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ANISOTROPIC NONLOCAL DIFFUSION OPERATORS FOR NORMAL AND ANOMALOUS DYNAMICS*

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Abstract. The Laplacian Δ is the infinitesimal generator of isotropic Brownian motion, being the limit process of normal diffusion, while the fractional Laplacian $\Delta^{\beta/2}$ serves as the infinitesimal generator of the limit process of isotropic Lévy process. Taking limit, in some sense, means that the operators can approximate the physical process well after sufficient long time. We introduce the nonlocal operators (being effective from the starting time), which describe the general processes undergoing anisotropic normal diffusion. For anomalous diffusion, we extend to the anisotropic fractional Laplacian $\Delta_m^{\beta/2}$ and the tempered one $\Delta_m^{\beta/2,\lambda}$ in $\mathbb{R}^n$. Their definitions are proved to be equivalent to an alternative one in Fourier space. Based on these new anisotropic diffusion operators, we further derive the deterministic governing equations of some interesting statistical observables of the very general jump processes with multiple internal states. Finally, we consider the associated initial and boundary value problems and prove their well-posedness of the Galerkin weak formulation in $\mathbb{R}^n$. To obtain the coercivity, we claim that the probability density function $\mathbf{Y}$ should be nondegenerate.

Key words. jump processes, nonlocal normal diffusion, anisotropic anomalous diffusion, tempered Lévy flight, multiple internal states, well-posedness

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1. Introduction. Diffusion phenomena are ubiquitous in the natural world, which describe the net movements of the microscopic molecules or atoms from a region of high concentration to a region of low concentration. The speed of diffusion can be characterized by the second moment of the particle trajectories $\langle x^2(t)\rangle \sim t^\alpha$. It is called normal diffusion if $\alpha = 1$ and anomalous diffusion [13, 30, 37] if $\alpha \neq 1$. The scaling limits of all the processes undergoing normal diffusion are Brownian motion. But without the scaling limits, most of the time, they are pure jump processes. For anomalous diffusion, the processes are always characterized by long-range correlation or broad distribution. The former includes fractional Brownian motion [28] and tempered fractional Brownian motion [8, 27], while the latter contains the processes with long-tailed waiting time or jump length. In the framework of continuous-time random walks (CTRWs) [23, 31], any one of the first moments of waiting time and the second moments of jump length diverging leads to the anomalous dynamics. If we extend to the processes with multiple internal states [42], then the diffusion phenomena will depend on the distribution of each internal state, transition matrix, and initial distribution, involving more complex dynamics.

There are many microscopic/stochastic models to describe normal and anomalous diffusions and many different ways of deriving the macroscopic/deterministic equa-

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tions governing the PDFs of some particular statistical observables of the stochastic processes. The commonly used stochastic model is CTRW, which consists of two important random variables, i.e., waiting time and jump length. Two of the important CTRW models undergoing superdiffusion are Lévy flight and Lévy walk. For Lévy flight, the second moment of jump length diverges, which implies the processes propagate with infinite speed. Therefore, the physical realizations of such processes are quite hard and then rare. Lévy walks [44] can remedy the infinite speed by coupling the distribution of waiting time and jump length. This gives rise to a class of space-time coupled processes. Different from Lévy walks, another idea to bound the second moment is to truncate the long-tailed probability distribution of jump length [28, 29], i.e., modify $|\mathbf{X}|^{-n-\beta}$ as $e^{-\lambda|\mathbf{X}|}|\mathbf{X}|^{-n-\beta}$ with λ being a small positive constant, leading to the tempered Lévy flights, which have the advantage of still being an infinitely divisible Lévy process. The Langevin-type equations are built based on Newton's second law with noise as random forces, and the CTRW models also have their corresponding Langevin pictures [19, 39]. Sometimes, it is convenient to use this type of model if the external potentials are considered [7].

Another way to describe anomalous diffusion is Lévy process, which is defined by its characteristic function and more convenient to deal with the stochastic process in high dimensions. According to the Lévy–Khintchine formula [1], the characteristic function of Lévy process has a specific form

$$\mathbf{E}(e^{i\mathbf{k}\cdot\mathbf{X}}) = \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{X}} p(\mathbf{X},t)d\mathbf{X} = e^{t\Phi(\mathbf{k})}, \tag{1}$$

where

$$\Phi(\mathbf{k}) = i\mathbf{k}\cdot\mathbf{b} - \frac{1}{2}\mathbf{k}\cdot\mathbf{a}\mathbf{k} + \int_{\mathbb{R}^n\backslash\{0\}} \left[e^{i\mathbf{k}\cdot\mathbf{Y}} - 1 - i\mathbf{k}\cdot\mathbf{Y}_{\chi_{\{|\mathbf{Y}|<1\}}}\right]\nu(d\mathbf{Y}) \tag{2}$$

with $\mathbf{b}\in\mathbb{R}^n$ and $\mathbf{a}$ is a positive definite symmetric $n\times n$ matrix, χ_I is the indicator function of the set I, and ν is a finite Lévy measure on $\mathbb{R}^n\backslash\{0\}$, implying that $\int_{\mathbb{R}^n\backslash\{0\}}\min\{1,|\mathbf{Y}|^2\}\nu(d\mathbf{Y}) < \infty$. If we take $\mathbf{a}$ and $\mathbf{b}$ to be zero and ν to be a rotationally symmetric (tempered) β-stable Lévy measure

$$\nu(d\mathbf{Y}) = c_{n,\beta}|\mathbf{Y}|^{-n-\beta}d\mathbf{Y} \quad \text{or} \quad \nu(d\mathbf{Y}) = c_{n,\beta,\lambda}e^{-\lambda|\mathbf{Y}|}|\mathbf{Y}|^{-n-\beta}d\mathbf{Y}, \tag{3}$$

then the corresponding PDF of the particles' positions solves

$$\frac{\partial p(\mathbf{X},t)}{\partial t} = \Delta^{\beta/2}\, p(\mathbf{X},t) \quad \text{or} \quad \frac{\partial p(\mathbf{X},t)}{\partial t} = \Delta^{\beta/2,\lambda}\, p(\mathbf{X},t), \tag{4}$$

where the operators $\Delta^{\beta/2}$ and $\Delta^{\beta/2,\lambda}$ are defined in [10, eq. (34)] by the Fourier transform $\hat{g}(\mathbf{k}) := \mathscr{F}[g(\mathbf{X})](\mathbf{k}) = \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{X}} g(\mathbf{X})d\mathbf{X}$ with

$$\begin{aligned}&\mathscr{F}[\Delta^{\beta/2}g(\mathbf{X})] = -|\mathbf{k}|^{\beta}\hat{g}(\mathbf{k}) \text{ and}\\ &\mathscr{F}[\Delta^{\beta/2,\lambda}g(\mathbf{X})] = (-1)^{\lceil\beta\rceil}\Big((\lambda^2+|\mathbf{k}|^2)^{\beta/2} - \lambda^{\beta} + \mathcal{O}(|\mathbf{k}|^2)\Big)\hat{g}(\mathbf{k});\end{aligned} \tag{5}$$

here $\beta\in(0,1)\cup(1,2)$, and $\lceil\beta\rceil$ denotes the smallest integer that is bigger than or equal to β. A similar operator $(\lambda^{1/\beta}+|\mathbf{k}|^2)^{\beta}-\lambda$ appears in [35, eq. (3)], where the only difference is the term $\mathcal{O}(|\mathbf{k}|^2)$. However, their physical background is completely different. The term $\mathcal{O}(|\mathbf{k}|^2)$ in (5) is strictly derived in [10, eq. (34)], where we consider the compound Poisson processes with tempered power law jump lengths, i.e., take the Lévy measure $\nu(d\mathbf{Y})$ to be $e^{-\lambda|\mathbf{Y}|}|\mathbf{Y}|^{-n-\beta}$. But the formula in [35, eq. (3)] is inspired by the Schrödinger operator with the free Hamiltonian of the form $H_0 = (\lambda^2-\Delta)^{1/2}-\lambda$ in [4] and naturally extended to the form $(\lambda^{1/\beta}+|\mathbf{k}|^2)^\beta-\lambda$ with fractional order β.

The two equations in (5) describe the isotropic movements of microscopic particles. However, the anisotropic motions are also very popular, especially in biological systems. The cytoplasm of biological cells is always crowded with various obstacles. These crowders are usually not uniformly distributed and provide the heterogeneous media for tracer particles in them. So we need to develop models to characterize the anisotropic feature. Compte [9] generalized the scheme of CTRWs and showed the diffusion-advection equation and the mean square displacement (MSD) in three kinds of shear flows. Meerschaert, Benson, and Bäumer [26] made an extension to high dimensions and provided an operator being a mixture of directional derivatives taken in each radial direction, where the operator was directly given in Fourier space and the associated fractional advection-dispersion equation was derived. Ervin and Roop [17] discussed directional integral and directional differential operators in two dimensions and defined the appropriate fractional directional derivative spaces. For more details, we refer the interested reader to these literature and the references cited therein.

In this paper, we start from the compound Poisson process to discuss more general anisotropic nonlocal normal diffusion and anomalous diffusion. We present in detail how to derive the macroscopic equations through the Lévy–Khintchine formula for a general anisotropic process. For normal diffusion, the exact macroscopic equations are given without taking a scaling limit. In this way, we find that the anisotropic dynamics significantly result in different PDFs even for normal diffusion. We also discuss the anomalous diffusion undergoing anisotropic movements in $\mathbb{R}^n$ and provide the anisotropic tempered fractional Laplacian operator $\Delta_m^{\beta/2,\lambda}$ (the subscript m means the dependence on directional measure $m(\theta)$ or $m(\mathbf{Y})$, first appearing in (15)). Despite the complexity of $\Delta_m^{\beta/2,\lambda}$, we derive its Fourier symbol, which looks more concise and understandable. Finally, we discuss the space fractional differential equations with the newly defined operator $\Delta_m^{\beta/2,\lambda}$ in $\mathbb{R}^n$, endowed with generalized Dirichlet and Neumann boundary conditions, and prove their well-posedness. We construct a new Hilbert space to include the solutions not vanishing at infinity and propose that the nondegenerate function $m(\mathbf{Y})$ guarantees the coercivity of the variational formulation of the corresponding equations.

All the models mentioned above are for diffusion with single internal state, implying that the processes have the same distributions of waiting time and jump length at each step. Intrigued by applications, e.g., particles moving in multiphase viscous liquid composed of materials with different chemical properties, we further generalize the processes with multiple internal states. In fact, the case of two internal states is considered in [20, 34] with applications, including trapping in amorphous semiconductors, electronic burst noise, movement in systems with fractal boundaries, the digital generation of $1/f$ noise, and ionic currents in cell membranes; Niemann, Barkai, and Kantz [33] investigated in detail a stochastic signal with multiple states, in which each state has an associated joint distribution for the signal's intensity and its holding time. Xu and Deng [42] extended the Fokker–Planck and Feynman–Kac equations [38, 40, 41] to cases with multiple temporal internal states. Here, we further present the fractional Fokker–Planck and Feynman–Kac equations with multiple internal states, both temporally and spatially. "Multiple internal state" implies a kind of inhomogeneous motion. We show how to combine it with an anisotropic directional measure $m(\mathbf{Y})$.

The rest of this paper is organized as follows. In section 2, we show two kinds of processes with Gaussian jumps, leading to different nonlocal macroscopic equations describing normal diffusion. More general anisotropic processes undergoing anomalous diffusions are discussed in section 3, and we also give two kinds of definitions of anisotropic (tempered) fractional Laplacian for two different motivations and prove their equivalences. In section 4, the fractional Fokker–Planck and Feynman–Kac equations of anisotropic (tempered) fractional Laplacian with multiple internal states are derived. The initial and boundary value problems with generalized Dirichlet and Neumann boundary conditions are given in section 5, and their well-posedness is proved in section 6. We conclude the paper with some discussion in the last section.

2. Nonlocal normal diffusion. As we all know, the paths of all Lévy processes are discontinuous except for Brownian motion with drift. From the viewpoint of [15, 10], the macroscopic equations governing the PDFs of these processes should be endowed with the generalized boundary conditions since the boundary $\partial\Omega$ itself cannot be hit by the majority of discontinuous sample trajectories. For nonlocal normal diffusion, it is a pure jump process with Gaussian jumps. Therefore, the boundary conditions should be specified on the domain $\mathbb{R}^n\backslash\Omega$. By the central limit theorem, the scaling limits of all these processes are Brownian motion. But without scaling limit, these processes are different and should be distinguished.

Now we consider the compound Poisson process with Gaussian jump length, in which Poisson process is taken as the renewal process. Let Poisson process $N(t)$ satisfy $P\{N(t)=n\} = \frac{(\zeta t)^n}{n!}e^{-\zeta t}$, where the rate $\zeta > 0$ denotes the mean number of jumps per unit time. Then the compound Poisson process is defined as $\mathbf{X}(t) = \sum_{j=0}^{N(t)} \mathbf{X}_j$, where $\mathbf{X}_j$ are i.i.d. random variables obeying Gaussian distribution. The characteristic function of $\mathbf{X}(t)$ has a specific form as [10, eq. (9)]

$$\mathbf{E}(e^{i\mathbf{k}\cdot\mathbf{X}}) = \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{X}} p(\mathbf{X},t)d\mathbf{X} = e^{\zeta t(\Phi_0(\mathbf{k})-1)}, \tag{6}$$

where $\Phi_0(\mathbf{k}) = \mathbf{E}(e^{i\mathbf{k}\cdot\mathbf{X}_j}), j = 0,1,\ldots,N(t)$. Denoting the probability measure of the jump length $\mathbf{X}_j$ by $\nu(d\mathbf{Y})$, we have

$$\Phi_0(\mathbf{k}) - 1 = \int_{\mathbb{R}^n} (e^{i\mathbf{k}\cdot\mathbf{Y}} - 1)\nu(d\mathbf{Y}), \tag{7}$$

which is the same as the Lévy–Khintchine formula (2) by taking $\mathbf{a} = 0$ and $\mathbf{b}' = 0$ ($\mathbf{b}'$ contains $\mathbf{b}$ and the third term in the integral of (2)). Although the length of $\mathbf{X}_j$ obeys Gaussian distribution, the distribution of the direction of the movement has many different choices. Here, we consider two specific cases in two-dimensional space and derive the corresponding deterministic equations. The first case is that the particles spread uniformly in all directions, while the second one is that the particles move only in horizontal and vertical directions. Considering the definition of the Fourier transform and (6), we have

$$\hat{p}(\mathbf{k},t) = e^{\zeta t(\Phi_0(\mathbf{k})-1)}, \tag{8}$$

which implies that the equation in $\mathbf{k}$ space is

$$\frac{\partial \hat{p}(\mathbf{k},t)}{\partial t} = \zeta(\Phi_0(\mathbf{k})-1)\hat{p}(\mathbf{k},t). \tag{9}$$

Next, we give the specific expressions of $\Phi_0(\mathbf{k})$ (or $\nu(d\mathbf{Y})$) for these two cases.

Case 1. Since the particles spread uniformly in all directions, $\nu(d\mathbf{Y})$ is taken as

$$\nu(d\mathbf{Y}) = \frac{1}{2\pi\sigma^2} e^{-\frac{|\mathbf{Y}|^2}{2\sigma^2}} d\mathbf{Y},$$

where σ^2 is the variance. Then we obtain

$$\Phi_0(\mathbf{k}) - 1 = e^{-\frac{1}{2}\sigma^2|\mathbf{k}|^2} - 1, \tag{10}$$

which implies

$$\frac{\partial p(\mathbf{X},t)}{\partial t} = -\frac{\zeta}{2\pi\sigma^2}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{X}-\mathbf{Y}|^2}{2\sigma^2}}(p(\mathbf{X},t) - p(\mathbf{Y},t))d\mathbf{Y} \tag{11}$$

by taking the inverse Fourier transform

$$\begin{aligned}\mathscr{F}^{-1}[(\Phi_0(\mathbf{k}) - 1)\hat{p}(\mathbf{k},t)] &= \mathscr{F}^{-1}[\Phi_0(\mathbf{k})\hat{p}(\mathbf{k},t)] - \mathscr{F}^{-1}[\hat{p}(\mathbf{k},t)] \\ &= \frac{1}{2\pi\sigma^2}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{X}-\mathbf{Y}|^2}{2\sigma^2}} p(\mathbf{Y},t)d\mathbf{Y} - p(\mathbf{X},t) \\ &= -\frac{1}{2\pi\sigma^2}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{X}-\mathbf{Y}|^2}{2\sigma^2}}(p(\mathbf{X},t) - p(\mathbf{Y},t))d\mathbf{Y}.\end{aligned}$$

Case 2. Since the particles spread in either horizontal or vertical direction, we take $\nu(d\mathbf{Y})$ to be

$$\nu(d\mathbf{Y}) = \frac{1}{2(2\pi\sigma^2)^{\frac{1}{2}}} e^{-\frac{|y_1|^2}{2\sigma^2}}\delta(y_2)d\mathbf{Y} + \frac{1}{2(2\pi\sigma^2)^{\frac{1}{2}}} e^{-\frac{|y_2|^2}{2\sigma^2}}\delta(y_1)d\mathbf{Y}.$$

Similar to Case 1, we obtain

$$\Phi_0(\mathbf{k}) - 1 = \frac{1}{2}e^{-\frac{1}{2}\sigma^2|k_1|^2} + \frac{1}{2}e^{-\frac{1}{2}\sigma^2|k_2|^2} - 1 \tag{12}$$

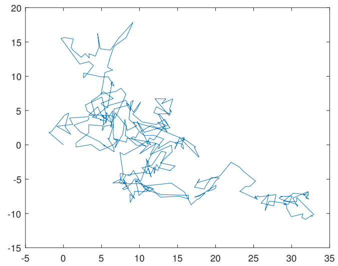
and the equation

$$\begin{aligned}\frac{\partial p(\mathbf{X},t)}{\partial t} = -\frac{\zeta}{2(2\pi\sigma^2)^{\frac{1}{2}}}\Big(&\int_{\mathbb{R}} e^{-\frac{|x_1-y_1|^2}{2\sigma^2}}(p(x_1,x_2,t) - p(y_1,x_2,t))dy_1 \\ &+ \int_{\mathbb{R}} e^{-\frac{|x_2-y_2|^2}{2\sigma^2}}(p(x_1,x_2,t) - p(x_1,y_2,t))dy_2\Big).\end{aligned} \tag{13}$$

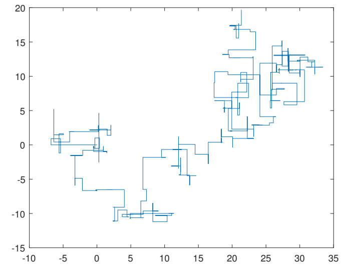
From (11) and (13), it can be noted that different ways of movement of microscopic particles lead to different macroscopic equations. These macroscopic equations are both nonlocal and should be endowed with the generalized boundary conditions. But if we take the scaling limits of the Gaussian jump processes, the two cases above both converge to Brownian motion. In fact, let $1/\zeta, \sigma^2 \to 0$, and keep the product $\zeta\sigma^2/2 \to K_1$ being a constant, where K_1 is the diffusion coefficient with unit [cm^2]/[s] [3]. Then, both (10) and (12) become, up to a multiplier, $\Phi_0(\mathbf{k}) - 1 = -\frac{1}{2}\sigma^2|\mathbf{k}|^2$, resulting in the classical heat equation

$$\frac{\partial p(\mathbf{X},t)}{\partial t} = K_1\Delta p(\mathbf{X},t), \tag{14}$$

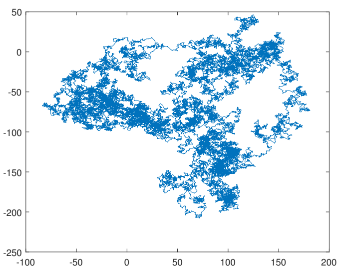
where Δ is the usual Laplacian in $\mathbb{R}^2$. To illustrate the relationship between Case 1 and Case 2, we simulate the trajectories of the particles undergoing Gaussian jumps in Fig-



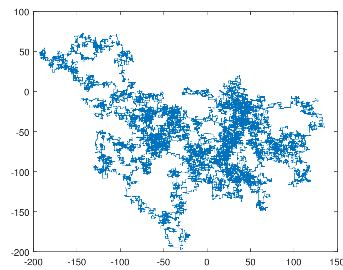
(a): Case 1 (400 steps)



(b): Case 2 (400 steps)



(c): Case 1 (40000 steps)



(d): Case 2 (40000 steps)

FIG. 1. *Random trajectories of Gaussian jumps in Case* 1 *and Case* 2 *with* 400 *steps in the top row and* 40000 *steps in the bottom row.*

ure 1. Two pictures in the top row are for the 400 jumps performed uniformly (a) and just in horizontal-vertical direction (b), while another two pictures in the bottom row display 40000 jumps, respectively. The differences between Case 1 and Case 2 are apparent for a relatively small number of jumps. But after many thousands of jumps, the differences gradually disappear, as both processes converge to the Brownian motion.

Besides the two cases above, more generally, the particles can move in a variety of different ways, depending on the environment. There may be more particles spreading in one direction or some particles spreading faster in another direction. This phenomenon is called anisotropic diffusion and can be characterized clearly by the Lévy measure $\nu(d\mathbf{Y})$. More precisely, still in two-dimensional space, by polar coordinate transformation, take $\nu(d\mathbf{Y})$ to be

$$\nu(d\mathbf{Y}) = c_m \exp\left[-\frac{r^2}{2\sigma_\theta^2}\right] m(\theta) r dr d\theta, \tag{15}$$

where $c_m > 0$ is the normalized parameter, $r \geq 0$, $\theta \in [0, 2\pi)$ denotes the different directions, $m(\theta)$ denotes the probability distribution of particles spreading in θ-direction satisfying $m(\theta) \geq 0$, $\int_0^{2\pi} m(\theta) d\theta = 1$, and σ_θ denotes the different variance or speed of particles spreading in θ-direction. Different from (3), this $\nu(d\mathbf{Y})$ contains a new PDF $m(\theta)$, which only depends on direction. Turning back to the Cartesian coordinate system and following (7), we have

$$\Phi_0(\mathbf{k}) - 1 = c_m \int_{\mathbb{R}^2} (e^{i\mathbf{k}\cdot\mathbf{Y}} - 1) \exp\left[-\frac{|\mathbf{Y}|^2}{2\sigma_{\mathbf{Y}}^2}\right] m(\mathbf{Y}) d\mathbf{Y},$$

where the PDF $m(\theta)$ is abused by $m(\mathbf{Y})$ and $\mathbf{Y} \in \mathbb{R}^n \backslash \{0\}$ is in the Cartesian coordinate system, while it really means $m(\frac{\mathbf{Y}}{|\mathbf{Y}|})$, only depending on the radial direction of $\mathbf{Y}$. The notation $m(\mathbf{Y})$ will be used in the subsequent sections. Then similar to (11) and (13), we can derive the equation

$$\frac{\partial p(\mathbf{X},t)}{\partial t} = -\zeta c_m \int_{\mathbb{R}^2} (p(\mathbf{X},t) - p(\mathbf{Y},t)) \exp\left[-\frac{|\mathbf{X}-\mathbf{Y}|^2}{2\sigma^2_{\mathbf{X}-\mathbf{Y}}}\right] m(\mathbf{X}-\mathbf{Y}) d\mathbf{Y}. \tag{16}$$

If we take $\sigma_\theta = \sigma$, $m(\theta) \equiv (2\pi)^{-1}$, or $m(\theta) = \frac{1}{4}(\delta(\theta) + \delta(\theta - \frac{\pi}{2}) + \delta(\theta - \pi) + \delta(\theta - \frac{3\pi}{2}))$ in (15), then (16) reduces to (11) and (13), respectively.

The discussions above, including the case of (15), are aiming to describe anisotropic jump processes. All the macroscopic equations are nonlocal and should be endowed with generalized boundary conditions [12].

3. Anisotropic anomalous diffusion. Now we discuss the anomalous diffusion with the property of anisotropy. Still based on the compound Poisson processes in the previous section but with the diffusion processes being anisotropic (tempered) β-stable, we try to derive their corresponding deterministic equations undergoing anomalous diffusion. Taking $\zeta = 1$ in (9) leads to

$$\frac{\partial \hat{p}(\mathbf{k},t)}{\partial t} = (\Phi_0(\mathbf{k}) - 1)\hat{p}(\mathbf{k},t), \tag{17}$$

where

$$\Phi_0(\mathbf{k}) - 1 = \int_{\mathbb{R}^n\setminus\{0\}} \left[e^{i\mathbf{k}\cdot\mathbf{Y}} - 1 - i\mathbf{k}\cdot\mathbf{Y}\chi_{\{|\mathbf{Y}|<1\}}\right]\nu(d\mathbf{Y}). \tag{18}$$

Here, different from (7), we add a term $i\mathbf{k}\cdot\mathbf{Y}$ to overcome the possible divergence of the integral of (18) because of the possible strong singularity of $\nu(d\mathbf{Y})$ at zero for the case of anomalous diffusion. For an isotropic β-stable anomalous diffusion process in n-dimensional space, its distribution of jump length is $c_\beta r^{-n-\beta}$, which means that

$$\nu(d\mathbf{Y}) = c_\beta |\mathbf{Y}|^{-n-\beta} d\mathbf{Y}. \tag{19}$$

When $0 < \beta < 1$, the term $i\mathbf{k}\cdot\mathbf{Y}$ can be omitted due to weak singularity (the integral in (18) is convergent at origin). If $1 \le \beta < 2$, though the singularity is strong, this term can also be omitted due to the possible symmetry of the Lévy measure $\nu(d\mathbf{Y})$, i.e., $\nu(d\mathbf{Y}) = \nu(-d\mathbf{Y})$ (the integral in (18) at origin can be understood in the sense of Cauchy principal value). Therefore, if $1 \le \beta < 2$ meets with the asymmetry of $\nu(d\mathbf{Y})$, this term is required. Based on the analyses above, we will keep the term $i\mathbf{k}\cdot\mathbf{Y}$ formally for $0 < \beta < 2$ in the following, though it vanishes in some appropriate situations.

Two special cases have been considered in [10], i.e., the isotropic one (19) and the horizontal-vertical one,

$$\begin{aligned} \nu(d\mathbf{Y}) = {} & c_{\beta_1}|y_1|^{-1-\beta_1}\delta(y_2)\delta(y_3)\cdots\delta(y_n)d\mathbf{Y} \\ & + c_{\beta_2}|y_2|^{-1-\beta_2}\delta(y_1)\delta(y_3)\cdots\delta(y_n)d\mathbf{Y} + \cdots \\ & + c_{\beta_n}|y_n|^{-1-\beta_n}\delta(y_1)\delta(y_2)\cdots\delta(y_{n-1})d\mathbf{Y}, \end{aligned} \tag{20}$$

where $\beta_i \in (0,2)$ and y_i is the component of $\mathbf{Y}$, i.e., $\mathbf{Y} = [y_1, y_2, \ldots, y_n]^T$. Their corresponding macroscopic equations are

$$\begin{aligned} \frac{\partial p(\mathbf{X},t)}{\partial t} &= \Delta^{\beta/2} p(\mathbf{X},t) \\ &= -c_{n,\beta}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{p(\mathbf{X},t) - p(\mathbf{Y},t)}{|\mathbf{X}-\mathbf{Y}|^{n+\beta}} d\mathbf{Y} \end{aligned} \tag{21}$$

and

$$\frac{\partial p(\mathbf{X},t)}{\partial t} = (\Delta_{x_1}^{\beta_1/2} + \Delta_{x_2}^{\beta_2/2} + \cdots + \Delta_{x_n}^{\beta_n/2}) p(\mathbf{X},t), \tag{22}$$

respectively, where $\Delta_{x_i}^{\beta_i/2}$ is the fractional Laplacian in $\mathbb{R}^1$ w.r.t. x_i. Besides the two cases, there are also a large number of irregular motions the microscopic particles perform. In general, we call it anisotropy. With the aid of the Lévy–Khintchine formula (2), the concrete form of $\nu(d\mathbf{Y})$ can be given.

Following (17) and (18), with the inverse Fourier transform, we have

$$\frac{\partial p(\mathbf{X},t)}{\partial t} = \int_{\mathbb{R}^n\setminus\{0\}} [p(\mathbf{X}-\mathbf{Y}) - p(\mathbf{X}) + (\mathbf{Y}\cdot\nabla_{\mathbf{X}} p(\mathbf{X}))\chi_{[|\mathbf{Y}|<1]}]\nu(d\mathbf{Y}), \tag{23}$$

where $\nabla_{\mathbf{X}} = [\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}]^T$. Taking

$$\nu(d\mathbf{Y}) = \frac{1}{|\Gamma(-\beta)|}\frac{m(\mathbf{Y})}{|\mathbf{Y}|^{n+\beta}} d\mathbf{Y} \tag{24}$$

gives

$$\begin{aligned}\frac{\partial p(\mathbf{X},t)}{\partial t} = \frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\setminus\{0\}} &\left[p(\mathbf{X}-\mathbf{Y}) - p(\mathbf{X}) + (\mathbf{Y}\cdot\nabla_{\mathbf{X}} p(\mathbf{X}))\chi_{[|\mathbf{Y}|<1]}\right] \\ &\times \frac{m(\mathbf{Y})}{|\mathbf{Y}|^{n+\beta}} d\mathbf{Y}.\end{aligned} \tag{25}$$

We can make the meaning of $m(\mathbf{Y})$ clear by transforming this equation into a polar coordinate system. In the two- and three-dimensional cases, (25) becomes, respectively,

$$\begin{aligned}\frac{\partial p(\mathbf{X},t)}{\partial t} = \frac{1}{|\Gamma(-\beta)|}\int_0^\infty \int_0^{2\pi} \Bigg[& p(x_1 - r\cos(\theta), x_2 - r\sin(\theta)) - p(x_1,x_2) \\ & + \left(r\cos(\theta)\frac{\partial p}{\partial x_1} + r\sin(\theta)\frac{\partial p}{\partial x_2}\right)_{\chi_{[r<1]}}\Bigg] \frac{m(\theta)}{r^{1+\beta}} d\theta dr\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial p(\mathbf{X},t)}{\partial t} \\ &= \frac{1}{|\Gamma(-\beta)|}\int_0^\infty\int_0^\pi\int_0^{2\pi}\Bigg[p(x_1 - r\sin(\theta)\cos(\phi), x_2 - r\sin(\theta)\sin(\phi), x_3 - r\cos(\theta)) \\ &\quad - p(x_1,x_2,x_3) + \left(r\sin(\theta)\cos(\phi)\frac{\partial p}{\partial x_1} + r\sin(\theta)\sin(\phi)\frac{\partial p}{\partial x_2} + r\cos(\theta)\frac{\partial p}{\partial x_3}\right)_{\chi_{[r<1]}}\Bigg] \\ &\quad \times \frac{m(\theta,\phi)\sin\theta}{r^{1+\beta}} d\phi d\theta dr,\end{aligned}$$

where the directional measure $m(\theta)$ or $m(\theta,\phi)$ specifies the distribution of particles spreading in the radial direction of $\mathbf{Y}$; among them, $m(\theta)$ is defined on $[0,2\pi]$, satisfying $\int_0^{2\pi} m(\theta)d\theta = 1$, while $m(\theta,\phi)$ is defined on a $[0,\pi]\times[0,2\pi]$ rectangular domain, satisfying $\int_0^\pi\int_0^{2\pi} m(\theta,\phi)d\phi d\theta = 1$. The situation becomes much more simple if the particles move in one dimension. It is like the biased CTRW model with asymmetric probability of jumping left or right [2, 23].

For the tempered Lévy flight, we can describe the movement of microscopic particles and derive the macroscopic equations by defining

$$\nu(d\mathbf{Y}) = \frac{1}{|\Gamma(-\beta)|}\,\frac{m(\mathbf{Y})}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}}d\mathbf{Y}, \tag{26}$$

and (23) becomes

$$\begin{aligned}\frac{\partial p(\mathbf{X},t)}{\partial t} &= \frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\setminus\{0\}}\left[p(\mathbf{X}-\mathbf{Y}) - p(\mathbf{X}) + (\mathbf{Y}\cdot\nabla_{\mathbf{X}}p(\mathbf{X}))_{\chi_{[|\mathbf{Y}|<1]}}\right] \\ &\quad\times\frac{m(\mathbf{Y})}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}}d\mathbf{Y}.\end{aligned} \tag{27}$$

We write (25) and (27), respectively, as

$$\frac{\partial p(\mathbf{X},t)}{\partial t} = \Delta_m^{\beta/2}p(\mathbf{X},t) \quad \text{and} \quad \frac{\partial p(\mathbf{X},t)}{\partial t} = \Delta_m^{\beta/2,\lambda}p(\mathbf{X},t), \tag{28}$$

where the notation $\Delta_m^{\beta/2}$ ($\Delta_m^{\beta/2,\lambda}$) denotes the anisotropic (tempered) fractional Laplacian in $\mathbb{R}^n$ and their definitions are the right-hand sides of (25) and (27).

Different from (25) and (27), an alternative definition of the anisotropic (tempered) fractional Laplacians can be given in Fourier space:

$$\mathscr{F}[\Delta_m^{\beta/2}p(\mathbf{X},t)] = (-1)^{\lceil\beta\rceil}\left[\int_{|\boldsymbol{\phi}|=1}(-i\mathbf{k}\cdot\boldsymbol{\phi})^\beta m(\boldsymbol{\phi})d\boldsymbol{\phi}\right]\hat{p}(\mathbf{k},t) \tag{29}$$

and

$$\mathscr{F}[\Delta_m^{\beta/2,\lambda}p(\mathbf{X},t)] = (-1)^{\lceil\beta\rceil}\left[\int_{|\boldsymbol{\phi}|=1}\left((\lambda - i\mathbf{k}\cdot\boldsymbol{\phi})^\beta - \lambda^\beta\right)m(\boldsymbol{\phi})d\boldsymbol{\phi}\right]\hat{p}(\mathbf{k},t). \tag{30}$$

The former one has been given in [26, eq. (2)]. It seems that these definitions are natural for the study of the governing equations since the symbol $(-i\mathbf{k}\cdot\boldsymbol{\phi})^\beta$ for $\beta\in(0,1)\cup(1,2)$ denotes a β-order fractional directional derivative. Now we consider the question of when the two ways of defining the operators are equivalent. To establish the relationship between them, we focus on two cases.

- **Case I.** $0<\beta<1$ or m is symmetric. Recall that here that the third term in (25) and (27) can be deleted:

$$\Delta_m^{\beta/2}p(\mathbf{X},t) = \frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\setminus\{0\}}\left[p(\mathbf{X}-\mathbf{Y}) - p(\mathbf{X})\right]\frac{m(\mathbf{Y})}{|\mathbf{Y}|^{n+\beta}}d\mathbf{Y}, \tag{31}$$

$$\Delta_m^{\beta/2,\lambda}p(\mathbf{X},t) = \frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\setminus\{0\}}\left[p(\mathbf{X}-\mathbf{Y}) - p(\mathbf{X})\right]\frac{m(\mathbf{Y})}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}}d\mathbf{Y}. \tag{32}$$

- **Case II.** $1<\beta<2$ and m is asymmetric. Recall that the integrals in (25) and (27) without the third terms can be understood in the Hadamard sense [36, eq. (5.65)], i.e.,

$$\begin{aligned}\Delta_m^{\beta/2}p(\mathbf{X},t) &= \text{p.f.}\ \frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\setminus\{0\}}\left[p(\mathbf{X}-\mathbf{Y}) - p(\mathbf{X})\right]\frac{m(\mathbf{Y})}{|\mathbf{Y}|^{n+\beta}}d\mathbf{Y} \\ &= \frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\setminus\{0\}}\left[p(\mathbf{X}-\mathbf{Y}) - p(\mathbf{X}) + (\mathbf{Y}\cdot\nabla_{\mathbf{X}}p(\mathbf{X}))\right] \\ &\quad\times\frac{m(\mathbf{Y})}{|\mathbf{Y}|^{n+\beta}}d\mathbf{Y},\end{aligned} \tag{33}$$

$$
\begin{aligned}
\Delta_m^{\beta/2,\lambda} p(\mathbf{X},t) &= \text{p.f.}\ \frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\backslash\{0\}} [p(\mathbf{X}-\mathbf{Y})-p(\mathbf{X})]\,\frac{m(\mathbf{Y})}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}}d\mathbf{Y} \\
&= \frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\backslash\{0\}} [p(\mathbf{X}-\mathbf{Y})-p(\mathbf{X})+(\mathbf{Y}\cdot\nabla_{\mathbf{X}}p(\mathbf{X}))] \\
&\quad \times \frac{m(\mathbf{Y})}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}}d\mathbf{Y} - \frac{1}{|\Gamma(-\beta)|}\Gamma(1-\beta)\lambda^{\beta-1}(\mathbf{b}\cdot\nabla_{\mathbf{X}}p(\mathbf{X})),
\end{aligned} \tag{34}
$$

where $\mathbf{b} = \int_{|\boldsymbol{\phi}|=1} \boldsymbol{\phi}\, m(\boldsymbol{\phi}) d\boldsymbol{\phi}$.

In Case II, since the high singularity makes the integral divergent, we use the notation p.f. to denote its finite part in the Hadamard sense.

Then we have the following theorem; see Appendix A for the proof, which further implies the equality (34).

Theorem 1. *Let $m(\mathbf{Y})$ be any directional measure on unit sphere and $\lambda \geq 0$. The definitions of the anisotropic (tempered) fractional Laplacians $\Delta_m^{\beta/2,\lambda}$ in both Case* I *and Case* II *are, respectively, equivalent to $\Delta_m^{\beta/2,\lambda}$ in* (29) *and* (30) *in $\mathbb{R}^n$.*

We have just defined the anisotropic (tempered) fractional Laplacian by extending the Lévy measure $\nu(d\mathbf{Y})$ with different probability distribution in different directions. More generally, another two variables, jump length exponent β and truncation exponent λ, can also be generalized to be anisotropic, i.e., $\beta(\boldsymbol{\phi})$ and $\lambda(\boldsymbol{\phi})$, sometimes abused by $\beta(\mathbf{Y})$ and $\lambda(\mathbf{Y})$ similar to $m(\mathbf{Y})$. Let $\beta(\boldsymbol{\phi}) \in (0,1)\cup(1,2)$ and $\lambda(\boldsymbol{\phi}) \geq 0$. Following (29), (30), (32), and (34), the definitions of new anisotropic (tempered) fractional Laplacian are, respectively, the following:

- Case I: $0 < \beta < 1$ or m is symmetric,

$$
\begin{aligned}
\tilde{\Delta}_m^{\beta/2,\lambda} p(\mathbf{X},t) &= \int_{\mathbb{R}^n\backslash\{0\}} [p(\mathbf{X}-\mathbf{Y})-p(\mathbf{X})] \\
&\quad \times \frac{m(\mathbf{Y})}{|\Gamma(-\beta(\mathbf{Y}))|e^{\lambda(\mathbf{Y})|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta(\mathbf{Y})}}d\mathbf{Y}.
\end{aligned} \tag{35}
$$

- Case II: $1 < \beta < 2$ and m is asymmetric,

$$
\begin{aligned}
\tilde{\Delta}_m^{\beta/2,\lambda} p(\mathbf{X},t) &= \text{p.f.} \int_{\mathbb{R}^n\backslash\{0\}} [p(\mathbf{X}-\mathbf{Y})-p(\mathbf{X})] \\
&\quad \times \frac{m(\mathbf{Y})}{|\Gamma(-\beta(\mathbf{Y}))|e^{\lambda(\mathbf{Y})|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta(\mathbf{Y})}}d\mathbf{Y} \\
&= \int_{\mathbb{R}^n\backslash\{0\}} [p(\mathbf{X}-\mathbf{Y})-p(\mathbf{X})+(\mathbf{Y}\cdot\nabla_{\mathbf{X}}p(\mathbf{X}))] \\
&\quad \times \frac{m(\mathbf{Y})}{|\Gamma(-\beta(\mathbf{Y}))|e^{\lambda(\mathbf{Y})|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta(\mathbf{Y})}}d\mathbf{Y} - (\mathbf{b}\cdot\nabla_{\mathbf{X}}p(\mathbf{X})),
\end{aligned} \tag{36}
$$

where $\mathbf{b} = \int_{|\boldsymbol{\phi}|=1} \Gamma(1-\beta(\boldsymbol{\phi}))\,\lambda(\boldsymbol{\phi})^{\beta(\boldsymbol{\phi})-1}\,\boldsymbol{\phi}\, m(\boldsymbol{\phi})/|\Gamma(-\beta(\boldsymbol{\phi}))| d\boldsymbol{\phi}$.

In Fourier space, the new operator has the form

$$
\mathscr{F}[\tilde{\Delta}_m^{\beta/2,\lambda} p(\mathbf{X},t)] = (-1)^{\lceil\beta\rceil} \int_{|\boldsymbol{\phi}|=1} \left((\lambda(\boldsymbol{\phi})-i\mathbf{k}\cdot\boldsymbol{\phi})^{\beta(\boldsymbol{\phi})} - \lambda(\boldsymbol{\phi})^{\beta(\boldsymbol{\phi})}\right) m(\boldsymbol{\phi}) d\boldsymbol{\phi}\, \hat{p}(\mathbf{k},t). \tag{37}
$$

We simulate the trajectories of the particles with the anisotropic movements. Figure 2 shows the random trajectories of tempered Lévy flights and the corresponding MSDs. All particles start the movements from origin. Compared with the isotropic

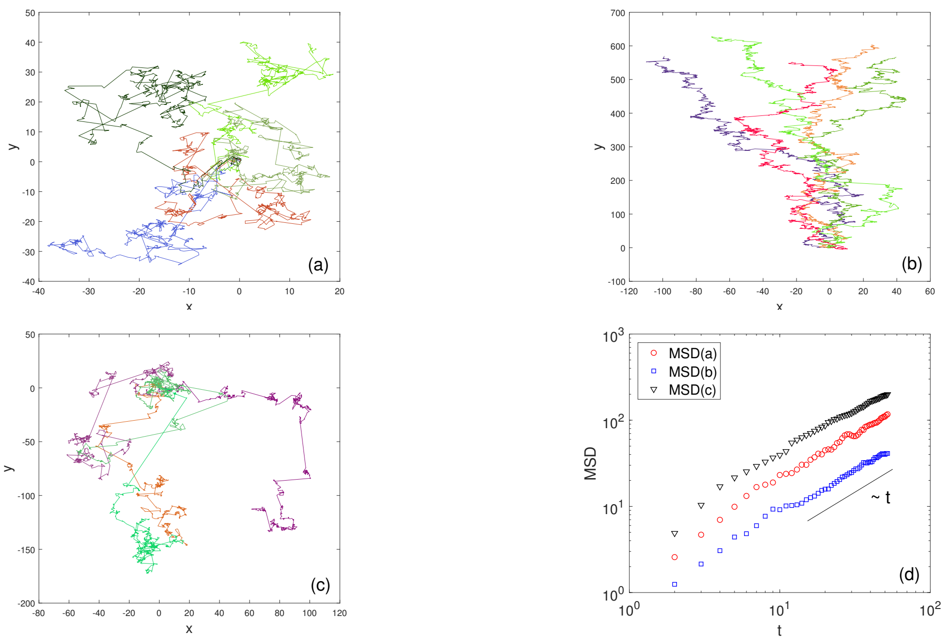


FIG. 2. *Several random trajectories and MSDs of tempered Lévy flights. The isotropic movements with* $\beta = 1.3$ *and* $\lambda = 0.1$ *are shown in* (a). *To describe the movements with anisotropic* $m(\boldsymbol{\phi})$, *we choose* $m(\boldsymbol{\phi}) = 3/(4\pi)$ *for* $\arg(\boldsymbol{\phi}) \in (0, \pi)$ *and* $m(\boldsymbol{\phi}) = 1/(4\pi)$ *for* $\arg(\boldsymbol{\phi}) \in (\pi, 2\pi)$ *in* (b), *while for the case with anisotropic truncation parameter* $\lambda(\boldsymbol{\phi})$, *we choose* $\lambda(\boldsymbol{\phi}) = 0.1$ *for* $\arg(\boldsymbol{\phi}) \in (0, \pi)$ *and* $\lambda(\boldsymbol{\phi}) = 0.01$ *for* $\arg(\boldsymbol{\phi}) \in (\pi, 2\pi)$ *in* (c). *The MSDs of the three kinds of tempered Lévy flights are shown in* (d).

movements in (a), the particles are more inclined to move upward in (b) due to the anisotropic $m(\boldsymbol{\phi})$. Some large downward jump lengths are found in (c) since the truncation parameter $\lambda = 0.01$ is smaller for $\arg(\boldsymbol{\phi}) \in (\pi, 2\pi)$. The MSDs of these three cases are presented in (d), both exhibiting normal diffusion. For the case with anisotropic $m(\boldsymbol{\phi})$, the bias resulting from the asymmetric probability of jumps suppresses the diffusion behavior, while for the anisotropic $\lambda(\boldsymbol{\phi})$, the weak tempering $\lambda = 0.01$ enhances the fluctuation and thus results in a large diffusivity. Besides the MSD, there are also many other interesting statistical quantities, such as first-passage time and escape probability [5, 11, 24]. With the accurate characterization of the anisotropic normal or anomalous diffusion processes in n dimensions, these quantities can be further considered in the near future. Some interesting phenomena should be observed due to the anisotropy.

Remark 3.1. In the practical problem, the directional measure may depend on the concentration gradient. To emphasize the effects caused by the directional gradient, the definition of the anisotropic (tempered) fractional Laplacian in (35) can be extended to

$$\tilde{\Delta}_m^{\beta/2,\lambda} p(\mathbf{X}, t) = (-1)^{\lceil \beta \rceil} \int_{\mathbb{R}^n \setminus \{0\}} [p(\mathbf{X} - \mathbf{Y}) - p(\mathbf{X})] \times \frac{m\left(\mathbf{Y}, \frac{\partial p(\mathbf{Y})}{\partial \mathbf{Y}}\right)}{|\Gamma(-\beta(\mathbf{Y}))| e^{\lambda(\mathbf{Y})|\mathbf{Y}|} |\mathbf{Y}|^{n+\beta(\mathbf{Y})}} d\mathbf{Y}, \tag{38}$$

where m should be an increasing function of directional gradient $\frac{\partial p(\mathbf{Y})}{\partial \mathbf{Y}}$.

As a complement to the definition of the anisotropic (tempered) fractional Laplacian (29) and (30), we also present the definition of the operator in the case that $\beta = 1$; i.e., let $\nu(d\mathbf{Y}) = \frac{m(\mathbf{Y})}{|\mathbf{Y}|^{n+1}} d\mathbf{Y}$, which still is a nonlocal operator. For the sake of simplicity, we assume that $m(\mathbf{Y})$ is symmetric; then the term $(\mathbf{Y} \cdot \nabla_{\mathbf{X}} p(\mathbf{X}))_{\chi_{[|\mathbf{Y}|<1]}}$ in (23) can be omitted. For the one-dimensional asymmetric operators with $\beta = 1$, see [22] for details.

PROPOSITION 2. *Let $\beta = 1$ and $\lambda > 0$. If the directional measure $m(\mathbf{Y})$ is symmetric, then the Fourier symbols of the anisotropic fractional Laplacian and the corresponding tempered one are, respectively,*

$$\mathscr{F}[\Delta_m^{1/2} p(\mathbf{X}, t)] = \frac{\pi}{2} \int_{|\boldsymbol{\phi}|=1} |(\mathbf{k} \cdot \boldsymbol{\phi})| \; m(\boldsymbol{\phi}) d\boldsymbol{\phi} \cdot \hat{p}(\mathbf{k}, t) \tag{39}$$

and

$$\begin{aligned}\mathscr{F}[\Delta_m^{1/2,\lambda} p(\mathbf{X}, t)] = \int_{|\boldsymbol{\phi}|=1} \Big[&(\mathbf{k} \cdot \boldsymbol{\phi}) \arctan\left(\frac{\mathbf{k} \cdot \boldsymbol{\phi}}{\lambda}\right) - \frac{\lambda}{2} \ln(\lambda^2 + (\mathbf{k} \cdot \boldsymbol{\phi})^2) \\ &+ \lambda \ln \lambda \Big] m(\boldsymbol{\phi}) d\boldsymbol{\phi} \cdot \hat{p}(\mathbf{k}, t).\end{aligned} \tag{40}$$

Proof. We first prove the tempered case. Taking the Fourier transform of the right-hand side of (27), we have

$$\begin{aligned}\mathscr{F}\left[\Delta_m^{1/2,\lambda} p(\mathbf{X}, t)\right](\mathbf{k}) &= \int_{\mathbb{R}^n} \frac{e^{i\mathbf{k}\cdot\mathbf{Y}} - 1}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+1}} m(\mathbf{Y}) d\mathbf{Y} \cdot \hat{p}(\mathbf{k}, t) \\ &= \left[\int_{\mathbb{R}^n} \frac{\cos(\mathbf{k}\cdot\mathbf{Y}) - 1}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+1}} m(\mathbf{Y}) d\mathbf{Y}\right] \cdot \hat{p}(\mathbf{k}, t),\end{aligned}$$

where the term $i \sin(\mathbf{k} \cdot \mathbf{Y})$ vanishes due to the symmetry of $m(\mathbf{Y})$. By the polar coordinate transformation, we have

$$\int_{\mathbb{R}^n} \frac{1 - \cos(\mathbf{k}\cdot\mathbf{Y})}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+1}} m(\mathbf{Y}) d\mathbf{Y} = \int_0^\infty \int_{|\boldsymbol{\phi}|=1} r^{-2} e^{-\lambda r} (1 - \cos(r\mathbf{k} \cdot \boldsymbol{\phi})) m(\boldsymbol{\phi}) d\boldsymbol{\phi} dr.$$

Denote $I(r) = \int_{|\boldsymbol{\phi}|=1} (1 - \cos(r\mathbf{k} \cdot \boldsymbol{\phi})) m(\boldsymbol{\phi}) d\boldsymbol{\phi}$ for simplicity. After performing integration by parts twice, the equation above becomes

$$\begin{aligned}\int_0^\infty r^{-2} e^{-\lambda r} I(r) dr &= \int_0^\infty r^{-1} e^{-\lambda r} (I'(r) - \lambda I(r)) dr \\ &= -\int_0^\infty \ln(r) e^{-\lambda r} (I''(r) - 2\lambda I'(r) + \lambda^2 I(r)) dr,\end{aligned} \tag{41}$$

where the boundary terms vanish due to $I(0) = I'(0) = 0$. Then (40) can be directly obtained by using the formulas [21]

$$\begin{aligned}\int_0^\infty e^{-qx} \sin(px) \ln(x) dx &= \frac{1}{p^2 + q^2} \left[q \arctan\frac{p}{q} - \frac{p}{2} \ln(p^2 + q^2) - p\mathcal{C}\right], \\ \int_0^\infty e^{-qx} \cos(px) \ln(x) dx &= -\frac{1}{p^2 + q^2} \left[p \arctan\frac{p}{q} + \frac{q}{2} \ln(p^2 + q^2) + q\mathcal{C}\right], \\ \int_0^\infty e^{-qx} \ln(x) dx &= -\frac{1}{q}(\mathcal{C} + \ln q),\end{aligned} \tag{42}$$

where $\mathcal{C}$ is the Euler constant.

For the proof of (39), taking $\lambda = 0$ in (40) leads to

$$\begin{aligned}\mathscr{F}[\Delta_m^{1/2}p(\mathbf{X},t)] &= \int_{|\boldsymbol{\phi}|=1}(\mathbf{k}\cdot\boldsymbol{\phi})\frac{\pi}{2}\mathrm{sgn}(\mathbf{k}\cdot\boldsymbol{\phi})\ m(\boldsymbol{\phi})d\boldsymbol{\phi}\cdot\hat{p}(\mathbf{k},t)\\ &= \frac{\pi}{2}\int_{|\boldsymbol{\phi}|=1}|(\mathbf{k}\cdot\boldsymbol{\phi})|\ m(\boldsymbol{\phi})d\boldsymbol{\phi}\cdot\hat{p}(\mathbf{k},t). \qquad \square\end{aligned}$$

Furthermore, if $m(\boldsymbol{\phi})$ is isotropic, then

$$\begin{aligned}\mathscr{F}[\Delta_m^{1/2}p(\mathbf{X},t)] &= \frac{\pi}{2\omega_n}\int_{|\boldsymbol{\phi}|=1}|(\mathbf{k}\cdot\boldsymbol{\phi})|\ d\boldsymbol{\phi}\cdot\hat{p}(\mathbf{k},t) = \frac{\pi}{2\omega_n}|\mathbf{k}|\int_{|\boldsymbol{\phi}|=1}|\cos(\theta_1)|\ d\boldsymbol{\phi}\cdot\hat{p}(\mathbf{k},t)\\ &= \frac{\pi}{2\omega_n}C_n|\mathbf{k}|\int_0^{\pi}\sin^{n-2}(\theta_1)|\cos(\theta_1)|d\theta_1\cdot\hat{p}(\mathbf{k},t)\\ &= \frac{1}{\omega_n}\frac{\pi}{n-1}C_n|\mathbf{k}|\cdot\hat{p}(\mathbf{k},t) = \frac{1}{\omega_n}\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}|\mathbf{k}|\cdot\hat{p}(\mathbf{k},t),\end{aligned}$$

where ω_n is the measure of the n-dimensional unit sphere, $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ if $n \geq 2$ and $\omega_n = 2$ when $n = 1$; the rotation invariance [32, Prop. 3.3] of the integrand is used in the second equality, and $\cos(\theta_1)$ denotes one of the components of vector $\boldsymbol{\phi}$:

$$C_n = \Big(\int_0^{\pi}\sin^{n-3}(\theta_2)d\theta_2\Big)\cdots\Big(\int_0^{\pi}\sin(\theta_{n-2})d\theta_{n-2}\Big)\Big(\int_0^{2\pi}d\theta_{n-1}\Big) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}.$$

Following (40), the Fourier symbol of the new anisotropic tempered fractional Laplacian when $\beta = 1$ is

$$\mathscr{F}[\tilde{\Delta}_m^{1/2,\lambda}] = \int_{|\boldsymbol{\phi}|=1}\Big[(\mathbf{k}\cdot\boldsymbol{\phi})\arctan\left(\frac{\mathbf{k}\cdot\boldsymbol{\phi}}{\lambda(\boldsymbol{\phi})}\right) - \frac{\lambda(\boldsymbol{\phi})}{2}\ln(\lambda(\boldsymbol{\phi})^2+(\mathbf{k}\cdot\boldsymbol{\phi})^2)+\lambda(\boldsymbol{\phi})\ln(\lambda(\boldsymbol{\phi}))\Big]\ m(\boldsymbol{\phi})d\boldsymbol{\phi}.$$

All the discussions above are based on compound Poisson processes with a different probability distribution of jump length for (tempered) Lévy flights, which render the deterministic governing equations with classical first derivative temporally. Instead, the fractional Poisson processes are taken as the renewal process, in which the time interval between each pair of events follows the power law distribution. Then the deterministic governing equations with a Caputo fractional derivative temporally can be derived. More precisely, let $S(t)$ be a nondecreasing subordinator [6] with Laplace exponent s^{α}, $\alpha \in (0, 1)$. Then consider a new process $\mathbf{Z}(t) = \mathbf{X}(E(t))$, where $\mathbf{X}(t)$ is the Lévy process discussed in (17) with Fourier symbol $\Phi_0(\mathbf{k})-1$ and the inverse subordinator $E(t) = \inf\{\tau > 0 : S(\tau) > t\}$. Then similar to [10, eqs. (16) and (17)], we have

$$p_z(\mathbf{Z},t) = \int_0^{\infty}p_x(\mathbf{Z},\tau)p_e(\tau,t)d\tau,$$

where $p_e(\tau, t)$ denotes the PDF of $E(t)$. Performing the Fourier–Laplace transform leads to

$$\tilde{\hat{p}}_z(\mathbf{k},s) = \frac{s^{\alpha-1}}{s^{\alpha}+1-\Phi_0(\mathbf{k})},$$

where the notation $\tilde{\cdot}$ denotes the Laplace transform from t to s. Arranging the terms and performing the inverse Laplace transform, one obtains

$$(43)\qquad {}_0^CD_t^{\alpha}\hat{p}_z(\mathbf{k},t) = (\Phi_0(\mathbf{k})-1)\hat{p}_z(\mathbf{k},t),$$

the only difference of which with (17) is the temporal derivative. Then, as a way

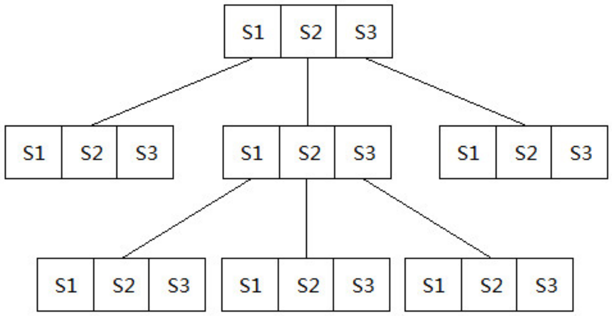


Fig. 3. *Three internal states in each step. Each internal state of $S1$, $S2$, and $S3$ contains different distributions of waiting time ξ and/or jump length η.*

of treating (17), taking the inverse Fourier transform results in the corresponding deterministic equations whose expressions depend on the specific $\nu(d\mathbf{Y})$.

4. Multiple internal states with anisotropic diffusion. Now, we derive the fractional Fokker–Planck and Feynman–Kac equations with multiple internal states both temporally and spatially, with the spatial operators being the anisotropic (tempered) fractional Laplacian $\Delta_m^{\beta/2,\lambda}$ presented in the above section. We first try to make it clear what multiple internal states mean. The motion of particles is characterized by waiting time ξ and jump length η in the CTRW framework. Assume the process only has three different possibilities of distributions of ξ and/or η at each step. We call it three internal states $S1$, $S2$, and $S3$, as in Figure 3. The information contained in each internal state $Si\,(i=1,2,3)$ is the distributions of ξ and η at the current step. More general models may contain more information and more internal states. In one step, each possibility of the three will yield the next step still with three different possibilities. So step after step, a Markov chain is formed. As long as the initial distribution $|\mathrm{init}\rangle$ and transition matrix M are given, the distribution of internal states of nth step can be easily obtained, denoted by $(M^T)^{n-1}|\mathrm{init}\rangle$. Here, the element m_{ij} of the matrix M denotes the transition probability from state i to state j, and the notations bras $\langle\cdot|$ and kets $|\cdot\rangle$ denote the row and column vectors, respectively.

The number of the internal states is taken as N for the fractional Fokker–Planck and Feynman–Kac equations, the derivation processes of which are similar to the ones given in [42]. Here we only provide the derivation of the Fokker–Planck equation. We denote the column vector by capital letter and its components by lowercase letters, e.g., $|G(\mathbf{X},t)\rangle$, with its components $g^i(\mathbf{X},t), i=1,2,\dots,N$ being the PDF of finding the particle, at time t, position $\mathbf{X}$ in n-dimensional space, and internal state i. Then define the waiting time distribution matrix $\Phi(t)=\mathrm{diag}(\phi^1(t),\phi^2(t),\dots,\phi^N(t))$ and the jump length one $\Lambda(\mathbf{X})=\mathrm{diag}(\lambda^1(\mathbf{X}),\lambda^2(\mathbf{X}),\dots,\lambda^N(\mathbf{X}))$, where $\phi^i(t)$ and $\lambda^i(\mathbf{X})$ are, respectively, the PDFs of waiting time and jump length at the ith internal state.

Let $|Q_n(\mathbf{X},t)\rangle$ be composed by $q_n^i(\mathbf{X},t), i=1,2,\dots,N$, representing the PDF of the particle that just arrives at position $\mathbf{X}$, time t, and ith internal state after n steps. Thus, the matrix of survival probability is

$$
\begin{aligned}
W(t) &= \mathrm{diag}\left(w^1(t),\dots,w^N(t)\right)\\
&= \mathrm{diag}\left(\int_t^\infty \phi^1(\tau)d\tau,\dots,\int_t^\infty \phi^N(\tau)d\tau\right) = I-\int_0^t \Phi(\tau)d\tau,
\end{aligned}
$$

where I denotes the identity matrix. This indicates that the Laplace transform of $W(t)$ is

$$\tilde{W}(s) = \frac{I - \tilde{\Phi}(s)}{s}.$$

For G and Q, there exists

$$|G(\mathbf{X}, t)\rangle = \int_0^t W(\tau) \sum_{n=0}^{\infty} |Q_n(\mathbf{X}, t - \tau)\rangle d\tau. \tag{44}$$

On the other hand, for each component q_n^i of Q_n, we have

$$q_n^i(\mathbf{X}, t) = \sum_{j=1}^{N} \int_0^t \int_{\mathbb{R}^n} m_{ji} \Lambda(\mathbf{X} - \mathbf{Y}) \Phi(t - \tau) q_{n-1}^j(\mathbf{Y}, \tau) d\mathbf{Y} d\tau.$$

Thus, Q satisfies

$$|Q_n(\mathbf{X}, t)\rangle = \int_0^t \int_{\mathbb{R}^n} M^T \Lambda(\mathbf{X} - \mathbf{Y}) \Phi(t - \tau) |Q_{n-1}(\mathbf{Y}, \tau)\rangle d\mathbf{Y} d\tau. \tag{45}$$

Taking the Fourier–Laplace transform to (44) and (45) leads to

$$|\hat{\tilde{G}}(\mathbf{k}, s)\rangle = \frac{I - \tilde{\Phi}(s)}{s} [I - M^T \hat{\Lambda}(\mathbf{k}) \tilde{\Phi}(s)]^{-1} |\text{init}\rangle. \tag{46}$$

The Fokker–Planck equation can be obtained by applying the inverse Fourier–Laplace transform on (46). Here, we take the waiting time distributions as asymptotic power laws; i.e., in Laplace space $\tilde{\Phi}(s) \sim I - \text{diag}(s^{\alpha_1}, \dots, s^{\alpha_N}), 0 < \alpha_1, \dots, \alpha_N < 1$. As for jump lengths, they obey the Lévy distributions; i.e., in Fourier space, each component of $\hat{\Lambda}(\mathbf{k})$ is the form of (30) with particular β_i and λ_i. Then, the Fokker–Planck equation with N internal states is

$$\begin{aligned} M^T \frac{\partial}{\partial t} |G(\mathbf{X}, t)\rangle = {} & (M^T - I) \text{diag}(D_t^{1-\alpha_1}, \dots, D_t^{1-\alpha_N}) |G(\mathbf{X}, t)\rangle \\ & + M^T \text{diag}(D_t^{1-\alpha_1} \Delta_m^{\beta_1/2, \lambda_1}, \dots, D_t^{1-\alpha_N} \Delta_m^{\beta_N/2, \lambda_N}) |G(\mathbf{X}, t)\rangle, \end{aligned} \tag{47}$$

where $D_t^{1-\alpha_i}$ is the Riemann–Liouville derivative defined as [36]

$$D_t^{1-\alpha_i} g^i(\mathbf{X}, t) = \frac{1}{\Gamma(\alpha_i)} \frac{\partial}{\partial t} \int_0^t \frac{g^i(\mathbf{X}, \tau)}{(t - \tau)^{1-\alpha_i}} d\tau \tag{48}$$

and $\Delta_m^{\beta_i/2, \lambda_i}$ denotes the anisotropic (tempered) fractional Laplacian with its Fourier transform $\hat{\lambda}^i(\mathbf{k})$.

For the Feynman–Kac equations, we define the functional $A = \int_0^t U(\mathbf{X}(\tau)) d\tau$, where U is a prespecified function. Let $G(\mathbf{X}, A, t)$ be the PDF of the functional A and position $\mathbf{X}$ and $\bar{G}(\mathbf{X}, \rho, t)$ be the Fourier transform from A to ρ. Then the forward Feynman–Kac equation is

$$\begin{aligned} M^T \frac{\partial}{\partial t} |\bar{G}(\mathbf{X}, \rho, t)\rangle = {} & (M^T - I) \text{diag}(\mathcal{D}_t^{1-\alpha_1}, \dots, \mathcal{D}_t^{1-\alpha_N}) |\bar{G}(\mathbf{X}, \rho, t)\rangle \\ & + M^T \text{diag}(\Delta_m^{\beta_1/2, \lambda_1} \mathcal{D}_t^{1-\alpha_1}, \dots, \Delta_m^{\beta_N/2, \lambda_N} \mathcal{D}_t^{1-\alpha_N}) |\bar{G}(\mathbf{X}, \rho, t)\rangle + i\rho U(\mathbf{X}) M^T |\bar{G}(\mathbf{X}, \rho, t)\rangle, \end{aligned}$$

where

$$\mathcal{D}_t^{1-\alpha_i}\bar{g}^i(\mathbf{X},\rho,t)=\frac{1}{\Gamma(\alpha_i)}\left(\frac{\partial}{\partial t}-i\rho U(\mathbf{X})\right)\int_0^t\frac{e^{i(t-\tau)\rho U(\mathbf{X})}}{(t-\tau)^{1-\alpha_i}}\bar{g}^i(\mathbf{X},\rho,\tau)d\tau,$$

and the backward version is

$$\begin{aligned}M^T\frac{\partial}{\partial t}|\bar{G}_{\mathbf{X}_0}(\rho,t)\rangle&=(M^T-I)\mathrm{diag}(\mathcal{D}_t^{1-\alpha_1},\dots,\mathcal{D}_t^{1-\alpha_N})|\bar{G}_{\mathbf{X}_0}(\rho,t)\rangle\\&+M^T\mathrm{diag}(\mathcal{D}_t^{1-\alpha_1}\Delta_{m,\mathbf{X}_0}^{\beta_1/2,\lambda_1},\dots,\mathcal{D}_t^{1-\alpha_N}\Delta_{m,\mathbf{X}_0}^{\beta_N/2,\lambda_N})|\bar{G}_{\mathbf{X}_0}(\rho,t)\rangle+i\rho U(\mathbf{X}_0)M^T|\bar{G}_{\mathbf{X}_0}(\rho,t)\rangle.\end{aligned}$$

5. Generalized boundary conditions. In this section, we mainly consider the initial and boundary value problems with the anisotropic tempered fractional Laplacian. The case for the anisotropic fractional Laplacian can be obtained by taking $\lambda=0$. Following the ideas of [10, 12, 15], the local boundary $\partial\Omega$ itself cannot be hit by the majority of discontinuous sample trajectories; based on this physical implication, these problems should be specified the generalized Dirichlet- and Neumann-type boundary conditions. For the sake of simplicity, we only discuss the anisotropic tempered fractional Laplacian $\Delta_m^{\beta/2,\lambda}p(\mathbf{X},t)$ defined in (32); i.e., λ and β are constant:

$$\Delta_m^{\beta/2,\lambda}p(\mathbf{X},t)=\frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\backslash\{0\}}[p(\mathbf{X}-\mathbf{Y})-p(\mathbf{X})]\frac{m(\mathbf{Y})}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}}d\mathbf{Y}. \tag{49}$$

Consider the time-dependent Dirichlet problem

$$\begin{cases}\dfrac{\partial p(\mathbf{X},t)}{\partial t}-\Delta_m^{\beta/2,\lambda}p(\mathbf{X},t)=f(\mathbf{X},t) & \text{in }\ \Omega,\\ p(\mathbf{X},t)=g(\mathbf{X},t) & \text{in }\ \mathbb{R}^n\backslash\Omega,\\ p(\mathbf{X},0)=p_0(\mathbf{X}) & \text{in }\ \Omega\end{cases} \tag{50}$$

and the Neumann problem

$$\begin{cases}\dfrac{\partial p(\mathbf{X},t)}{\partial t}-\Delta_m^{\beta/2,\lambda}p(\mathbf{X},t)=f(\mathbf{X},t) & \text{in }\ \Omega,\\ \Delta_m^{\beta/2,\lambda}p(\mathbf{X},t)=g(\mathbf{X},t) & \text{in }\ \mathbb{R}^n\backslash\Omega,\\ p(\mathbf{X},0)=p_0(\mathbf{X}) & \text{in }\ \Omega.\end{cases} \tag{51}$$

Remark 5.1. If we consider the model with a Caputo fractional derivative in time, like (43), its Dirichlet problem can be similarly formulated as above, while its Neumann problem should be

$$\begin{cases}{}_0^CD_t^{\alpha}p(\mathbf{X},t)-\Delta_m^{\beta/2,\lambda}p(\mathbf{X},t)=f(\mathbf{X},t) & \text{in }\ \Omega,\\ D_t^{1-\alpha}\Delta_m^{\beta/2,\lambda}p(\mathbf{X},t)=g(\mathbf{X},t) & \text{in }\ \mathbb{R}^n\backslash\Omega,\\ p(\mathbf{X},0)=p_0(\mathbf{X}) & \text{in }\ \Omega,\end{cases} \tag{52}$$

where $D_t^{1-\alpha}$ is the Riemann-Liouville derivative defined in (48). It should be noted that the Neumann boundary condition $g(\mathbf{X},t)$ is time dependent in both (51) and (52), meaning that the numerical flux of diffusing particles across the boundary $\partial\Omega$ is time dependent.

Remark 5.2. For the problem (51) with homogeneous Neumann boundary conditions $g = 0$ and source term $f = 0$, if the directional measure $m(\mathbf{Y})$ is symmetric, we can prove the property of conservation of mass inside Ω.

More specifically, from the symmetry of $m(\mathbf{Y})$, we have

$$\begin{aligned}&\iint_{\Omega\times\Omega} \frac{p(\mathbf{X}) - p(\mathbf{Y})}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} m(\mathbf{X}-\mathbf{Y})d\mathbf{X}d\mathbf{Y} \\ &= \iint_{\Omega\times\Omega} \frac{p(\mathbf{Y}) - p(\mathbf{X})}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} m(\mathbf{X}-\mathbf{Y})d\mathbf{X}d\mathbf{Y} = 0.\end{aligned}$$

Therefore, for (51) with $f = g = 0$,

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega p\,d\mathbf{X} &= \int_\Omega \Delta_m^{\beta/2,\lambda} p(\mathbf{X})\,d\mathbf{X} \\ &= -\frac{1}{|\Gamma(-\beta)|}\int_\Omega\int_{\mathbb{R}^n} \frac{p(\mathbf{X}) - p(\mathbf{Y})}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} m(\mathbf{X}-\mathbf{Y})d\mathbf{Y}d\mathbf{X} \\ &= -\frac{1}{|\Gamma(-\beta)|}\int_\Omega\int_{\mathbb{R}^n\setminus\Omega} \frac{p(\mathbf{X}) - p(\mathbf{Y})}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} m(\mathbf{X}-\mathbf{Y})d\mathbf{Y}d\mathbf{X} \\ &= -\frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\setminus\Omega}\int_\Omega \frac{p(\mathbf{X}) - p(\mathbf{Y})}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} m(\mathbf{X}-\mathbf{Y})d\mathbf{X}d\mathbf{Y} \\ &= -\frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\setminus\Omega}\int_{\mathbb{R}^n} \frac{p(\mathbf{X}) - p(\mathbf{Y})}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} m(\mathbf{X}-\mathbf{Y})d\mathbf{X}d\mathbf{Y} \\ &= -\frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n\setminus\Omega} \Delta_m^{\beta/2,\lambda} p(\mathbf{Y})\,d\mathbf{Y} = 0.\end{aligned}$$

Thus, the quantity $\int_\Omega p d\mathbf{X}$ does not depend on t, which means the conservation of mass inside Ω.

Based on the definition of $\Delta_m^{\beta/2,\lambda} p(\mathbf{X},t)$ in (49), there is no need for the solution $p(\mathbf{X},t)$ to vanish at infinity. To guarantee the convergence of the integral in (49), the solution $p(\mathbf{X},t)$ should satisfy that there exist positive M and C such that when $|\mathbf{X}| > M$,

$$|p(\mathbf{X},t)| < Ce^{(\lambda-\epsilon)|\mathbf{X}|} \qquad \text{for positive small } \epsilon.$$

This is an essential difference from Riesz fractional derivatives [43], which must vanish at infinity. A special example is that $p(\mathbf{X},t) \equiv 1$ and $\Delta_m^{\beta/2,\lambda} 1 \equiv 0$. Indeed, that $p(\mathbf{X},t)$ does not vanish at infinity still has some clear physical meaning, e.g., escape probability [11]. Considering the case of $\beta = 2$ in (30), we have

$$\mathscr{F}[\Delta_m^{1,\lambda} p(\mathbf{X},t)] = \int_{|\boldsymbol{\phi}|=1} \Big[-(\mathbf{k}\cdot\boldsymbol{\phi})^2 - 2\lambda(i\mathbf{k}\cdot\boldsymbol{\phi})\Big] m(\boldsymbol{\phi})d\boldsymbol{\phi}\cdot\hat{p}(\mathbf{k},t). \tag{53}$$

In this case, $m(\boldsymbol{\phi})$ determines the covariance matrix $\mathbf{a}$ in (2) [26]. If $m(\boldsymbol{\phi})$ is symmetric, the term containing $i\mathbf{k}$, corresponding to the first-order derivative, vanishes. If not, from (53),

$$\mathscr{F}[\Delta_m^{1,\lambda} p(\mathbf{X},t)] = \big((i\mathbf{k})^T A(i\mathbf{k}) - 2\lambda(i\mathbf{k})^T\mathbf{b}\big)\,\hat{p}(\mathbf{k},t), \tag{54}$$

where the matrix $A = (a_{ij})_{n\times n}$ with $a_{ij} = \int_{|\boldsymbol{\phi}|=1} \boldsymbol{\phi}_i\boldsymbol{\phi}_j\, m(\boldsymbol{\phi})d\boldsymbol{\phi}$ and the vector $\mathbf{b} = (b_j)_{n\times 1}$ with $b_j = \int_{|\boldsymbol{\phi}|=1} \boldsymbol{\phi}_j\, m(\boldsymbol{\phi})d\boldsymbol{\phi}$. This implies

$$\Delta_m^{1,\lambda} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial \mathbf{X}_i\partial \mathbf{X}_j} + 2\lambda\sum_{j=1}^{n} b_j\frac{\partial}{\partial \mathbf{X}_j}. \tag{55}$$

Then the weak solution $p \in H^1(\mathbb{R}^n)$ of (51) satisfies, for all $q \in H^1(\mathbb{R}^n)$,

$$\int_\Omega \frac{\partial p}{\partial t}qd\mathbf{X}+\int_{\mathbb{R}^n}\sum_{i,j=1}^{n} a_{ij}\frac{\partial p}{\partial \mathbf{X}_i}\frac{\partial q}{\partial \mathbf{X}_j}d\mathbf{X}-2\lambda\int_{\mathbb{R}^n}\sum_{j=1}^{n} b_j\frac{\partial p}{\partial \mathbf{X}_j}\,qd\mathbf{X} = \int_\Omega fqd\mathbf{X}-\int_{\mathbb{R}^n\backslash\Omega} gqd\mathbf{X}.$$

For the Neumann boundary conditions in (51), we have

$$\begin{aligned}\int_{\mathbb{R}^n\backslash\Omega} gqd\mathbf{X} &= \int_{\mathbb{R}^n\backslash\Omega}\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 p}{\partial \mathbf{X}_i\partial \mathbf{X}_j}\,qd\mathbf{X} + 2\lambda\int_{\mathbb{R}^n\backslash\Omega}\sum_{j=1}^{n} b_j\frac{\partial p}{\partial \mathbf{X}_j}\,qd\mathbf{X}\\ &= -\int_{\partial\Omega}\sum_{i,j=1}^{n} a_{ij}\frac{\partial p}{\partial \mathbf{n}_i}\,qds - \int_{\mathbb{R}^n\backslash\Omega}\sum_{i,j=1}^{n} a_{ij}\frac{\partial p}{\partial \mathbf{X}_i}\frac{\partial q}{\partial \mathbf{X}_j}d\mathbf{X} + 2\lambda\int_{\mathbb{R}^n\backslash\Omega}\sum_{j=1}^{n} b_j\frac{\partial p}{\partial \mathbf{X}_j}\,qd\mathbf{X}.\end{aligned}$$

Then

$$\begin{aligned}&\int_\Omega \frac{\partial p}{\partial t}qd\mathbf{X} + \int_\Omega\sum_{i,j=1}^{n} a_{ij}\frac{\partial p}{\partial \mathbf{X}_i}\frac{\partial q}{\partial \mathbf{X}_j}d\mathbf{X} - 2\lambda\int_\Omega\sum_{j=1}^{n} b_j\frac{\partial p}{\partial \mathbf{X}_i}\,qd\mathbf{X}\\ &= \int_\Omega fqd\mathbf{X} + \int_{\partial\Omega}\sum_{i,j=1}^{n} a_{ij}\frac{\partial p}{\partial \mathbf{n}_i}\,qds,\end{aligned}$$

which means that the usual Neumann boundary condition is recovered. Similarly, for the Dirichlet boundary condition in (50), when $\beta = 2$, $\Delta_m^{1,\lambda}$ becomes a local operator. Then only the information of $g(\mathbf{X}, t)$ on the boundary $\partial\Omega$ is used to solve the problem, implying that the usual Dirichlet boundary condition is recovered.

6. Well-posedness and regularity. Here we show the well-posedness of the problems provided in the previous section. First, we define the fractional Sobolev space for $s \in (0, 1)$,

$$H^s(\Omega) := \left\{v \in L^2(\Omega) : |v|_{H^s(\Omega)} < \infty\right\},$$

where

$$|v|_{H^s(\Omega)} = \left(\iint_{\Omega\times\Omega}\frac{(v(x)-v(y))^2}{|x-y|^{n+2s}}dxdy\right)^{1/2}$$

is the Aronszajn–Slobodeckij seminorm. The space $H^s(\Omega)$ is a Banach space, endowed with the norm

$$\|v\|_{H^s(\Omega)} := \left(\|v\|^2_{L^2(\Omega)} + |v|^2_{H^s(\Omega)}\right)^{1/2}.$$

Equivalently, the space $H^s(\Omega)$ can be regarded as the restriction to Ω of functions in $H^s(\mathbb{R}^n)$. We define $H^s_0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. Consider the space

$$\tilde{H}^s_0(\Omega) = \{v \in H^s(\mathbb{R}^n) :\ v = 0 \text{ in } \mathbb{R}^n\backslash\Omega\}$$

equipped with the $H^s(\mathbb{R}^n)$ norm. The dual space of $\tilde{H}^s_0(\Omega)$ is denoted by $H^{-s}(\Omega)$ or $\tilde{H}^s_0(\Omega)'$.

If $g \in L^2(0,T;H^{\beta/2}(\mathbb{R}^n)) \cap H^1(0,T;H^{-\beta/2}(\mathbb{R}^n))$ and $f \in L^2(0,T;H^{-\beta/2}(\Omega))$, then the weak formulation of (50) is to find $p = \tilde{p}+g$ such that $\tilde{p} \in L^2(0,T;\tilde{H}^{\beta/2}_0(\Omega)) \cap H^1(0,T;H^{-\beta/2}(\Omega)) \hookrightarrow C([0,T];L^2(\Omega))$ and

$$\int_0^T\int_\Omega \partial_t\tilde{p}\, q\, d\mathbf{X}dt + \frac{1}{2|\Gamma(-\beta)|}\int_0^T a(\tilde{p},q)dt = \int_0^T\int_\Omega (f + \Delta_m^{\beta/2,\lambda}g - \partial_t g)\, q\, d\mathbf{X}dt \tag{56}$$

for all $q \in L^2(0,T;\tilde{H}^{\beta/2}_0(\Omega))$, where

$$\begin{aligned} a(\tilde{p},q) &= 2|\Gamma(-\beta)|\left(-\Delta_m^{\beta/2,\lambda}\tilde{p}, q\right) \\ &= 2\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(\tilde{p}(\mathbf{X})-\tilde{p}(\mathbf{Y}))}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} q(\mathbf{X})m(\mathbf{X}-\mathbf{Y})d\mathbf{X}d\mathbf{Y} \\ &= 2\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(\tilde{p}(\mathbf{Y})-\tilde{p}(\mathbf{X}))}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} q(\mathbf{Y})m(\mathbf{Y}-\mathbf{X})d\mathbf{X}d\mathbf{Y} \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(\tilde{p}(\mathbf{X})-\tilde{p}(\mathbf{Y}))(q(\mathbf{X})m(\mathbf{X}-\mathbf{Y})-q(\mathbf{Y})m(\mathbf{Y}-\mathbf{X}))}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}}d\mathbf{X}d\mathbf{Y}. \end{aligned} \tag{57}$$

To show the well-posedness of the weak formulation (56), the main task is to prove the continuity and coercivity of bilinear form $a(\tilde{p},q)$ [18, 45], while $l(q) := \int_\Omega (f + \Delta_m^{\beta/2,\lambda}g - \partial_t g)\, q\, d\mathbf{X}$ is a continuous linear functional on $L^2(0,T;\tilde{H}^{\beta/2}_0(\Omega))$ evidently. Here, the bilinear form $a(\tilde{p},q)$ is based on (32). For (34), the bilinear form becomes a little bit complex. But the well-posedness is still valid since we mainly prove it in Fourier space.

Lemma 3. *The bilinear form $a(p,q)$ is continuous on $H^{\beta/2}(\mathbb{R}^n)\times H^{\beta/2}(\mathbb{R}^n)$.*

Proof. We prove the continuity in the Fourier space. Using the Parseval equality and Theorem 1, we have

$$\begin{aligned} a(p,q) &= 2|\Gamma(-\beta)|(\mathscr{F}[-\Delta_m^{\beta/2,\lambda}p], \mathscr{F}[q]) \\ &= 2\Gamma(-\beta)\int_{\mathbb{R}^n}\int_{|\boldsymbol{\phi}|=1}\left(\lambda^\beta - (\lambda^2+(\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\beta/2}e^{-i\beta\eta}\right) m(\boldsymbol{\phi})d\boldsymbol{\phi}\ \hat{p}(\mathbf{k})\overline{\hat{q}(\mathbf{k})}d\mathbf{k}, \end{aligned} \tag{58}$$

where $\eta = \arctan\left(\frac{\mathbf{k}\cdot\boldsymbol{\phi}}{\lambda}\right)$. Then because of $(\lambda^2+|\mathbf{k}\cdot\boldsymbol{\phi}|^2)^{\beta/2} \le 2^{\beta/2}(\lambda^\beta + |\mathbf{k}\cdot\boldsymbol{\phi}|^\beta)$,

$$\begin{aligned} |a(p,q)| &\le C\int_{\mathbb{R}^n}\int_{|\boldsymbol{\phi}|=1}(1+|\mathbf{k}\cdot\boldsymbol{\phi}|^\beta)m(\boldsymbol{\phi})d\boldsymbol{\phi}\ |\hat{p}(\mathbf{k})||\hat{q}(\mathbf{k})|d\mathbf{k} \\ &\le C\int_{\mathbb{R}^n}\int_{|\boldsymbol{\phi}|=1}(1+|\mathbf{k}|^\beta)m(\boldsymbol{\phi})d\boldsymbol{\phi}\ |\hat{p}(\mathbf{k})||\hat{q}(\mathbf{k})|d\mathbf{k} \\ &= C\int_{\mathbb{R}^n}(1+|\mathbf{k}|^\beta)\ |\hat{p}(\mathbf{k})||\hat{q}(\mathbf{k})|d\mathbf{k} \\ &= C\|p\|_{H^{\beta/2}(\mathbb{R}^n)}\cdot\|q\|_{H^{\beta/2}(\mathbb{R}^n)}, \end{aligned} \tag{59}$$

which completes the proof. ∎

Before proving the coercivity of the bilinear form $a(q,q)$, we show a lemma first. Because of the Parseval equality, there exists

$$a(q,q) = 2\Gamma(-\beta)\int_{\mathbb{R}^n} d(\mathbf{k})\ |\hat{q}(\mathbf{k})|^2 d\mathbf{k}, \tag{60}$$

where $\eta = \arctan(\frac{\mathbf{k}\cdot\boldsymbol{\phi}}{\lambda})$ and $d(\mathbf{k}) = \int_{|\boldsymbol{\phi}|=1}(\lambda^\beta - (\lambda^2 + (\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\beta/2}e^{-i\beta\eta})\, m(\boldsymbol{\phi})d\boldsymbol{\phi}$. Thus, the complex conjugate of $d(\mathbf{k})$ satisfies $\overline{d(\mathbf{k})} = d(-\mathbf{k})$, which implies that $\Im[d(\mathbf{k})]$ is an odd function. On the other hand, since $\hat{q}(\mathbf{k}) = \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{X}}q(\mathbf{X})d\mathbf{X}$ and $q(\mathbf{X})$ is a real function, we have that $\hat{q}(\mathbf{k}) = \overline{\hat{q}(-\mathbf{k})}$ and $|\hat{q}(\mathbf{k})|^2$ is an even function by

$$|\hat{q}(\mathbf{k})|^2 = \hat{q}(\mathbf{k})\overline{\hat{q}(\mathbf{k})} = \overline{\hat{q}(-\mathbf{k})}\hat{q}(-\mathbf{k}) = |\hat{q}(-\mathbf{k})|^2.$$

Therefore, $\Im[a(q,q)] = 0$ and

$$a(q,q) = 2\Gamma(-\beta)\int_{\mathbb{R}^n}\int_{|\boldsymbol{\phi}|=1}\left(\lambda^\beta - (\lambda^2 + (\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\beta/2}\cos(\beta\eta)\right) m(\boldsymbol{\phi})d\boldsymbol{\phi}\ |\hat{q}(\mathbf{k})|^2 d\mathbf{k}. \tag{61}$$

For the isotropic case, $m(\boldsymbol{\phi})$ is a constant, and

$$\mathscr{F}[-\Delta^{\beta/2,\lambda}q(\mathbf{X})] = \frac{(-1)^{\lceil\beta\rceil}}{\omega_n}\int_{|\boldsymbol{\phi}|=1}\left(\lambda^\beta - (\lambda^2 + (\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\frac{\beta}{2}}\cos(\beta\eta)\right) d\boldsymbol{\phi}\cdot\hat{q}(\mathbf{k}). \tag{62}$$

In the following, we show that under some reasonable assumptions on $m(\boldsymbol{\phi})$, there exists a constant $C > 0$ such that

$$\Re[\mathscr{F}[-\Delta_m^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})}] \geq C\mathscr{F}[-\Delta^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})} \qquad \forall\,\mathbf{k}\in\mathbb{R}^n, \tag{63}$$

where

$$\Re[\mathscr{F}[-\Delta_m^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})}] = (-1)^{\lceil\beta\rceil}\int_{|\boldsymbol{\phi}|=1}\left(\lambda^\beta - (\lambda^2+(\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\frac{\beta}{2}}\cos(\beta\eta)\right) m(\boldsymbol{\phi})d\boldsymbol{\phi}\cdot|\hat{q}(\mathbf{k})|^2.$$

DEFINITION 4. *The directional measure $m(\boldsymbol{\phi})$ on the unit sphere in $\mathbb{R}^n$ is said to be nondegenerate if the set $A_m(\boldsymbol{\phi}) := \{\boldsymbol{\phi}; m(\boldsymbol{\phi}) \neq 0\}$ can span the whole space $\mathbb{R}^n$.*

LEMMA 5. *Let $\beta \in (0,1)\cup(1,2)$. For the operator $-\Delta_m^{\beta/2,\lambda}$, the nondegeneration of the directional measure $m(\boldsymbol{\phi})$ on the unit sphere is equivalent to* (63)*.*

Proof. Denote $f(\mathbf{k}\cdot\boldsymbol{\phi}) = (-1)^{\lceil\beta\rceil}(\lambda^\beta - (\lambda^2 + (\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\frac{\beta}{2}}\cos(\beta\eta))$. Then $f' \geq 0$ and $f_{\min} = f(0) = 0$ [46, App.], which implies that $\mathscr{F}[-\Delta^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})} \geq 0$ and $\Re[\mathscr{F}[-\Delta_m^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})}] \geq 0$. If $\mathbf{k} = \mathbf{0}$, then (63) holds. If $\mathbf{k} \neq \mathbf{0}$, then (63) is equivalent to

$$\frac{\Re[\mathscr{F}[-\Delta_m^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})}]}{\mathscr{F}[-\Delta^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})}} \geq C > 0 \qquad \forall\,\mathbf{k}\in\mathbb{R}^n.$$

First, we prove the sufficiency. If the probability density function $m(\boldsymbol{\phi})$ is degenerate, i.e., $\mathrm{span}\{A_m(\boldsymbol{\phi})\}$ is the strict subspace of $\mathbb{R}^n$, then there exists $\mathbb{Q}$ being the orthogonal complement of $\mathrm{span}\{A_m(\boldsymbol{\phi})\}$ in $\mathbb{R}^n$, satisfying $\forall\mathbf{k}\in\mathbb{Q}$ and $\forall\boldsymbol{\phi}\in A_m(\boldsymbol{\phi})$, $(\mathbf{k}\cdot\boldsymbol{\phi}) = 0$. In this case, there exist $\mathbf{k},\boldsymbol{\phi}\in\mathbb{Q}\subset\mathbb{R}^n$ s.t. $(\mathbf{k}\cdot\boldsymbol{\phi}) > 0$. It means that $\mathscr{F}[-\Delta^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})} > 0$ but $\Re[\mathscr{F}[-\Delta_m^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})}] = 0$. Then (63) does not hold.

On the contrary, for necessity, we assume that $m(\boldsymbol{\phi})$ is nondegenerate. If $q(\mathbf{X})$ does not equal zero but $\Re[\mathscr{F}[-\Delta_m^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})}] = 0$, then for any $\boldsymbol{\phi}$ and $\mathbf{k}$, $f(\mathbf{k}\cdot\boldsymbol{\phi})m(\boldsymbol{\phi}) = 0$ almost everywhere. Since $m(\boldsymbol{\phi})$ is nondegenerate, $\mathbf{k}$ must be orthogonal to the space $\mathrm{span}\{A_m(\boldsymbol{\phi})\}(=\mathbb{R}^n)$. So $\mathbf{k}$ must be a zero vector, which means that $\Re[\mathscr{F}[-\Delta_m^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})}] = 0$ has the only zero point $\mathbf{k} = \mathbf{0}$ if $q(\mathbf{X})$ is not zero. By a simple calculation, both $\Re[\mathscr{F}[-\Delta_m^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})}]$ and $\mathscr{F}[-\Delta^{\beta/2,\lambda}q(\mathbf{X})]\cdot\overline{\hat{q}(\mathbf{k})}$ are $\mathcal{O}(|\mathbf{k}|^2)$ when $|\mathbf{k}|\to 0$ and $\mathcal{O}(|\mathbf{k}|^\beta)$ when $|\mathbf{k}|\to\infty$. Then (63) holds. ∎

LEMMA 6. *Let $q \in \tilde{H}_0^{\beta/2}(\Omega)$. If the directional measure $m(\boldsymbol{\phi})$ is nondegenerate, then the bilinear form $a(q,q) \geq C|q|^2_{H^{\beta/2}(\mathbb{R}^n)}$; i.e., it is coercive in $H^{\beta/2}(\mathbb{R}^n)$.*

Proof. The coercivity is proved in two steps. The first step is to show that $a(q,q)$ can bound the bilinear form $\tilde{a}(q,q)$ with isotropic $m(\boldsymbol{\phi})$, i.e., $a(q,q) \geq C\tilde{a}(q,q)$, where

$$\tilde{a}(p,q) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(p(\mathbf{X}) - p(\mathbf{Y}))(q(\mathbf{X}) - q(\mathbf{Y}))}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} d\mathbf{X}d\mathbf{Y}. \tag{64}$$

In the second step, we prove that $\tilde{a}(q,q)$ can be bounded by the norm $\|q\|^2_{H^{\beta/2}(\mathbb{R}^n)}$.

In the first step, we prove it in the Fourier space like Lemma 3. It suffices to prove that there exists a positive constant C such that

$$\Re[\mathscr{F}[-\Delta_m^{\beta/2,\lambda} q(\mathbf{X})] \cdot \overline{\hat{q}(\mathbf{k})}] \geq C\mathscr{F}[-\Delta^{\beta/2,\lambda} q(\mathbf{X})] \cdot \overline{\hat{q}(\mathbf{k})} \qquad \forall\, \mathbf{k} \in \mathbb{R}^n, \tag{65}$$

which can be guaranteed if $m(\boldsymbol{\phi})$ is nondegenerate from Lemma 5. See [17, eq. (5.11)] for some specific expressions of $m(\boldsymbol{\phi})$, where the two-dimensional case is discussed but without tempering. In the second step, we adopt the common technique of splitting the nonlocal seminorm into "near" and "far" pieces in the nonlocal literatures [16, 25, 46]. More precisely, take a sufficiently big ball B_ρ centering at the origin with radius ρ so that $\Omega \subset B_\rho$. Denote $\delta > 0$ as the distance between Ω and ∂B_ρ, $\delta = \inf_{\mathbf{X}\in\Omega, \mathbf{Y}\in\partial B_\rho} |\mathbf{X}-\mathbf{Y}|$. Then for $q \in \tilde{H}_0^{\beta/2}(\Omega)$,

$$\begin{aligned} |q|^2_{H^{\beta/2}(B_\rho)} &\geq \int_\Omega q^2(\mathbf{X}) \int_{B_\rho\backslash\Omega} \frac{1}{|\mathbf{X}-\mathbf{Y}|^{n+\beta}} d\mathbf{Y}d\mathbf{X} \\ &\geq (2\rho)^{-n-\beta}|B_\rho\backslash\Omega| \int_\Omega q^2(\mathbf{X})d\mathbf{X} \\ &= C\|q\|^2_{L^2(\Omega)} = C\|q\|^2_{L^2(\mathbb{R}^n)} \end{aligned} \tag{66}$$

and

$$\begin{aligned} |q|^2_{H^{\beta/2}(\mathbb{R}^n)} &= |q|^2_{H^{\beta/2}(B_\rho)} + 2\int_\Omega \int_{\mathbb{R}^n\backslash B_\rho} \frac{q^2(\mathbf{X})}{|\mathbf{X}-\mathbf{Y}|^{n+\beta}} d\mathbf{Y}d\mathbf{X} \\ &\leq |q|^2_{H^{\beta/2}(B_\rho)} + 2\int_\Omega q^2(\mathbf{X})d\mathbf{X} \int_{\mathbb{R}^n\backslash B_\delta} |\mathbf{Y}|^{-n-\beta} d\mathbf{Y} \\ &= |q|^2_{H^{\beta/2}(B_\rho)} + \frac{2\omega_n \delta^{-\beta}}{\beta}\|q\|^2_{L^2(\Omega)} \\ &\leq C|q|^2_{H^{\beta/2}(B_\rho)}. \end{aligned} \tag{67}$$

Therefore,

$$\begin{aligned} \tilde{a}(q,q) &\geq \iint_{B_\rho\times B_\rho} \frac{(q(\mathbf{X}) - q(\mathbf{Y}))^2}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} d\mathbf{X}d\mathbf{Y} \\ &\geq e^{-2\lambda\rho}|q|^2_{H^{\beta/2}(B_\rho)} \\ &\geq C\|q\|^2_{H^{\beta/2}(\mathbb{R}^n)}. \end{aligned} \tag{68}$$

The proof is completed. ∎

THEOREM 7 (existence and uniqueness of weak solutions). *Let $p_0 \in L^2(\Omega)$, $f \in L^2(0,T; H^{-\beta/2}(\Omega))$, and $g \in L^2(0,T; H^{\beta/2}(\mathbb{R}^n)) \cap H^1(0,T; H^{-\beta/2}(\mathbb{R}^n))$. If the directional measure $m(\boldsymbol{\phi})$ is nondegenerate, there exists a unique weak solution of* (50) *in the sense of* (56).

Proof. The continuity and coercivity of bilinear form $a(\tilde{p}, q)$ have been obtained. Furthermore, $l(q)$ is a continuous linear functional. Then the original initial boundary value problem (50) has a unique solution. ∎

For the Neumann problem (51), first we define the tempered fractional space

$$H^{\beta/2,\lambda}(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) : |v|_{H^{\beta/2,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where the seminorm

$$|v|_{H^{\beta/2,\lambda}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(\mathbf{X}) - v(\mathbf{Y}))^2}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} d\mathbf{X} d\mathbf{Y} \right)^{1/2} \tag{69}$$

and the norm

$$\|v\|_{H^{\beta/2,\lambda}(\mathbb{R}^n)} = \left(\|v\|^2_{L^2(\mathbb{R}^n)} + |v|^2_{H^{\beta/2,\lambda}(\mathbb{R}^n)} \right)^{1/2}. \tag{70}$$

The main difference of the Neumann problem with the Dirichlet one is that essentially it is an unbounded problem. There are also some interesting properties for the operator $\Delta_m^{\beta/2,\lambda}$ defined in unbounded domain, e.g., $\Delta_m^{\beta/2,\lambda} 1 = 0$ for constant 1, which may produce some dedicated/complicated issues for the choice of function spaces, ways of proving the well-posedness, etc. For example, for the bilinear form $a(\cdot,\cdot)$ in (57),

$$|a(p,q)| \nleq C\, |p|_{H^{\beta/2,\lambda}(\mathbb{R}^n)} \cdot |q|_{H^{\beta/2,\lambda}(\mathbb{R}^n)}. \tag{71}$$

In fact, take $n = 1$, $q(x) \equiv 1$ and

$$p(x) = \begin{cases} -1 & x < 0, \\ 0, & x \geq 0, \end{cases} \qquad m(x) = \begin{cases} 0 & x = -1, \\ 1 & x = 1. \end{cases}$$

Then the right-hand side of (71) equals 0, while the left-hand side

$$\begin{aligned} a(p,q) &= 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p(x) - p(y)}{e^{\lambda|x-y|}|x-y|^{1+\beta}} m(\mathrm{sgn}(x-y)) dx dy \\ &= 2 \int_{-\infty}^{\infty} \int_{y}^{\infty} \frac{p(x) - p(y)}{e^{\lambda|x-y|}|x-y|^{1+\beta}} dx dy \\ &= 2 \int_{-\infty}^{0} \int_{0}^{\infty} \frac{1}{e^{\lambda|x-y|}|x-y|^{1+\beta}} dx dy > 0. \end{aligned}$$

In the following, we just focus on the case that the probability density function $m(\mathbf{Y})$ is symmetric. We define the function space, containing the functions that may not vanish at infinity,

$$\mathbb{V} = \{ p \in L^2(\Omega) : |p|_{H_m^{\beta/2,\lambda}(\mathbb{R}^n)} < \infty \}, \tag{72}$$

furnished with the norm

$$\begin{aligned} \|p\|_{\mathbb{V}} &= \left(\|p\|^2_{L^2(\Omega)} + |p|^2_{H_m^{\beta/2,\lambda}(\mathbb{R}^n)} \right)^{1/2}, \\ |p|_{H_m^{\beta/2,\lambda}(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(p(\mathbf{X}) - p(\mathbf{Y}))^2}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} m(\mathbf{X}-\mathbf{Y}) d\mathbf{X} d\mathbf{Y} \right)^{1/2}. \end{aligned} \tag{73}$$

PROPOSITION 8. $\mathbb{V}$ *is a Hilbert space with the norm defined in* (73).

Proof. We first verify that the norm in (73) is well defined. Let $\|p\|_{\mathbb{V}} = 0$. It can be easily obtained that $p = 0$ a.e. in Ω from $\|p\|_{L^2(\Omega)} = 0$. Then from $|p|_{H_m^{\beta/2,\lambda}(\mathbb{R}^n)} = 0$, one gets that $(p(\mathbf{X})-p(\mathbf{Y}))^2 m(\mathbf{X}-\mathbf{Y}) = 0$ a.e. in $\mathbb{R}^n$. Since m is nondegenerate, there exist n linearly independent nonzero vectors $\mathbf{r}_i, i = 1, \ldots, n$, satisfying $m(\mathbf{r}_i) \neq 0$. Therefore, the differences of p along the directions $\mathbf{r}_i$ are zero, that is, $\delta p_i = p(\mathbf{X} + h\mathbf{r}_i) - p(\mathbf{X}) = 0$ for any constant h. Then we consider a normal unit vector ε_i with the ith component being unit. Since the sets $\mathbf{r}_i$ can span the whole n-dimensional space, the difference of p along the direction ε_i is the linear combination of those in direction $\mathbf{r}_i$, i.e.,

$$p(\mathbf{X} + h\varepsilon_i) - p(\mathbf{X}) = \sum_{i=1}^{n} a_i \delta p_i = 0 \quad \text{for any } h, \tag{74}$$

which implies that each component of ∇p is zero. Therefore, $\nabla p = \mathbf{0}$. Combining with $p = 0$ a.e. in Ω, we obtain $p = 0$ a.e. in $\mathbb{R}^n$.

Then we prove that $\mathbb{V}$ is complete by imitating the proof of [14, Prop. 3.1]. Take a Cauchy sequence p_k with respect to the norm in (73). In particular, p_k is a Cauchy sequence in $L^2(\Omega)$, and therefore, up to a subsequence, we suppose that p_k converges to some p in $L^2(\Omega)$ and a.e. in Ω. On the other hand, for any $(\mathbf{X}, \mathbf{Y}) \in \mathbb{R}^{2n}$, define

$$E_{p_k}(\mathbf{X}, \mathbf{Y}) := (p_k(\mathbf{X}) - p_k(\mathbf{Y})) \frac{m^{1/2}(\mathbf{X} - \mathbf{Y})}{e^{\lambda|\mathbf{X}-\mathbf{Y}|/2}|\mathbf{X} - \mathbf{Y}|^{(n+\beta)/2}}. \tag{75}$$

Accordingly, since p_k is a Cauchy sequence in $\mathbb{V}$, for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that if $k, k' \geq N_\varepsilon$, then

$$\begin{aligned}\varepsilon^2 &\geq \int_{\mathbb{R}^{2n}} |(p_k - p_{k'})(\mathbf{X}) - (p_k - p_{k'})(\mathbf{Y})|^2 \frac{m(\mathbf{X} - \mathbf{Y})}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X} - \mathbf{Y}|^{n+\beta}} d\mathbf{X} d\mathbf{Y} \\ &= \|E_{p_k} - E_{p'_k}\|^2_{L^2(\mathbb{R}^{2n})},\end{aligned}$$

which means that E_{p_k} is a Cauchy sequence in $L^2(\mathbb{R}^{2n})$. Up to a subsequence, we assume that E_{p_k} converges to some E in $L^2(\mathbb{R}^{2n})$ and a.e. in $\mathbb{R}^{2n}$.

Fixing $\mathbf{X}_0 \in \Omega$, there exists $\lim_{k\to\infty} p_k(\mathbf{X}_0) = p(\mathbf{X}_0)$; then for any given $\mathbf{Y} \in \mathbb{R}^n \backslash \Omega$, we have that

$$\lim_{k\to\infty} E_{p_k}(\mathbf{X}_0, \mathbf{Y}) = E(\mathbf{X}_0, \mathbf{Y}).$$

Noticing that

$$E_{p_k}(\mathbf{X}_0, \mathbf{Y}) := (p_k(\mathbf{X}_0) - p_k(\mathbf{Y})) \frac{m^{1/2}(\mathbf{X}_0 - \mathbf{Y})}{e^{\lambda|\mathbf{X}_0-\mathbf{Y}|/2}|\mathbf{X}_0 - \mathbf{Y}|^{(n+\beta)/2}},$$

there exists

$$\begin{aligned}\lim_{k\to\infty} p_k(\mathbf{Y}) &= \lim_{k\to\infty} \left(p_k(\mathbf{X}_0) - \frac{e^{\lambda|\mathbf{X}_0-\mathbf{Y}|/2}|\mathbf{X}_0 - \mathbf{Y}|^{(n+\beta)/2}}{m^{1/2}(\mathbf{X}_0 - \mathbf{Y})} E_{p_k}(\mathbf{X}_0, \mathbf{Y}) \right) \\ &= p(\mathbf{X}_0) - \frac{e^{\lambda|\mathbf{X}_0-\mathbf{Y}|/2}|\mathbf{X}_0 - \mathbf{Y}|^{(n+\beta)/2}}{m^{1/2}(\mathbf{X}_0 - \mathbf{Y})} E(\mathbf{X}_0, \mathbf{Y})\end{aligned}$$

for a.e. $\mathbf{Y} \in \mathbb{R}^n \backslash \Omega$. This means that p_k converges to some p a.e. in $\mathbb{R}^n$. So, using that p_k is a Cauchy sequence in $\mathbb{V}$, any fixed $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that, for any $k' \geq N_\varepsilon$,

$$\begin{aligned}
\varepsilon^2 &\geq \liminf_{k\to\infty} \|p_k - p_{k'}\|_{\mathbb{V}}^2 \\
&= \liminf_{k\to\infty} \int_\Omega (p_k - p_{k'})^2 d\mathbf{X} \\
&\quad + \liminf_{k\to\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |(p_k - p_{k'})(\mathbf{X}) - (p_k - p_{k'})(\mathbf{Y})|^2 \frac{m(\mathbf{X}-\mathbf{Y})}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} d\mathbf{X}d\mathbf{Y} \\
&\geq \int_\Omega (p - p_k')^2 d\mathbf{X} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} ((p - p_{k'})(\mathbf{X}) - (p - p_{k'})(\mathbf{Y}))^2 \frac{m(\mathbf{X}-\mathbf{Y})}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} d\mathbf{X}d\mathbf{Y} \\
&= \|p - p_{k'}\|_{\mathbb{V}}^2,
\end{aligned}$$

where Fatou's lemma is used. This says that p_k' converges to p in $\mathbb{V}$, showing that $\mathbb{V}$ is complete. ∎

Then the weak formulation of (51) is to find $p \in L^2(0,T;\mathbb{V}) \cap H^1(0,T;\mathbb{V}')$ satisfying

$$\int_0^T\int_\Omega \frac{\partial p}{\partial t} q d\mathbf{X}dt + \frac{1}{2|\Gamma(-\beta)|}\int_0^T a(p,q)dt = \int_0^T\int_\Omega f\, q d\mathbf{X}dt - \int_0^T\int_{\mathbb{R}^n\setminus\Omega} gq\, d\mathbf{X}dt \tag{76}$$

for all $q \in L^2(0,T;\mathbb{V})$, where

$$a(p,q) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(p(\mathbf{X}) - p(\mathbf{Y}))(q(\mathbf{X}) - q(\mathbf{Y}))}{e^{\lambda|\mathbf{X}-\mathbf{Y}|}|\mathbf{X}-\mathbf{Y}|^{n+\beta}} m(\mathbf{X}-\mathbf{Y}) d\mathbf{X}d\mathbf{Y}. \tag{77}$$

Similar to [10, Thm. 4.2], we have the following.

THEOREM 9 (existence and uniqueness of weak solutions). *Let $p_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$, and $g \in L^2(0,T;\mathbb{V}')$. If $m(\mathbf{Y})$ is nondegenerate, then there exists a unique weak solution of* (51) *in the sense of* (76).

Proof. Let $t_k = k\tau$, $k = 0,1,\ldots,N$, be a partition of the time interval $[0,T]$ with step size $\tau = T/N$, and define

$$f_k(\mathbf{X}) := \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f(\mathbf{X},t)dt, \quad g_k(\mathbf{X}) := \frac{1}{\tau}\int_{t_{k-1}}^{t_k} g(\mathbf{X},t)dt, \quad k = 1,\ldots,N.$$

Then consider the time discrete problem: For a given $p_{k-1} \in \mathbb{V}$, find $p_k \in \mathbb{V}$ such that

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega p_k(\mathbf{X})q(\mathbf{X})d\mathbf{X} + \frac{1}{2|\Gamma(-\beta)|} a(p_k(\mathbf{X}), q(\mathbf{X})) \\
&= \frac{1}{\tau}\int_\Omega p_{k-1}(\mathbf{X})q(\mathbf{X})d\mathbf{X} + \int_\Omega f_k(\mathbf{X})q(\mathbf{X})d\mathbf{X} - \int_{\mathbb{R}^n\setminus\Omega} g_k(\mathbf{X})q(\mathbf{X})d\mathbf{X} \quad \forall q \in \mathbb{V}.
\end{aligned} \tag{78}$$

From the definition of $\mathbb{V}$ in (73), the continuity and coercivity of $a(p,q)$ of the left-hand side of (78) on $\mathbb{V}$ is evident. For the last term on the right-hand side, we define $g(\mathbf{X}) = 0, \mathbf{X} \in \Omega$ for supplementary. Then $g_k(\mathbf{X}) = 0, \mathbf{X} \in \Omega$, and

$$\left|\int_{\mathbb{R}^n\setminus\Omega} g_k(\mathbf{X})q(\mathbf{X})d\mathbf{X}\right| = \left|\int_{\mathbb{R}^n} g_k(\mathbf{X})q(\mathbf{X})d\mathbf{X}\right| \le \|g_k(\mathbf{X})\|_{\mathbb{V}'}\|q(\mathbf{X})\|_{\mathbb{V}}.$$

Thus, the right-hand side of (78) satisfies

$$\begin{aligned}\text{RHS} &\le C\|p_{k-1}\|_{L^2(\Omega)} \cdot \|q\|_{L^2(\Omega)} + \|f_k\|_{L^2(\Omega)} \cdot \|q\|_{L^2(\Omega)} + \|g_k\|_{\mathbb{V}'}\|q\|_{\mathbb{V}} \\ &\le C\Big(\|p_{k-1}\|_{L^2(\Omega)} + \|f_k\|_{L^2(\Omega)} + \|g_k\|_{\mathbb{V}'}\Big) \cdot \|q\|_{\mathbb{V}},\end{aligned}$$

which implies that the right-hand side is a continuous linear functional on $\mathbb{V}$. Therefore, by the Lax–Milgram lemma, there exists a unique solution $p_k \in \mathbb{V}$ for (78). Then using the technique in [10, Thm. 4.2], there exists a unique solution p satisfying (76). ∎

7. Conclusion. This is a companion paper with the latest one [10]. The main generalizations come from five aspects: 1. We show how to derive the macroscopic equations through the Lévy–Khintchine formula for a general anisotropic process in n dimensions. 2. The anisotropic diffusion operators characterizing normal and anomalous diffusion behavior in nonhomogeneous media are proposed. 3. The tempered anisotropic diffusion operators are introduced by two different ways with different motivations, and they are proved to be equivalent. 4. The well-posedness and regularity of the anisotropic diffusion equations are discussed. 5. The models for the anisotropic anomalous diffusion with multiple internal states are built, including the Fokker–Planck and Feynman–Kac equations, respectively, governing the PDF of positions of particles and the PDF of the functional of the particles' trajectories. Wider applications and numerical methods for the newly built various models will be discussed in detail in the near future.

Appendix A. Proof of Theorem 1.

Proof. We mainly prove the equivalence of the anisotropic tempered fractional Laplacian in (34) of Case II to the alternative definition (30). The equivalence of the anisotropic fractional Laplacian of Case I and definition (29) can be obtained similarly. Taking the Fourier transform of the right-hand side of (34) leads to

$$\begin{aligned}&\mathscr{F}\Big[\Delta_m^{\beta/2,\lambda} p(\mathbf{X},t)\Big](\mathbf{k}) \\ &= \frac{1}{|\Gamma(-\beta)|}\int_{\mathbb{R}^n} \frac{e^{i\mathbf{k}\cdot\mathbf{Y}} - 1 - i\mathbf{k}\cdot\mathbf{Y}}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}} m(\mathbf{Y})d\mathbf{Y} \cdot \hat{p}(\mathbf{k},t) \\ &\quad - \frac{1}{|\Gamma(-\beta)|}\Gamma(1-\beta)\lambda^{\beta-1}(-i\mathbf{k}\cdot\mathbf{b})\hat{p}(\mathbf{k},t) \\ &= \frac{1}{|\Gamma(-\beta)|}\left[\int_{\mathbb{R}^n} \frac{\cos(\mathbf{k}\cdot\mathbf{Y}) - 1}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}} m(\mathbf{Y})d\mathbf{Y} + i\int_{\mathbb{R}^n} \frac{\sin(\mathbf{k}\cdot\mathbf{Y}) - \mathbf{k}\cdot\mathbf{Y}}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}} m(\mathbf{Y})d\mathbf{Y}\right] \cdot \hat{p}(\mathbf{k},t) \\ &\quad + \frac{1}{|\Gamma(-\beta)|}\Gamma(1-\beta)\lambda^{\beta-1}\int_{|\boldsymbol{\phi}|=1} (i\mathbf{k}\cdot\boldsymbol{\phi})m(\boldsymbol{\phi})d\boldsymbol{\phi} \cdot \hat{p}(\mathbf{k},t).\end{aligned}$$

Since $\beta \in (1,2)$ in this case, by polar coordinate transformation and integration by parts, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1-\cos(\mathbf{k}\cdot\mathbf{Y})}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}} m(\mathbf{Y})d\mathbf{Y} \\
&= \int_0^\infty \int_{|\boldsymbol{\phi}|=1} r^{-1-\beta} e^{-\lambda r}(1-\cos(r\mathbf{k}\cdot\boldsymbol{\phi}))m(\boldsymbol{\phi})d\boldsymbol{\phi}dr \\
&= \frac{\lambda^2}{(-\beta)(1-\beta)} \int_0^\infty r^{1-\beta} e^{-\lambda r} \int_{|\boldsymbol{\phi}|=1} (1-\cos(r\mathbf{k}\cdot\boldsymbol{\phi}))m(\boldsymbol{\phi})d\boldsymbol{\phi}dr \\
&\quad - \frac{2\lambda}{(-\beta)(1-\beta)} \int_0^\infty r^{1-\beta} e^{-\lambda r} \int_{|\boldsymbol{\phi}|=1} (\mathbf{k}\cdot\boldsymbol{\phi}) \sin(r\mathbf{k}\cdot\boldsymbol{\phi})m(\boldsymbol{\phi})d\boldsymbol{\phi}dr \\
&\quad + \frac{1}{(-\beta)(1-\beta)} \int_0^\infty r^{1-\beta} e^{-\lambda r} \int_{|\boldsymbol{\phi}|=1} (\mathbf{k}\cdot\boldsymbol{\phi})^2 \cos(r\mathbf{k}\cdot\boldsymbol{\phi})m(\boldsymbol{\phi})d\boldsymbol{\phi}dr \\
&= \int_{|\boldsymbol{\phi}|=1} (I_1+I_2+I_3)\ m(\boldsymbol{\phi})d\boldsymbol{\phi}.
\end{aligned}$$

Then using the formulas [21, eq. (3.944(5–6))] and taking $\eta = \arctan\frac{(\mathbf{k}\cdot\boldsymbol{\phi})}{\lambda}$, we get

$$\begin{aligned}
I_1 &= \Gamma(-\beta)\lambda^\beta - \Gamma(-\beta)(\lambda^2+(\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\frac{\beta}{2}-1}\cdot\lambda^2\cos((2-\beta)\eta), \\
I_2 &= -2\Gamma(-\beta)(\lambda^2+(\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\frac{\beta}{2}-1}\cdot(\mathbf{k}\cdot\boldsymbol{\phi})\lambda\sin((2-\beta)\eta), \\
I_3 &= \Gamma(-\beta)(\lambda^2+(\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\frac{\beta}{2}-1}\cdot(\mathbf{k}\cdot\boldsymbol{\phi})^2\cos((2-\beta)\eta),
\end{aligned}$$

which results in

$$I_1+I_2+I_3 = \Gamma(-\beta)\Big(\lambda^\beta - (\lambda^2+(\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\frac{\beta}{2}}\cos(\beta\eta)\Big).$$

Then

$$\int_{\mathbb{R}^n} \frac{\cos(\mathbf{k}\cdot\mathbf{Y})-1}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}} m(\mathbf{Y})d\mathbf{Y} = -\Gamma(-\beta)\int_{|\boldsymbol{\phi}|=1}\Big(\lambda^\beta - (\lambda^2+(\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\frac{\beta}{2}}\cos(\beta\eta)\Big)\ m(\boldsymbol{\phi})d\boldsymbol{\phi}. \tag{79}$$

Similarly,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\sin(\mathbf{k}\cdot\mathbf{Y})-\mathbf{k}\cdot\mathbf{Y}}{e^{\lambda|\mathbf{Y}|}|\mathbf{Y}|^{n+\beta}} m(\mathbf{Y})d\mathbf{Y} \\
&= \int_0^\infty \int_{|\boldsymbol{\phi}|=1} r^{-1-\beta} e^{-\lambda r}(\sin(r\mathbf{k}\cdot\boldsymbol{\phi})-r\mathbf{k}\cdot\boldsymbol{\phi})m(\boldsymbol{\phi})d\boldsymbol{\phi}dr \\
&= \frac{\lambda^2}{(-\beta)(1-\beta)} \int_0^\infty r^{1-\beta} e^{-\lambda r} \int_{|\boldsymbol{\phi}|=1} (\sin(r\mathbf{k}\cdot\boldsymbol{\phi})-r\mathbf{k}\cdot\boldsymbol{\phi})m(\boldsymbol{\phi})d\boldsymbol{\phi}dr \\
&\quad - \frac{2\lambda}{(-\beta)(1-\beta)} \int_0^\infty r^{1-\beta} e^{-\lambda r} \int_{|\boldsymbol{\phi}|=1} (\mathbf{k}\cdot\boldsymbol{\phi})(\cos(r\mathbf{k}\cdot\boldsymbol{\phi})-1)m(\boldsymbol{\phi})d\boldsymbol{\phi}dr \\
&\quad - \frac{1}{(-\beta)(1-\beta)} \int_0^\infty r^{1-\beta} e^{-\lambda r} \int_{|\boldsymbol{\phi}|=1} (\mathbf{k}\cdot\boldsymbol{\phi})^2 \sin(r\mathbf{k}\cdot\boldsymbol{\phi})m(\boldsymbol{\phi})d\boldsymbol{\phi}dr \\
&= -\Gamma(-\beta)\int_{|\boldsymbol{\phi}|=1} (\lambda^2+(\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\frac{\beta}{2}}\sin(\beta\eta)\ m(\boldsymbol{\phi})d\boldsymbol{\phi} \\
&\quad - \Gamma(1-\beta)\lambda^{\beta-1}\int_{|\boldsymbol{\phi}|=1}(\mathbf{k}\cdot\boldsymbol{\phi})m(\boldsymbol{\phi})d\boldsymbol{\phi}.
\end{aligned} \tag{80}$$

Combining (79) and (80) leads to the anisotropic tempered fractional Laplacian in Fourier space

$$\begin{aligned}\mathscr{F}[\Delta_m^{\beta/2,\lambda}p(\mathbf{X},t)] &= (-1)^{\lceil\beta\rceil}\int_{|\boldsymbol{\phi}|=1}\left((\lambda^2+(\mathbf{k}\cdot\boldsymbol{\phi})^2)^{\frac{\beta}{2}}e^{-i\beta\eta}-\lambda^\beta\right)m(\boldsymbol{\phi})d\boldsymbol{\phi}\cdot\hat{p}(\mathbf{k},t),\\ &= (-1)^{\lceil\beta\rceil}\int_{|\boldsymbol{\phi}|=1}\left((\lambda-i\mathbf{k}\cdot\boldsymbol{\phi})^\beta-\lambda^\beta\right)m(\boldsymbol{\phi})d\boldsymbol{\phi}\cdot\hat{p}(\mathbf{k},t),\end{aligned}$$

which equals to (30). ∎

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