\boldsymbol{\ell})\setminus I_0(\boldsymbol{v}_s-\boldsymbol{h}_s)}A^0(v_{s,j},h_{s,j})\cdot\prod_{j=1,j\notin I_0(\boldsymbol{\ell})}^{d}A^1(v_{s,j},h_{s,j},\ell_j)\right\}\\
&\le \sum_{r=r_s}^{d-1}\sum_{\boldsymbol{\ell}\in J_r}\left\{\prod_{j=1,j\notin I_0(\boldsymbol{\ell})}^{d}A^1(v_{s,j},h_{s,j},\ell_j)\right\}\\
&\le \sum_{r=r_s}^{d-1}2^{d-r}C_{d-r_s}^{r-r_s}\cdot\sum_{\substack{\boldsymbol{\ell}\in J_r^*\\ \boldsymbol{\ell}>0}}\left\{\prod_{\substack{j=1\\ j\notin\{j_1^*,j_2^*,\dots,j_r^*\}}}^{d}\frac{(\eta!)^2}{\pi}g_1(\ell_j,2\eta+1)\right\}\\
&< \sum_{r=r_s}^{d-1}\frac{(\eta!)^{2(d-r)}}{\pi^{d-r}}C_{d-r_s}^{r-r_s}\cdot\sum_{\beta=0}^{d-r}(2\eta)^\beta C_{d-r}^{\beta}g_2.
\end{aligned}$$

Obviously, the upper bound mentioned above is tighter than the bound given in (2.30) since

$$\begin{aligned}
\sum_{r=r_s}^{d-1}\frac{(\eta!)^{2(d-r)}}{\pi^{d-r}}C_{d-r_s}^{r-r_s}\cdot\sum_{\beta=0}^{d-r}(2\eta)^\beta C_{d-r}^{\beta}g_2 &\le \sum_{r=d_m}^{d-1}\frac{(\eta!)^{2(d-r)}}{\pi^{d-r}}C_{d-d_m}^{r-d_m}\cdot\sum_{\beta=0}^{d-r}(2\eta)^\beta C_{d-r}^{\beta}g_2\\
&\le \sum_{r=0}^{d-1}\frac{(\eta!)^{2(d-r)}}{\pi^{d-r}}C_{d}^{r}\cdot\sum_{\beta=0}^{d-r}(2\eta)^\beta C_{d-r}^{\beta}g_2.
\end{aligned}$$

(ii) When $s\neq t$ and from (2.26), the upper bound of $|u_{ts}-\mathfrak{u}_{ts}|$ is

$$|u_{ts}-\mathfrak{u}_{ts}|<\sum_{r=r_s}^{d-1}\frac{(\eta!)^{2(d-r)}}{\pi^{d-r}}C_{d-r_s}^{r-r_s}\cdot\sum_{\beta=0}^{d-r}(2\eta)^\beta C_{d-r}^{\beta}g_3.$$

The proof is completed. □

The bound $\|U-\mathcal{U}\|_1$ is always finite when $G_j\gg L_j$. Lemma 2.4 has presented a sufficient condition to guarantee the invertibility of $\mathcal{U}$ and $\|\mathcal{U}_D^{-1}\|_1\cdot\|\mathcal{U}_O\|_1<1$. In the following, we derive an upper bound for $\|\mathcal{U}^{-1}\|_1$ by decomposing $\mathcal{U}$ into diagonal part $\mathcal{U}_D$ and nondiagonal part $\mathcal{U}_O$.

THEOREM 2.7. *Under Assumption* 1.3, *then* $\mathcal{U}$ *is invertible and*

$$\|\mathcal{U}^{-1}\|_1\le\frac{x_1}{1-x_1x_2}, \tag{2.31}$$

where

$$\begin{cases}x_1=\frac{5^{d-d_m}}{3^{d-d_m}(\eta!)^{2(d-d_m)}}(\frac{1}{2}+\eta)^{2\eta(d-d_m)},\\ x_2=(D-\zeta-1)\frac{(\eta!)^{2d}}{\pi^{(d-d_M)}}g_1((2\eta+1)(d-d_M)),\end{cases}$$

d_m *and* d_M *are defined in* (2.18).

Proof. Lemma 2.4 implies that $\mathcal{U}$ is invertible. For $\boldsymbol{v}_t \in \mathbb{R}^d \backslash \mathbb{Q}^d$,

$$\mathfrak{u}_{tt} = \phi^{\eta}_{e^{i2\pi \boldsymbol{v}_t \cdot \boldsymbol{x}/\boldsymbol{L}}, \boldsymbol{L}}(\boldsymbol{h}_t/\boldsymbol{L}) \neq 0.$$

This means that $\mathcal{U}_D$ is invertible. Let's prove the inequality (2.31) for $I_{in}(\boldsymbol{v}) = \emptyset$ and $I_{in}(\boldsymbol{v}) \neq \emptyset$, respectively.

(1) When $I_{in}(\boldsymbol{v}) = \emptyset$, from the definition (2.9) of ϕ, we have

$$\begin{aligned}
|\mathfrak{u}_{tt}^{-1}| = \left|[\phi^{\eta}_{e^{i2\pi \boldsymbol{v}_t \cdot \boldsymbol{x}/\boldsymbol{L}}, \boldsymbol{L}}(\boldsymbol{h}_t/\boldsymbol{L})]^{-1}\right| &= \left| \prod_{j=1}^{d} \frac{i2\pi \Pi_{j_1=-\eta}^{\eta}[(v_{t,j} - h_{t,j}) + j_1]}{(-1)^{\eta}(\eta!)^2(e^{i2\pi(v_{t,j}-h_{t,j})} - 1)} \right| \\
&= \prod_{j=1}^{d} \frac{2\pi \Pi_{j_1=-\eta}^{\eta} \left| v_{t,j} - h_{t,j} + j_1 \right|}{(\eta!)^2 \left| 2\sin[\pi(v_{t,j} - h_{t,j})] \right|} \\
&\leq \frac{1}{(\eta!)^{2d}} (\|\boldsymbol{v}_t - \boldsymbol{h}_t\|_{\ell^\infty} + \eta)^{2\eta d} \left(\frac{1 + \|\boldsymbol{v}_t - \boldsymbol{h}_t\|^2_{\ell^\infty}}{1 - \|\boldsymbol{v}_t - \boldsymbol{h}_t\|^2_{\ell^\infty}} \right)^d.
\end{aligned}$$

The last inequality is true according to the following inequality:

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \leq \frac{\sin x}{x} \leq \left(\frac{\pi^2 - x^2}{\pi^2 + x^2} \right)^{\pi^2/12}, \ x \in (0, \pi).$$

Since $\mathcal{U}_D^{-1} = (\mathfrak{u}_{tt}^{-1})$ and the function $f(x) = (1+x)/(1-x)$ is monotonically increasing, it follows that

$$\begin{aligned}
\|\mathcal{U}_D^{-1}\|_1 = \max_{\zeta+1 \leq t \leq D} |\mathfrak{u}_{tt}^{-1}| &\leq \max_{\zeta+1 \leq t \leq D} \frac{1}{(\eta!)^{2d}} (\|\boldsymbol{v}_t - \boldsymbol{h}_t\|_{\ell^\infty} + \eta)^{2\eta d} \left(\frac{1 + \|\boldsymbol{v}_t - \boldsymbol{h}_t\|^2_{\ell^\infty}}{1 - \|\boldsymbol{v}_t - \boldsymbol{h}_t\|^2_{\ell^\infty}} \right)^d \\
&< \frac{5^d}{3^d(\eta!)^{2d}} \left(\frac{1}{2} + \eta \right)^{2\eta d}.
\end{aligned}$$

For the nondiagonal elements of $\mathcal{U}$, from (2.20) and (2.22), we can obtain

$$\begin{aligned}
|\mathfrak{u}_{st}| = \left| \phi^{\eta}_{e^{i2\pi \boldsymbol{v}_t \cdot \boldsymbol{x}/\boldsymbol{L}}, \boldsymbol{L}}(\boldsymbol{h}_s/\boldsymbol{L}) \right| &= \left| \prod_{j=1}^{d} \frac{(-1)^{\eta}(\eta!)^2[e^{i2\pi(v_{t,j}-h_{s,j})} - 1]}{i2\pi \Pi_{j_1=-\eta}^{\eta}[(v_{t,j} - h_{s,j}) + j_1]} \right| \\
&= \prod_{\substack{j=1, \\ j \notin I_0(\boldsymbol{v}_s - \boldsymbol{v}_t)}}^{d} A^0(v_{t,j}, h_{s,j}) \cdot \prod_{j \in I_0(\boldsymbol{v}_s - \boldsymbol{v}_t)} A^0(v_{s,j}, h_{s,j}) \\
&< \frac{(\eta!)^{2(d-\alpha_{st})}}{\pi^{d-\alpha_{st}}} g_0((2\eta + 1)(d - \alpha_{st})),
\end{aligned}$$

where α_{st} is defined in (2.17).

From Assumption 2.2, we can obtain

$$\frac{L_{min} C_a}{(2N)^{2+\tau}} - \frac{1}{2} - \eta \geq 1.$$

Therefore, we have

$$\begin{aligned}\|\mathcal{U}_O\|_1 &= \max_{\zeta+1\le t\le D}\sum_{s=\zeta+1,s\neq t}^{D}|\mathfrak{u}_{st}| \\ &< \max_{\zeta+1\le t\le D}\sum_{s=\zeta+1,s\neq t}^{D}\frac{(\eta!)^{2(d-\alpha_{st})}}{\pi^{d-\alpha_{st}}}g_0((2\eta+1)(d-\alpha_{st})) \\ &< (D-\zeta-1)\frac{(\eta!)^{2d}}{\pi^{d-d_M}}g_0((2\eta+1)(d-d_M))\end{aligned}$$

with $d_M=\max_{s,t}\{\alpha_{st}\}\le d-1$. Applying Lemma 2.3, the conclusion of this theorem holds for $I_{in}(\boldsymbol{v})=\emptyset$.

(2) When $I_{in}(\boldsymbol{v})\neq\emptyset$, we can also provide the upper bound on $\|\mathcal{U}^{-1}\|_1$.

(i) For $s=t$, it follows that

$$\begin{aligned}|\mathfrak{u}_{ss}^{-1}| = \left|[\phi^{\eta}_{e^{i2\pi\boldsymbol{v}_s\cdot\boldsymbol{x}/\boldsymbol{L}},\boldsymbol{L}}(\boldsymbol{h}_s/\boldsymbol{L})]^{-1}\right| &= \left|\prod_{\substack{j=1\\ j\neq I_{in}(\boldsymbol{v}_s)}}^{d}\frac{i2\pi\Pi_{j_1=-\eta}^{\eta}[(v_{s,j}-h_{s,j})+j_1]}{(-1)^{\eta}(\eta!)^2(e^{i2\pi(v_{s,j}-h_{s,j})}-1)}\right| \\ &< \frac{5^{d-r_s}}{3^{d-r_s}(\eta!)^{2(d-r_s)}}\left(\frac{1}{2}+\eta\right)^{2\eta(d-r_s)},\end{aligned}$$

where r_s is defined in (2.17). Due to $\eta\ge 1$, we easily find

$$1\le\frac{5}{3(\eta!)^2}\left(\frac{1}{2}+\eta\right)^{2\eta},$$

then

$$\frac{5^{d-r_s}}{3^{d-r_s}(\eta!)^{2(d-r_s)}}\left(\frac{1}{2}+\eta\right)^{2\eta(d-r_s)}\le\frac{5^d}{3^d(\eta!)^{2d}}\left(\frac{1}{2}+\eta\right)^{2\eta d}.$$

Therefore, we can obtain

$$\|\mathcal{U}_D^{-1}\|_1=\max_{\zeta+1\le s\le D}|\mathfrak{u}_{ss}^{-1}|<\frac{5^{d-d_m}}{3^{d-d_m}(\eta!)^{2(d-d_m)}}\left(\frac{1}{2}+\eta\right)^{2\eta(d-d_m)}\le\frac{5^d}{3^d(\eta!)^{2d}}\left(\frac{1}{2}+\eta\right)^{2\eta d}$$

with $d_m=\min_s\{r_s\}\le d-1$.

(ii) For $s\neq t$ and from (2.23), $A^0(v_{t,j},h_{s,j})=0$ for $j\in I_0(\boldsymbol{v}_t-\boldsymbol{h}_t)\cap I_0^c(\boldsymbol{v}_s-\boldsymbol{v}_t)$, which implies $\mathfrak{u}_{st}=0$. Therefore, we only consider $I_0(\boldsymbol{v}_t-\boldsymbol{h}_t)\subset I_0(\boldsymbol{v}_s-\boldsymbol{v}_t)$,

$$\begin{aligned}|\mathfrak{u}_{st}| &= \prod_{\substack{j=1,\\ j\notin I_0(\boldsymbol{v}_s-\boldsymbol{v}_t)}}^{d}A^0(v_{t,j},h_{s,j})\cdot\prod_{j\in I_0(\boldsymbol{v}_s-\boldsymbol{v}_t)\setminus I_{in}(\boldsymbol{v}_s)}A^0(v_{s,j},h_{s,j}) \\ &< \frac{(\eta!)^{2d}}{\pi^{(d-d_M)}}g_0((2\eta+1)(d-d_M))\end{aligned}$$

with $d_M=\max_{s,t}\{\alpha_{st}\}\le d-1$. Consequently, we have

$$\|\mathcal{U}_O\|_1<(D-\zeta-1)\frac{(\eta!)^{2d}}{\pi^{d-d_M}}g_0((2\eta+1)(d-d_M)).$$

From the abve analysis, inequality (2.31) holds. ∎

According to Theorem 2.6 and (2.31), we can derive the upper bound of $\|U^{-1}\|_1$.

THEOREM 2.8. *Under Assumption* 1.3, U *is nonsingular and*

$$\|U^{-1}\|_1 \leq \frac{x_1}{1 - x_1(x_2 + x_3)},$$

where

$$\begin{aligned} x_1 &= \frac{5^{d-d_m}}{3^{d-d_m}(\eta!)^{2(d-d_m)}}\left(\frac{1}{2}+\eta\right)^{2\eta(d-d_m)}, \\ x_2 &= (D-\zeta-1)\frac{(\eta!)^{2d}}{\pi^{d-d_M}} g_0((2\eta+1)(d-d_M)), \\ x_3 &= \sum_{r=d_m}^{d-1} \frac{(\eta!)^{2(d-r)}}{\pi^{(d-r)}} C_{d-d_m}^{r-d_m} \cdot \sum_{\beta=0}^{d-r} (2\eta)^\beta C_{d-r}^\beta [g_2 + (D-\zeta-1)g_3], \end{aligned}$$

d_m *and* d_M *are defined by* (2.18).

Proof. Lemma 2.4 implies that U is invertible. According to Lemma 2.3, we have

$$\|U^{-1}\|_1 \leq \frac{\|\mathcal{U}^{-1}\|_1}{1 - \|\mathcal{U}^{-1}\|_1 \cdot \|U - \mathcal{U}\|_1} := \bar{F}(\|\mathcal{U}^{-1}\|_1, \|U - \mathcal{U}\|_1).$$

$\bar{F}$ monotonously increases with respect to $\|\mathcal{U}^{-1}\|_1$ and $\|U - \mathcal{U}\|_1$ since $\partial\bar{F}/\partial(\|\mathcal{U}^{-1}\|_1) > 0$ and $\partial\bar{F}/\partial(\|U - \mathcal{U}\|_1) > 0$. Combining with Theorem 2.6 and (2.31), the proof can be completed. ∎

Proof of Lemma 2.4. From the above analysis, we know that $x_1(x_2 + x_3) < 1$ implies U and $\mathcal{U}$ are invertible. We will now demonstrate that this inequality is true when Assumption 2.2 is satisfied for $\boldsymbol{L}$ and $\boldsymbol{G}$.

(1) When $I_{in}(\boldsymbol{v}) = \emptyset$, denote $x_3 = (D-\zeta-1)C_O + C_D$ where

$$C_O = \sum_{r=0}^{d-1} \frac{(\eta!)^{2(d-r)}}{\pi^{(d-r)}} C_d^r \cdot \sum_{\beta=0}^{d-r} (2\eta)^\beta C_{d-r}^\beta g_3, \;\; C_D = \sum_{r=0}^{d-1} \frac{(\eta!)^{2(d-r)}}{\pi^{(d-r)}} C_d^r \cdot \sum_{\beta=0}^{d-r} (2\eta)^\beta C_{d-r}^\beta g_2,$$

and

$$\frac{1}{x_1} = \frac{3^d(\eta!)^{2d}}{5^d}\left(\frac{1}{2}+\eta\right)^{-2\eta d} = C(d,\eta).$$

Since $d \geq 1$ and $L_j > 0$, then $C_D < C_O$. The inequality $x_1(x_2 + x_3) < 1$ becomes

$$(D-\zeta-1)C_O + C_D + (D-\zeta-1)\frac{(\eta!)^{2d}}{\pi^{d-d_M}} g_0((2\eta+1)(d-d_M)) < C(d,\eta). \tag{2.32}$$

Hence, we need to prove that the inequality (2.32) holds.

First, for a given positive number ϵ, since L_{max} satisfies

$$\frac{(2N)^{2+\tau}}{C_a} \cdot \left\{ \left[\frac{\pi^{d_M}}{\epsilon}\right]^{\frac{1}{(2\eta+1)(d-d_M)}} + \frac{1}{2} + \eta \right\} < L_{max},$$

we have

$$\left[\frac{\pi^{d_M}}{\epsilon}\right]^{\frac{1}{(2\eta+1)(d-d_M)}} + \frac{1}{2} + \eta < \frac{L_{max} C_a}{(2N)^{2+\tau}}.$$

It follows that

$$\frac{\pi^{d_M}}{\left[\frac{L_{max}C_a}{(2N)^{2+\tau}}-\frac{1}{2}-\eta\right]^{(2\eta+1)(d-d_M)}}=\pi^{d_M}g_0((2\eta+1)(d-d_M))<\epsilon.$$

Second, for positive numbers ϵ_r $(r=0,1,\dots,d-1)$, since G_{min} satisfies

$$G_{min}>\max_{0\le r\le d-1}\left\{2L_{max}\|\boldsymbol{P}\|_1N+\frac{1}{2}+\eta+\left\{\left[\frac{C_d^r\cdot\sum_{\beta=0}^{d-r}(2\eta)^\beta C_{d-r}^\beta}{\epsilon_r}\right]^{\frac{1}{d-r}}\cdot\frac{1}{\eta}\right\}^{\frac{1}{2\eta+1}}\right\},$$

i.e.,

$$\left[\frac{C_d^r\cdot\sum_{\beta=0}^{d-r}(2\eta)^\beta C_{d-r}^\beta}{\epsilon_r}\right]^{\frac{1}{d-r}}<\eta\left(G_{min}-2L_{max}\|\boldsymbol{P}\|_1N-\frac{1}{2}-\eta\right)^{2\eta+1},$$

we can obtain

$$\begin{aligned}C_O&=\sum_{r=0}^{d-1}\frac{(\eta!)^{2(d-r)}}{\pi^{(d-r)}}C_d^r\cdot\sum_{\beta=0}^{d-r}(2\eta)^\beta C_{d-r}^\beta\left[\frac{1}{\eta(G_{min}-2L_{max}\|\boldsymbol{P}\|_1N-\frac{1}{2}-\eta)^{2\eta+1}}\right]^{d-r}\\&<\sum_{r=0}^{d-1}\frac{(\eta!)^{2(d-r)}}{\pi^{(d-r)}}\epsilon_r.\end{aligned}$$

Moreover, $C_D<C_O$ means that $C_D<\sum_{r=0}^{d-1}\frac{(\eta!)^{2(d-r)}}{\pi^{(d-r)}}\epsilon_r$. If positive numbers ϵ_r $(r=0,1,\dots,d-1)$ and ϵ satisfy the inequality (2.19), we have

$$\begin{aligned}&(D-\zeta-1)C_O+C_D+(D-\zeta-1)\frac{(\eta!)^{2d}}{\pi^{d-d_M}}g_0((2\eta+1)(d-d_M))\\&\quad<(D-\zeta)\sum_{r=0}^{d-1}\frac{(\eta!)^{-2r}}{\pi^{-r}}\epsilon_r+(D-\zeta-1)\epsilon<C(d,\eta).\end{aligned}$$

Therefore, the inequality (2.32) is true and U is nonsingular. Moreover, we can obtain

$$(D-\zeta-1)\frac{(\eta!)^{2d}}{\pi^{d-d_M}}g_0((2\eta+1)(d-d_M))<C(d,\eta),$$

which means $x_1x_2<1$ and $\mathcal{U}$ is nonsingular. The condition that guarantees U is nonsingular is sufficient to ensure that $\mathcal{U}$ is also nonsingular.

(2) When $I_{in}(\boldsymbol{v})\neq\emptyset$, based on the analysis in Theorem 2.6 and (2.31), we know that $x_3=(D-\zeta-1)\hat{C}_O+\hat{C}_D$ where

$$\begin{aligned}\hat{C}_O&=\sum_{r=d_m}^{d-1}\frac{(\eta!)^{2(d-r)}}{\pi^{d-r}}C_{d-d_m}^{r-d_m}\cdot\sum_{\beta=0}^{d-r}(2\eta)^\beta C_{d-r}^\beta g_3,\\\hat{C}_D&=\sum_{r=d_m}^{d-1}\frac{(\eta!)^{2(d-r)}}{\pi^{d-r}}C_{d-d_m}^{r-d_m}\cdot\sum_{\beta=0}^{d-r}(2\eta)^\beta C_{d-r}^\beta g_2.\end{aligned}$$

Note that $\hat{C}_O\le C_O$, $\hat{C}_D\le C_D$, and $\hat{C}(d,\eta)>C(d,\eta)$. Hence, if the inequality (2.32) holds, we have

$$(D-\zeta-1)\hat{C}_O+\hat{C}_D+(D-\zeta-1)\frac{(\eta!)^{2d}}{\pi^{d-d_M}}g_0((2\eta+1)(d-d_M))<\hat{C}(d,\eta).$$

This means that U and $\mathcal{U}$ are nonsingular. Consequently, the proof of Lemma 2.4 is completed. ∎

Subproblem 2: The bound of $\|M_{12}\|_1$. Based on the analysis in Remark 2.1, we can directly give the upper bound of $\|M_{12}\|_1$. Based on the proof of Theorem 2.6 and (2.31), it follows that

$$\|M_{12}\|_1 < (\zeta+1)(x_2+y_2),$$

where

(2.33)

$$x_2 = \frac{(\eta!)^{2d}}{\pi^{d-d_M}} g_0((2\eta+1)(d-d_M)),\ y_2 = \sum_{r=d_m}^{d-1} \frac{(\eta!)^{2(d-r)}}{\pi^{d-r}} C_{d-d_m}^{r-d_m} \cdot \sum_{\beta=0}^{d-r} (2\eta)^{\beta} C_{d-r}^{\beta} g_3,$$

with d_m, d_M defined by (2.18).

2.2. Analysis of Diophantine approximation matrix $\Delta\mathbf{V}$. In this subsection, we analyze the approximation rate of Diophantine approximation error $\|\Delta \boldsymbol{V}\|_e$ and discuss the periodic approximation function sequence. From the definition of $\|\Delta \boldsymbol{V}\|_e$, we can derive

$$\begin{aligned}\|\Delta \boldsymbol{V}\|_e &= \|(\boldsymbol{h}_1-\boldsymbol{v}_1, \boldsymbol{h}_2-\boldsymbol{v}_2, \ldots, \boldsymbol{h}_D-\boldsymbol{v}_D)\|_e\\ &= \sum_{\ell=1}^{D}\sum_{j=1}^{d} |h_{\ell,j}-v_{\ell,j}| \le d \max_{1\le j\le d} \sum_{\ell=1}^{D} |h_{\ell,j}-v_{\ell,j}|,\end{aligned}$$

which is equivalent to the simultaneous approximation of $\boldsymbol{u}_j = (v_{\ell,j})_{\ell=1}^{D}$, $j=1,2,\ldots,d$. Denote

$$\hat{R}(\boldsymbol{h}) = (h_{\ell,j}-v_{\ell,j})_{\ell=1,j=1}^{D,d},\ R(\boldsymbol{v}) = ([v_{\ell,j}]-v_{\ell,j})_{\ell=1,j=1}^{D,d}.$$

Now we can show that $\|\hat{R}(\boldsymbol{h})\|_{\ell^\infty} < 1/2$ if and only if $h_{\ell,j} = [v_{\ell,j}]$. When $\hat{R}(\boldsymbol{h}) \ne R(\boldsymbol{v})$ and $\|R(\boldsymbol{v})\|_{\ell^\infty} < 1/2$, we have

$$\|\hat{R}(\boldsymbol{h}) - R(\boldsymbol{v})\|_{\ell^\infty} = \|(h_{\ell,j}-[v_{\ell,j}])_{\ell=1,j=1}^{D,d}\|_{\ell^\infty} \ge 1.$$

This means that $\|\hat{R}(\boldsymbol{h})\|_{\ell^\infty} \ge \|\hat{R}(\boldsymbol{h}) - R(\boldsymbol{v})\|_{\ell^\infty} - \|R(\boldsymbol{v})\|_{\ell^\infty} > 1/2$, which is obviously contradictory.

Since we assume that there are only integers and irrational elements in $\boldsymbol{u}_j$ ($j = 1,2,\ldots,d$), then $h_{\ell,j} = v_{\ell,j}$ when $v_{\ell,j}$ is an integer. According to Theorem 1.2 and Definition 1.1, it follows that

$$\|\Delta \boldsymbol{V}\|_e = \sum_{j=1}^{d}\sum_{k=1}^{s_j} |h_{\ell_k,j}-v_{\ell_k,j}| \le \sum_{j=1}^{d} C_{s_j} L_j^{-1/s_j},\ \ L_j \in \mathcal{T}_j(Y_D^d),$$

where s_j represents the number of different irrational Fourier frequency elements in $\boldsymbol{u}_j$ ($j=1,2,\ldots,d$), respectively. As a consequence, the above expression provides an upper bound of $\|\Delta \boldsymbol{V}\|_e$ rather than a supremum, and demonstrates that $\|\Delta \boldsymbol{V}\|_e$ is inversely proportional to L_j. Nevertheless, the uniform decrease of $\|\Delta \boldsymbol{V}\|_e$ with a gradually increasing L_j cannot be guaranteed due to the property of irrational number. It is possible that a gradual increase of L_j may increase $\|\Delta \boldsymbol{V}\|_e$.

Remark 2.3.

(i) When the vector $\boldsymbol{u}_j$ only contains an irrational number λ_{j_1}, then $\{t_1, t_2, \ldots\}$ is a continued fraction expansion of λ_{j_1}.

(ii) When the vector $\boldsymbol{u}_j$ contains more than one irrational number, finding the simultaneous approximation sequence $\{q_{1,j}, q_{2,j}, \ldots\}$ is an NP-hard problem [23, 24].

2.3. Summary. We put previous results together and give the bound of the rational approximation error

$$\begin{aligned} \|f_p - \mathcal{P}_N f\|_\infty &< \underbrace{b_{max}\|M^{-1}\|_1\|M_p - M\|_e + 2\pi b_{max}\|\Delta \boldsymbol{V}\|_e}_{\varepsilon_1} \\ &< \underbrace{2\pi b_{max}\Big[D\frac{[1+(\zeta+1)(x_2+y_2)]x_1}{1-x_1(x_2+x_3)}+1\Big]\|\Delta \boldsymbol{V}\|_e}_{\varepsilon_2}, \end{aligned} \tag{2.34}$$

where the definitions of x_1, x_2, x_3, and y_2 are given in Theorem 2.8 and (2.33), respectively.

The main result of this work is summarized as follows.

THEOREM 2.9. *Under Assumption* 1.3*, and assuming the quasiperiodic function* $f \in H^\alpha_{QP}(\mathbb{R}^d)$*, the error in approximating* f *with* $f_p \in H^\alpha(\mathbb{T}^d)$ *is given by*

$$\|f_p - f\|_\infty \lesssim \max_{1\le j\le d} L_j^{-s_j} + N^{\kappa-\alpha}|f|_\alpha,$$

where $\alpha > \kappa > d/2$*,* s_j *is the number of different irrational elements in the* j*th dimension of* Y_D^d*, and* $L_j \in \mathcal{T}_j(Y_D^d)$ *is the corresponding best simultaneous approximation sequence.*

2.4. Discussion.

2.4.1. On the error bound. From the above analysis, we can offer some discussion on the approximation error.

(i) When the quasiperiodic function $f(\boldsymbol{x})$ is known, we can use (2.3) to directly obtain the approximate error $\|f_p - f\|_\infty$.

(ii) When the Fourier exponents of $f(\boldsymbol{x})$ and the periodic approximation function $f_p(\boldsymbol{x})$ are given, we can calculate an error bound ε_1 defined in (2.34). Moreover, by solving (2.3), we can obtain the Fourier coefficient vector $\boldsymbol{y}$.

(iii) When the quasiperiodic function $f(\boldsymbol{x})$ is unknown, we can use Theorem 2.9 to obtain an upper bound of $\|f_p - f\|_\infty$. Moreover, the error bounds of $\|f_p - \mathcal{P}_N f\|_\infty$ have a relationship

$$\|f_p - \mathcal{P}_N f\|_\infty < \varepsilon_1 < \varepsilon_2.$$

2.4.2. On the best approximation rate. Although directly computing the best simultaneous approximation sequence $\mathcal{T}$ can be challenging, we can still discuss its growth rate. The sequence $\mathcal{T}$ increases at a rate of at least [21]

$$\liminf_{k\to\infty}(t_k)^{1/k} \ge 1 + \frac{1}{2^{s+1}}.$$

The sequence $\mathcal{T}$ grows at a rate of at most [22]

$$\limsup_{k\to\infty}\frac{1}{k}\ln t_k \le C$$

for almost all $\boldsymbol{u} = (u_\ell)_{\ell=1}^D \in \mathbb{R}^D$ and C is a contant. Here, "almost all" refers to the Lebesgue measure on $\mathbb{R}^D$.

3. Some examples. This section offers two examples for $d = 1$ and $d = 3$ to support our theoretical results. These examples involve only finite trigonometric summations, so there is no truncation error. Denote $\varepsilon_0 = \|f_p - \mathcal{P}_N f\|_\infty$ and $e(L_j) = \sum_{\ell=1}^D |L_j\lambda_{\ell,j} - [L_j\lambda_{\ell,j}]|$, $j = 1, 2, \ldots, d$. The d-dimensional quasiperiodic function $f(\boldsymbol{x})$ has the following expansion: $f(\boldsymbol{x}) = \sum_{\ell=1}^D a_\ell e^{i2\pi(\boldsymbol{L}\circ\boldsymbol{\lambda}_\ell)\cdot\boldsymbol{x}/\boldsymbol{L}}$. The corresponding periodic approximation function $f_p(\boldsymbol{x})$ is given in (1.4) with a fundamental domain $[0, L_1) \times [0, L_2) \times \cdots \times [0, L_d)$. We will show the rational approximation error ε_0 and two theoretical upper error bounds ε_1, ε_2. To derive a more accurate upper bound ε_2, x_1 in (2.34) is calculated by

$$x_1 = \max_{\zeta+1\le s\le D}\left\{(\|\boldsymbol{v}_s - \boldsymbol{h}_s\|_{\ell^\infty} + \eta)^{2\eta(d-r_s)}\left(\frac{1 + \|\boldsymbol{v}_s - \boldsymbol{h}_s\|_{\ell^\infty}^2}{1 - \|\boldsymbol{v}_s - \boldsymbol{h}_s\|_{\ell^\infty}^2}\right)^{d-r_s}\right\}.$$

Lemma 2.4 gives a sufficient condition that M is invertible. In the following examples, this condition can be weakened as

$$\frac{L_{min}C_a}{(2N)^{2+\tau}} - \frac{1}{2} - \eta > \max\left\{1, \left[\frac{(\eta!)^2}{\pi}\right]^{\frac{1}{2\eta+1}}\right\},\ G_{min} - 2L_{max}\|\boldsymbol{P}\|_1 N - \frac{1}{2} > \eta. \tag{3.1}$$

Let $\eta = 1$, then $[(\eta!)^2/\pi]^{\frac{1}{2\eta+1}} = (1/\pi)^{\frac{1}{3}} < 1$. Inequality (3.1) becomes

$$L_{min} > \frac{5}{2}\cdot\frac{(2N)^{2+\tau}}{C_a},\ G_{min} > 2L_{max}\|\boldsymbol{P}\|_1 N + \frac{3}{2}.$$

Example 3.1. Consider a one-dimensional quasiperiodic function $f(x)$ with four Fourier exponents $(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4) = (1,\ \sqrt{2},\ 2+\sqrt{2},\ 1+2\sqrt{2})$. The corresponding Fourier coefficients are $a_1 = 0.02 - 0.2i$, $a_2 = 0.1$, $a_3 = 0.03 + 0.1i$, $a_4 = 0.02$.

In this example, the projection matrix is $\boldsymbol{P} = (1 \quad \sqrt{2})$. The reciprocal lattice vectors are

$$\begin{pmatrix}\boldsymbol{k}_1 & \boldsymbol{k}_2 & \boldsymbol{k}_3 & \boldsymbol{k}_4\end{pmatrix} = \begin{pmatrix}1 & 0 & 2 & 1\\ 0 & 1 & 1 & 2\end{pmatrix}$$

with $N = 2$. Let Diophantine parameters $C_a = 2$ and $\tau = 0.2$. We can verify that λ_1, λ_2, λ_3, λ_4 satisfy the Diophantine condition and $\zeta = 1$, $I_{in}(\lambda_j) = \emptyset$ ($j = 2, 3, 4$), $d_M = 0$, $\|\boldsymbol{P}\|_1 = \sqrt{2}$. Therefore, $L > 20$ and $G > 4\sqrt{2}L + 3/2$. Here, we choose $G = 10L$.

We can obtain the periodic approximation function $f_p(x)$ from (2.3). For example, when $L = 13860$, Fourier exponents of $f_p(x)$ are

$$h_1 = 13860,\ h_2 = 19601,\ h_3 = 53062,\ h_4 = 47321.$$

The corresponding Fourier coefficient vector $\boldsymbol{y}_p$ is

$$\begin{aligned}\boldsymbol{y}_p = {} & 0.0200 - 0.2000i, 0.1000 - (8.0139e-07)i, 0.0300 + 0.1000i, 0.0200\\ & - (1.6028e-07)i^T.\end{aligned}$$

Figure 2 illustrates that as L increases, $e(L)$ decreases, but the decrease is not uniform. To compute the error results, we present the first eight terms of the optimal approximation sequence $\mathcal{T}$, which correspond to the first column in Table 1.

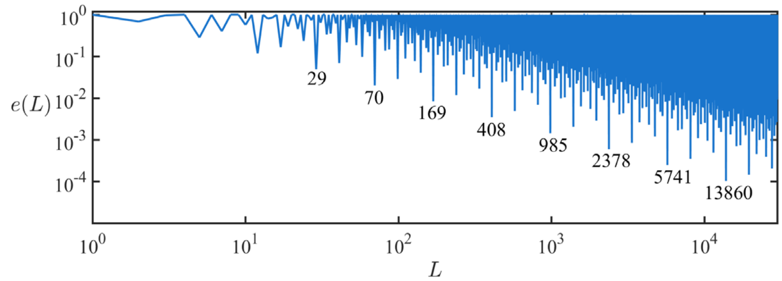


FIG. 2. *The change of Diophantine approximation error $e(L)$ with an increase of L. The sequence in the figure corresponds to the best simultaneous approximation sequence.*

TABLE 1
Error results of the one-dimensional case given in Example 3.1.

L	$\|\Delta \boldsymbol{V}\|_e$	ε_0	ε_1	ε_2
29	4.8773e-02	2.7689e-02	9.2404e-02	3.2843e-01
70	2.0203e-02	1.1549e-02	3.8271e-02	1.2979e-01
169	8.3682e-03	4.7935e-03	1.5852e-02	5.3200e-02
408	3.4662e-03	1.9877e-03	6.5662e-03	2.1948e-02
985	1.4357e-03	8.2512e-04	2.7198e-03	9.0765e-03
2378	5.9471e-04	3.4217e-04	1.1266e-03	3.7571e-03
5741	2.4634e-04	1.4180e-04	4.6665e-04	1.5558e-03
13860	1.0201e-04	5.8747e-05	1.9329e-04	6.4436e-04

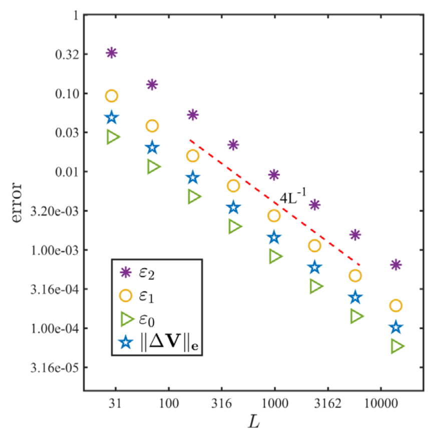


FIG. 3. *In Example* 3.1*, the relationship between errors $\|\Delta \boldsymbol{V}\|_e$, ε_0, ε_1, ε_2, and L.*

Table 1 presents the rational approximation error ε_0 and the corresponding theoretical upper bounds ε_1 and ε_2. The relationship of $\varepsilon_0 < \varepsilon_1 < \varepsilon_2$ is consistent with the discussion in subsection 2.4.

Figure 3 displays the reduction rates of these four errors when L is selected as the best approximation sequence. It is evident that all four errors decrease at the rate of $O(L^{-1})$. Furthermore, we observe that the error ε_0 depends not only on the error $\|\Delta \boldsymbol{V}\|_e$ between Fourier exponents but also on the error $\|\Delta \boldsymbol{y}\|$ between Fourier coefficients.

Example 3.2. We consider two three-dimensional quasiperiodic functions.

Case (i). The Fourier exponents are

$$\boldsymbol{\Lambda}_1 = \begin{pmatrix} 1 & \sqrt{3}/2 \\ 0 & \sqrt{2}/2 \\ \sqrt{3}/2 & 0 \end{pmatrix}.$$

The corresponding Fourier coefficients are $a_1 = 0.2+0.1i$, $a_2 = 0.1+0.2i$. Here, $d_M = 0$.

Case (ii). The Fourier exponents are

$$\boldsymbol{\Lambda}_2 = \begin{pmatrix} 1 & 0 & \sqrt{5}/4 \\ 0 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{3}/2 & \sqrt{3}/2 \end{pmatrix}.$$

The corresponding Fourier coefficients are $a_1 = 0.2 + 0.1i$, $a_2 = 0.1 + 0.2i$, $a_3 = 0.02 - 0.02i$. Here, $d_M = 1$.

For Case (i), the projection matrix is

$$\boldsymbol{P}_1 = \begin{pmatrix} 1 & 0 & 0 & \sqrt{3}/2 \\ 0 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{pmatrix},$$

and $N = 1$. The reciprocal lattice vectors are $\boldsymbol{k}_1 = (1,0,1,0)^T$ and $\boldsymbol{k}_2 = (0,1,0,1)^T$. When Diophantine parameters $C_a = 2$ and $\tau = 0.1$, $\zeta = 0$, $I_{in}(\boldsymbol{\lambda}_j) \neq \emptyset$, $(j = 1,2)$, $d_m = 1$, $\|\boldsymbol{P}_1\|_1 = 1$. Then $L_{min} > 5$ and $G_{min} > 2L_{max} + 3/2$. Here, we choose $G_{min} = 2L_{max} + 10$.

It is evident that the second dimension is associated with $\sqrt{2}$, while the first and third dimensions are related to $\sqrt{3}$. Consequently, Figures 4(a)–(b) show the relationship between $e(L_j)$ and L_j with $j = 1,2$, respectively. Note that $L_1 = L_3$.

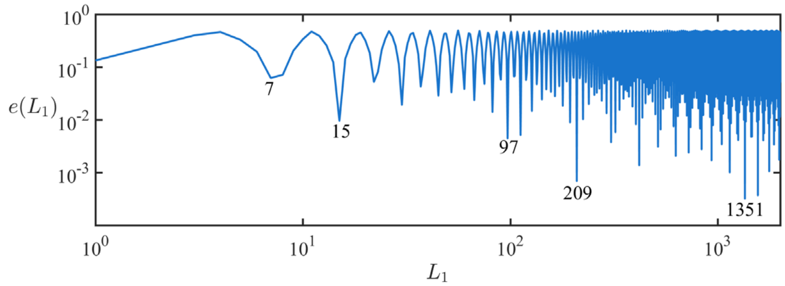
For Case (ii), the projection matrix is

$$\boldsymbol{P}_2 = \begin{pmatrix} 1 & 0 & 0 & \sqrt{5}/4 \\ 0 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{pmatrix}.$$

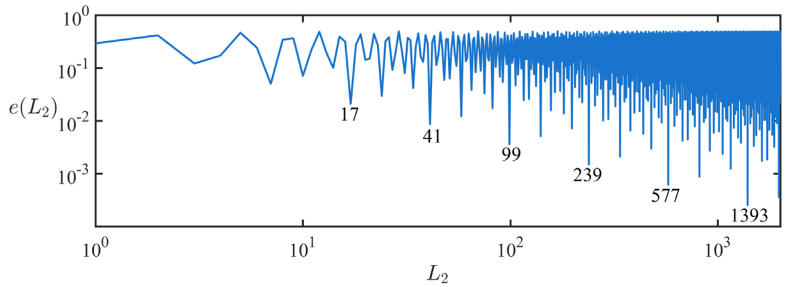
The reciprocal lattice vectors are $\boldsymbol{k}_1 = (1,0,0,0)^T$, $\boldsymbol{k}_2 = (0,1,1,0)^T$, and $\boldsymbol{k}_3 = (0,0,1,1)^T$ where $N = 1$. When Diophantine parameters $C_a = 2$ and $\tau = 0.2$, $\zeta = 1$, $I_{in}(\boldsymbol{\lambda}_j) \neq \emptyset$, $(j = 2,3)$, $d_m = 1$, $\|\boldsymbol{P}_2\|_1 = 1$. Then $L_{min} > 10$ and $G_{min} > 2L_{max} + 3/2$. Here, we choose $G_{min} = 2L_{max} + 10$.

Similarly, we show the relationship between $e(L_j)$ and L_j as well as the best approximation sequence for each dimension. The second and third dimensions are presented in Figure 4. Figure 5 illustrates this relationship for the first dimension in Case (ii).

For convenience, let $G_1 = G_2 = G_3 = G_{min}$ in these two cases. The errors of three-dimensional quasiperiodic functions are presented in Table 2. The table compares three error bounds and clearly demonstrates the consistency of our theoretical findings. Since the degrees of freedom of Case (i) and Case (ii) are given by $D = G_{min}^3$, the computational cost becomes significantly high as the area of $\boldsymbol{L}$ increases. Therefore, we restrict the calculation area to $(0,209] \times (0,239] \times (0,209]$ and $(0,127] \times (0,99] \times (0,209]$ in Table 2, respectively.



(a) The relationship between $e(L_1)$ and L_1.



(b) The relationship between $e(L_2)$ and L_2.

FIG. 4. *In Case* (i) *of Example* 3.2*, when the projection matrix is* $\boldsymbol{P}_1$*, Diophantine approximation error* $e(L_j)$ *changes with increasing* L_j *and* $j=1,2$*. The sequence in the subfigure corresponds to the best simultaneous approximation sequence in the corresponding dimension.*

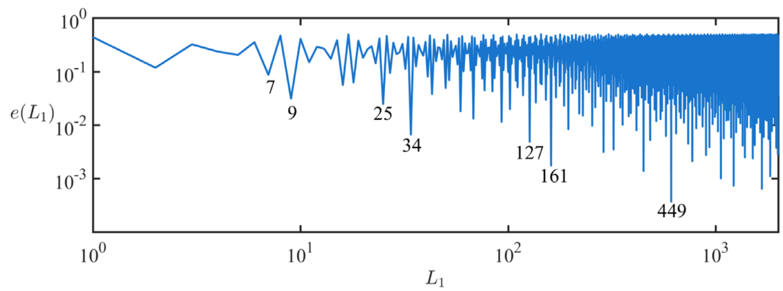


FIG. 5. *In Case* (ii)*, the change of Diophantine approximation error* $e(L_1)$ *with an increase of* L_1*.*

TABLE 2
Error results of two three-dimensional cases given in Example 3.2*.*

Fourier exponents	(L_1, L_2, L_3)	$\|\Delta \boldsymbol{V}\|_e$	ε_0	ε_1	ε_2
	(7,17,7)	1.4517e-01	2.9179e-01	3.0510e-01	7.3318e-01
$\boldsymbol{\Lambda}_1$	(15,41,15)	2.7860e-02	5.7767e-02	5.8709e-02	1.2095e-01
	(97,99,97)	1.2500e-02	2.6158e-02	2.6342e-02	5.3508e-02
	(209,239,209)	2.8605e-03	6.0135e-03	6.0284e-03	1.2147e-02
	(25,41,15)	5.2435e-02	4.6980e-02	1.1050e-01	3.1908e-01
$\boldsymbol{\Lambda}_2$	(34,99,97)	1.9077e-02	1.9775e-02	4.0205e-02	1.0976e-01
	(127,99,209)	9.7943e-03	6.0208e-03	1.9171e-02	5.6053e-02

4. Conclusion. This paper presents a comprehensive theoretical error analysis of approximating an arbitrary-dimensional quasiperiodic function with a periodic function. The approximation error of this problem includes two parts: rational approximation error and truncation error. If the quasiperiodic function exhibits certain regularity, the rational approximation error dominates the approximation error. Meanwhile, we investigate the approximation rates of both the rational approximation error and the best periodic approximation sequence. Finally, we further verify the correctness of the theoretical analysis by several examples.

There are still many problems worth studying, including applying the PAM to solve the quasiperiodic differential equations/operators and providing the corresponding mathematical analysis based on these results presented here. Furthermore, we will develop the new method to analyze the approximation error of quasiperiodic functions with non-Diophantine frequencies by periodic functions.

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